Robust and Adaptive Boundary Control of a Stretched String on a Moving Transporter

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Abstract—**Suppression of vibration is an important engineering problem. In this note, control problem of a flexible system that includes a stretched string supported on a transporter is defined and solved. Such a system may be encountered in device manufacturing and process automation. Robust and adaptive control is designed to damp out transverse oscillation of the string via compensating for possible uncertainties in string dynamics and transporter motion. Standard robust control design based on a straightforward Lyapunov argument commonly seen in control design for rigid-body systems is extended to the flexible system. Asymptotically/exponentially and robustly stabilizing controls are found.**

Index Terms—**Adaptive control, flexible system, Lyapunov functional, robust control, string system.**

NOMENCLATURE

$\{x_0, y_0, z_0\}, \{x, y, z\}, t$	Inertia frame, coordinate system fixed onto the transporter, and time.
x, dx	Axial coordinate along the equilibrium of the string, and an element along the x axis.
$y(x, t)$	Transverse displacement with respect to the equilibrium of the string (w.r.t. the transporter).
$y_t, y_x, y_{tt}, y_{xt}, y_{xx}$	$(\partial y(x, t))/(\partial t)$, $(\partial y(x, t))/(\partial x)$, $(\partial^2 y(x, t))/(\partial t^2)$, $(\partial^2 y(x, t))/(\partial t \partial x)$, $(\partial^2 y(x, t))/(\partial x^2)$.
$A(x), \rho(x)$	Cross-section area, linear density of the string, and the mass per unit length.
E, l	Elastic modulus, and axial length between supports.
$T_0(x), T(x, t)$	String initial tension, and nonlinear tension in the string.
$y_b(t)$, $\dot{y}_b(t)$ [or $v_b(t)$], $\ddot{y}_b(t)$, and M_b	Position, velocity, acceleration, and mass of the moving transporter.
$p_0(t), p_l(t)$	Positions of the control mechanism (at $x = 0, l$ and of mass M_0, M_l) w.r.t. $\{x, y, z\}$.
b_0, b_l	Dynamic friction coefficients between the control mechanism and the transporter.
$f_0(t), f_l(t)$	Boundary control forces.

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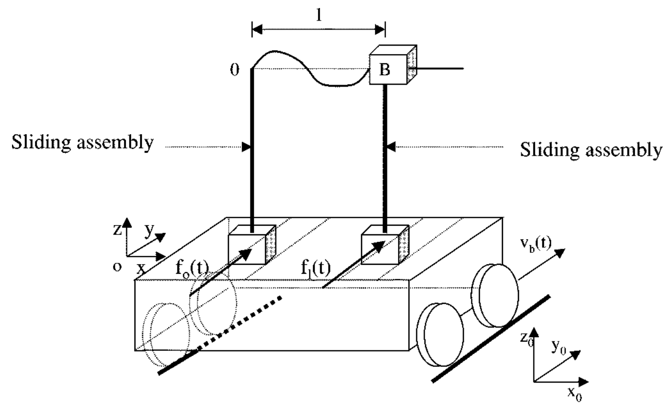

Fig. 1. A stretched string on a transporter.

I. INTRODUCTION

A string is a model that can be used to represent and understand dynamic behavior of many continuous time flexible systems. For example, strings have been used for modeling telephone wires, cables, conveyor belts, and even human DNA. String models and their boundary controls have been studied for decades, for example, [8], [10], [1], [2], [9], and the references cited therein. Although a majority of these results are based on linear models and perfect knowledge, nonlinearities in string dynamics are considered in recent results such as [15] and [16]. Nonlinear models are also used in [4] to design adaptive control that compensates for unknown friction and in [5] to design variable structure modal control. In the case that boundary mass is present or that advanced control schemes (such as adaptive control) are pursued, controls can be designed but more feedback information than boundary velocity are typically required; for instance, those developed in [9], [4] also need boundary slope and boundary slope rate, and that in [5] needs modal displacements and velocities.

This note addresses a general robust and adaptive control problem for string systems. Compared to the existing work, the following advances and extensions are made in the proposed result. First, nonlinear dynamics and their uncertainties are admitted in the model. For example, the string under consideration does not have to be uniform, and its tension can be a nonlinear function of both the transverse gradient and the axial coordinate. To compensate for the nonlinear dynamics and uncertainties, an everywhere-stabilizing[1] robust control is proposed. Second, the proposed robust control design is done by a straightforward Lyapunov argument (parallel to that for rigid-body systems). Third, a new control setting is considered here, in which the string system is supported on a transporter whose motion is uncertain, for which a combined robust and adaptive control is designed. Finally, an adaptive control requiring only boundary velocity feedback is proposed to compensate for unknown dynamic friction. As in the previous results, when boundary mass is present, robust and adaptive controls can be designed, but more boundary feedback information than boundary rates are required.

[1]All controls proposed in this note are stabilizing everywhere in the region in which the nonlinear string model, given by (1), holds.

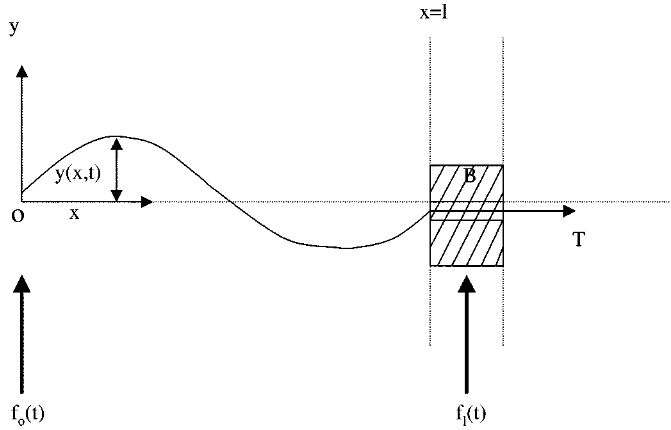

Fig. 2. Transverse vibration of a stretched string.

II. Problem Statement

In this note, a control problem extracted from device manufacturing and process automation is considered. A system under consideration, specifically that sketched in Fig. 1, belongs to the class of nonlinear flexible systems whose main characteristics are those of a string. As shown in the figure, the string system is being moved from one processing station to another on a transporter. The motion of transporter is characterized by a constant cruising speed plus a (possibly uncertain) variation. The establishment of cruising speed and the presence of its variation may cause the string to have transverse vibration, as shown by Fig. 2, where $\{x, y, z\}$ is a fixed frame on the transporter. To suppress the vibration with respect to the inertia frame $\{x_0, y_0, z_0\}$, force control is applied at the two supporting assemblies that are actuated on parallel sliding tracks on the transporter.

A. Dynamics of a String on a Transporter

Dynamic equation that governs the motion of the string system in Fig. 1 can be derived using either continuous limit of a discrete formulation, Hamilton's principle, or Newton's law with a free body diagram. It has been shown in [3] that, assuming small displacements and, thus, keeping Taylor series expansion at the first order, one can obtain the following equation of motion:

$$m(x)y_{tt}(x, t) = \frac{\partial}{\partial x}\left[T(x, t)y_x(x, t)\right]. \tag{1}$$

In the system under consideration, the string is supported and controlled on a moving transporter. As argued in [3] for beam dynamics, the motion equation for the string with a moving base is the same as (1) except that $y(x, t)$ is replaced by $Y_0(x, t) \triangleq y(x, t) + y_b(t)$. Thus, the dynamic equation becomes

$$m(x)\frac{\partial^2 Y_0(x,t)}{\partial t^2} = \frac{\partial}{\partial x}\left\{T(x, t)\frac{\partial Y_0(x,t)}{\partial x}\right\}$$

where $Y_0(x, t)$ is the position of the string in the inertia frame. Since $y_b(t)$ is only a function of time, the string equation modified to account for base motion can also be rewritten as

$$m(x)y_{tt} = \frac{\partial}{\partial x}\left[T(x, t)y_x(x, t)\right] - m(x)\ddot{y}_b(t)$$

which can be rewritten as

$$m(x)\frac{\partial^2 Y(x,t)}{\partial t^2} = \frac{\partial}{\partial x}\left\{T(x, t)\frac{\partial Y(x,t)}{\partial x}\right\} \tag{2}$$

where $Y(x, t) = Y_0(x, t) - c_b t$ for any constant c_b. In essence, a constant cruising speed of the base does not induce any vibration in the string.

It can be assumed that the tension in the string is of form

$$T(x, t) = T_0(x) + w(x)y_x^2(x, t) \tag{3}$$

where $T_0(x) > 0$ is the tension in the undisturbed string, and $w(x) \geq 0$ (for all $x \in [0, l]$) is the weighting that, together with $y_x^2(x, t)$, accounts for the strain in the displaced string. In the case where the uniformity of the string is assumed, and the tension is assumed to be independent of x, we have $T_0(x) = T_0$ and $w(x) = 0.5AE$, which are used in [8] and [9]. If function $w(x)$ is set to be zero, the string tension is a function of only x, and it includes the model in [4] as a special case.

Substituting the tension expression (3) into dynamic model (2) yields

$$\begin{aligned} m(x)y_{tt}(x, t) &- \left[T_0(x) + 3w(x)y_x^2(x, t)\right] y_{xx}(x, t) \\ &- \frac{\partial T_0(x)}{\partial x}y_x(x, t) - \frac{\partial w(x)}{\partial x}y_x^3(x, t) = -m(x)\ddot{y}_b(t) \end{aligned}$$

which provides the detailed expression of the model used in this paper for the system in Fig. 1. The initial conditions for displacement and velocity of the string are

$$y(x, 0) = c_1(x) \quad \text{and} \quad y_t(x, 0) = c_2(x) \tag{4}$$

and boundary conditions needed for solving the above motion equation are

$$y(0, t) = p_0(t), \qquad y(l, t) = p_l(t) \tag{5}$$

where $p_0(t)$, $p_l(t)$, and $y_b(t)$ are described by the following dynamic equations for control mechanism and transporter:

$$M_0[\ddot{p}_0(t) + \ddot{y}_b(t)] = f_0(t) - T(0, t)y_x(0, t) - b_0\dot{p}_0(t) \tag{6}$$

$$M_l[\ddot{p}_l(t) + \ddot{y}_b(t)] = f_l(t) + T(l, t)y_x(l, t) - b_l\dot{p}_l(t) \tag{7}$$

and

$$M_b\ddot{y}_b = \text{sum of all forces exerted onto the transporter.}$$

In (6) and (7), $f_0(t)$ and $f_l(t)$ are the two boundary control forces at points $x = 0, l$ and in the direction of the y axis. It is worth noting that, if $M_0 = M_l = b_0 = b_l = 0$, boundary conditions in (5) for solving equation (2) should be replaced by

$$f_0(t) = T(0, t)y_x(0, t), \quad \text{and} \quad f_l(t) = -T(l, t)y_x(l, t)$$

which was the case studied in the earlier version [14] of this note.

B. Robust and Adaptive Control Problem

Using forces $f_0(t)$ and $f_l(t)$ as the control variables, we define our robust and adaptive control problem in terms of the following assumptions and design objective.

Assumption 1: Motion profile of the transporter can be expressed as

$$\dot{y}_b(t) = c_b + \delta_b(t) \tag{8}$$

where c_b is a constant cruising speed, δ_b represents a speed variation of form

$$\delta_b(t) = \eta_1 \sin(w_b t + \eta_2)$$

w_b is a known oscillation frequency, η_1 and η_2 represent unknown magnitude and phase angle, respectively. In addition, friction coefficients b_0 and b_l are unknown, but constant.

Assumption 2: Functions such as $m(x)$, $T_0(x)$, and $w(x)$ may be uncertain to the control designer, but they are bounded by known, constant lower and upper bounds as follows: for all $x \in [0, l]$

$$\underline{m} \le m(x) \le \overline{m}$$
$$\underline{c}_{T_0} \le T_0(x) \le \overline{c}_{T_0}$$
$$\underline{w} \le w(x) \le \overline{w}.$$

Assumption 3: General size information on partial derivatives of functions $m(x)$, $T_0(x)$ and $w(x)$ with respect to x is available. That is, values of

$$\frac{\partial m(x)}{\partial x} \quad \frac{\partial T_0(x)}{\partial x} \qquad \frac{\partial w(x)}{\partial x}$$

are known to be within a certain range.

Design Objective: Under Assumptions 1–3, find boundary controls $f_0(t)$ and $f_l(t)$ using boundary measurements [including $Y_t(0, t)$ and $Y_t(l, t)$ measured in the inertia frame] such that, with respect to the inertia frame $\{x_0, y_0, z_0\}$, the stretched string will asymptotically (or exponentially) converge to its equilibrium (i.e., $Y(x, \infty) = 0$ for all $x \in [0, l]$).

Remark 2.2.1: It follows from Fourier series expansion that Assumption 1 can be relaxed to admit any unknown, periodic function that has a known oscillation frequency by setting

$$\delta_b(t) = \sum_{j=1}^{j_{\max}} [\eta_{j1} \sin(jw_b t + \eta_{j2}) + \eta_{j3} \cos(jw_b t + \eta_{j4})]$$

where $j_{\max} w_b$ is the maximum frequency worth considering. ⋄

Remark 2.2.2: As can be seen from model (2), constant cruising speed has no steady state impact on string vibration. Unknown base motion defined in (8) could come from imperfectly circular wheels of the transporter, or their actuators, or tracks. If the cruising speed is also changing, boundary control can be designed similarly to compensate directly for its impact on string oscillation. Alternatively, the impact of the short-term transient in establishing a new cruising speed can be embedded into the above design problem through nonzero initial conditions of the string transverse motion. ⋄

C. Robust Control Design Using Knowledge of Transporter Motion

Robust boundary control will be synthesized using Lyapunov's direct method. To this end, consider the following Lyapunov function candidate for the string:

$$V_s(t) = \int_0^l m(x) \Big\{ [y_t(x, t) + \delta_b(t)]^2 + \frac{T_0(x)}{m(x)} y_x^2(x, t) + \frac{w(x)}{2m(x)} y_x^4(x, t) + \frac{\alpha(x)x}{l} \times [y_t(x, t) + \delta_b(t)] y_x(x, t) \Big\} \, dx \tag{9}$$

where its initial condition can be computed using the initial conditions in (4), and $\alpha(x)$ is a positive scalar function satisfying the following inequalities: for all $x \in [0, l]$ and for some constant $\epsilon > 0$

$$x^2 \alpha^2(x) \overline{m} < \underline{c}_{T_0} l^2 \tag{10}$$
$$11\alpha^2(l)\overline{m} < 32\underline{c}_{T_0} \tag{11}$$
$$11\alpha^2(l)\overline{m} \le \left[4\sqrt{2\underline{c}_{T_0}} + \sqrt{32\underline{c}_{T_0} - 11\alpha^2(l)\overline{m}}\right]^2 \tag{12}$$
$$\frac{\partial[\alpha(x)m(x)x]}{\partial x} > \epsilon \tag{13}$$
$$\frac{\partial[\alpha(x)x]}{\partial x} T_0(x) > \alpha(x)x \frac{\partial T_0(x)}{\partial x} + \epsilon \tag{14}$$

and

$$3\frac{\partial[\alpha(x)x]}{\partial x} w(x) > \alpha(x)x\frac{\partial w(x)}{\partial x} + 2\epsilon. \tag{15}$$

Remark 3.1.1: Inequalities (10)–(12) all imply that magnitude of weighting function $\alpha(x)$ in Lyapunov function $V(t)$ should be chosen to be small and be based upon bounds on system parameters or upon bounding functions on system dynamics. Inequalities (13)–(15) can be satisfied if $\alpha(x)$ is chosen to be a highly increasing function. The two sets of inequalities can be simultaneously met by setting $\alpha(x) = \alpha_1 e^{\alpha_2 x}$ with $\alpha_1 > 0$ being sufficiently small and $\alpha_2 > 0$ being large. For example, if $m(x) = m_0 + \Delta m \sin(2\pi x/l)$ with $|\Delta m| \le m_1 < m_0$, inequality (13) can be met by setting $\alpha_2 > 2\pi m_1/[l(m_0 - m_1)]$.⋄

Remark 3.1.2: Inequalities (13)–(15) can be restated as that, through the choice of $\alpha(x)$, functions

$$\alpha(x)m(x)x \quad \frac{\alpha(x)x}{T_0(x)} \quad \text{and} \quad \frac{\alpha^3(x)x^3}{w(x)}$$

are all strictly increasing with respect to x, which can be guaranteed by chosen $\alpha(x)$ provided that, as stated in Assumption 3, such minimum information as general trends of functions $m(x)$, $T_0(x)$ and $w(x)$ are available. In case that $T_0(x)$ and $w(x)$ are constants, inequalities (13)–(15) become trivial by choosing $\alpha(x)$ such that $\alpha(x)m(x)$ and $\alpha(x)$ are nondecreasing. ⋄

The property of Lyapunov function $V_s(t)$ is summarized by the following lemma. The proof of the lemma can easily be done using scalar inequality $a^2 + b^2 \ge 2ab$.

Lemma 1: Under condition (10), Lyapunov function for the string is positive definite with respect to $Y_t(x, t)$ and $Y_x(x, t)$ as

$$V_s(t) \ge \tfrac{1}{2} \min\{\underline{m}, \underline{c}_{T_0}\} \int_0^l \left\{[y_t(x, t) + \delta_b(t)]^2 + y_x^2(x, t)\right\} dx$$

and

$$V_s(t) \le \int_0^l \max\left\{\overline{m} + 0.5\alpha^2(x)\overline{m}, \overline{c}_{T_0} + 0.5\overline{m}, 0.5\overline{w}\right\} \times \left\{[y_t(x, t) + \delta_b(t)]^2 + y_x^2(x, t) + y_x^4(x, t)\right\} dx. \tag{16}$$

It is obvious that, if $V_s(t)$ converges to zero (which can be achieved by making $\dot{V}_s(t)$ negative definite through a control design), the string will be at its equilibrium (which either stands still or moves at a constant speed in the inertia frame) as both $Y_t(x, t) = y_t(x, t) + \delta_b(t)$ and $Y_x(x, t) = y_x(x, t)$ converge to zero for all x.

The proposed results on robust boundary control are summarized by the following theorem and its corollary.

Theorem 1: Consider system (2) with boundary conditions (5). If Assumptions 2 and 3 hold, if there is a scalar function $\alpha(x)$ satisfying inequalities (10)–(15), if parameters η_1, η_2, b_0, and b_l are known, and if $M_0 = M_l = 0$, the following boundary controls are robust and exponentially stabilizing [measured by exponential stability of $V_s(t)$] with respect to the equilibrium of the string:

$$f_0(t) = k_0 Y_t(0, t) + b_0 [Y_t(0, t) - \delta_b(t)]$$
$$f_l(t) = k_l(l) Y_t(l, t) + b_l [Y_t(l, t) - \delta_b(t)] \tag{17}$$

where control gains are $k_0 \ge 0$ and $k_l(l) \in [\underline{k}_l(l), \overline{k}_l(l)]$

$$\underline{k}_l(l) = \frac{4\alpha(l)\overline{m}}{8 + \sqrt{64 - 22\dfrac{\alpha^2(l)\overline{m}}{\underline{c}_{T_0}}}} \tag{18}$$

and

$$\overline{k}_l(l) = \frac{16\underline{c}_{T_0} + \sqrt{256\underline{c}_{T_0}^2 - 88\alpha^2(l)\overline{m}\underline{c}_{T_0}}}{11\alpha(l)}.$$

The stability result holds everywhere in the region in which model (1) is valid.

Proof: It follows from dynamic equation (2) that the time derivative of $V_s(t)$ is

$$\begin{aligned}\dot{V}_s(t) = &\int_0^l \Big\{2m(x)Y_t(x,t)Y_{tt}(x,t) + 2T(x,t)y_x(x,t)\\ &\times y_{xt}(x,t) + \frac{\alpha(x)m(x)x}{l}Y_t(x,t)y_{xt}(x,t)\\ &+\frac{\alpha(x)m(x)x}{l}Y_{tt}(x,t)y_x(x,t)\Big\}\,dx\\ = &\int_0^l \Big\{2Y_t(x,t)\Big[\big[T_0(x)+3w(x)y_x^2(x,t)\big]y_{xx}(x,t)\\ &+\frac{\partial T_0(x)}{\partial x}y_x(x,t) + \frac{\partial w(x)}{\partial x}y_x^3(x,t)\Big]\\ &+2T(x,t)y_x(x,t)y_{xt}(x,t) + \frac{\alpha(x)m(x)x}{l}\\ &\times Y_t(x,t)y_{xt}(x,t) + \frac{\alpha(x)x}{l}y_x(x,t)\\ &\times\Big[\big[T_0(x)+3w(x)y_x^2(x,t)\big]y_{xx}(x,t)\\ &+\frac{\partial T_0(x)}{\partial x}y_x(x,t) + \frac{\partial w(x)}{\partial x}y_x^3(x,t)\Big]\Big\}\,dx\\ = &\int_0^l \Big\{2\frac{\partial\{T(x,t)y_x(x,t)Y_t(x,t)\}}{\partial x} + \frac{1}{2}\frac{\alpha(x)m(x)x}{l}\\ &\times\frac{\partial Y_t^2(x,t)}{\partial x} + \frac{1}{2}\frac{\alpha(x)x}{l}\frac{\partial\big[T_0(x)y_x^2(x,t)\big]}{\partial x}\\ &+\frac{3}{4}\frac{\alpha(x)x}{l}\frac{\partial\big[w(x)y_x^4(x,t)\big]}{\partial x} + \frac{1}{2}\frac{\alpha(x)x}{l}\\ &\times\frac{\partial T_0(x)}{\partial x}y_x^2(x,t) + \frac{1}{4}\frac{\alpha(x)x}{l}\frac{\partial w(x)}{\partial x}y_x^4(x,t)\Big\}\,dx.\end{aligned}$$

Integrating by part yields

$$\begin{aligned}\dot{V}_s(t) = &-2T(0,t)y_x(0,t)Y_t(0,t) + 2T(l,t)y_x(l,t)\\ &\times\big[Y_t(l,t)+\tfrac{3}{8}\alpha(l)y_x(l,t)\big] - \tfrac{1}{4}\alpha(l)T_0(l)y_x^2(l,t)\\ &-\int_0^l \frac{1}{2l}\frac{\partial[\alpha(x)m(x)x]}{\partial x}Y_t^2(x,t)\,dx - \int_0^l\frac{1}{2l}\\ &\times\Big[\frac{\partial[\alpha(x)x]}{\partial x}T_0(x) - \alpha(x)x\frac{\partial T_0(x)}{\partial x}\Big]y_x^2(x,t)\,dx\\ &-\int_0^l\frac{1}{4l}\Big[3\frac{\partial[\alpha(x)x]}{\partial x}w(x) - \alpha(x)x\frac{\partial w(x)}{\partial x}\Big]\\ &\times y_x^4(x,t)\,dx + \frac{1}{2}\alpha(l)m(l)Y_t^2(l,t)\\ \le &-2T(0,t)y_x(0,t)Y_t(0,t) + 2T(l,t)y_x(l,t)\\ &\times\big[Y_t(l,t)+\tfrac{3}{8}\alpha(l)y_x(l,t)\big] - \tfrac{1}{4}\alpha(l)T_0(l)y_x^2(l,t)\\ &+\frac{1}{2}\alpha(l)m(l)Y_t^2(l,t) - \frac{\epsilon}{2l}\int_0^l\\ &\times\big\{Y_t^2(x,t)+y_x^2(x,t)+y_x^4(x,t)\big\}\,dx \qquad (19)\end{aligned}$$

in which the last inequality is obtained by applying properties (13)–(15) of function $\alpha(x)$.

Since $M_0 = M_l = 0$, it follows from (6) and (7) that, under boundary controls given by (17)

$$T(0,t)y_x(0,t) = k_0Y_t(0,t)$$

and

$$T(l,t)y_x(l,t) = -k_lY_t(l,t).$$

Therefore, we have

$$\begin{aligned}&\tfrac{3}{4}\alpha(l)T(l,t)y_x^2(l,t) - \tfrac{1}{4}\alpha(l)T_0(l)y_x^2(l,t)\\ &\quad= \tfrac{1}{2}\alpha(l)T_0(l)y_x^2(l,t) + \tfrac{3}{4}\alpha(l)w(l)y_x^4(l,t)\\ &\quad= \frac{1}{2}\alpha(l)T_0(l)\frac{k_l^2(l)Y_t^2(l,t)}{[T_0(l)+w(l)y_x^2(l,t)]^2}\\ &\qquad+\frac{3}{4}\alpha(l)w(l)\frac{k_l^2(l)Y_t^2(l,t)y_x^2(l,t)}{[T_0(l)+w(l)y_x^2(l,t)]^2}\\ &\quad\le \frac{11k_l^2(l)}{16T_0(l)}\alpha(l)Y_t^2(l,t).\end{aligned}$$

Substituting the above expression, and invoking properties (11), (12), and (16), yields

$$\begin{aligned}\dot{V}_s(t) \le &-2k_0Y_t^2(0,t)\\ &-\Big[2k_l(l) - \frac{1}{2}\alpha(l)m(l) - \frac{11}{16T_0(l)}\alpha(l)k_l^2(l)\Big]Y_t^2(l,t)\\ &-\frac{\epsilon}{2l}\int_0^l\big\{Y_t^2(x,t)+y_x^2(x,t)+y_x^4(x,t)\big\}\,dx\\ \le &-2k_0Y_t^2(0,t) - k_l'(l)Y_t^2(l,t)\\ &-\frac{\epsilon}{2l}\int_0^l\big\{Y_t^2(x,t)+y_x^2(x,t)+y_x^4(x,t)\big\}\,dx\\ \le &-\epsilon_{v_s}V_s \qquad (20)\end{aligned}$$

where $k_l'(l) = k_l(l) - \underline{k}_l(l) \ge 0$, $\underline{k}_l(l)$ is that defined in (18), and

$$\epsilon_{v_s} = \frac{\epsilon}{2l\max\{\overline{m}+0.5\max_{x\in[0,l]}\alpha^2(x)\overline{m},\ \overline{c}_{T_0}+0.5\overline{m},\ 0.5\overline{w}\}}.$$

The solution to the above differential inequality is

$$V_s(t) \le V_s(t_0)e^{-\epsilon_{v_s}(t-t_0)}$$

which demonstrates exponential stability. □

Corollary 1: Consider system (2) with boundary conditions (5). If Assumptions 2 and 3 hold, if there is a scalar function $\alpha(x)$ satisfying inequalities (10), and (13)–(15) and

$$\alpha^2(l)\overline{m} < \tfrac{16}{9}\underline{c}_{T_0} \qquad (21)$$

and if parameters η_1, η_2, b_0, and b_l are known, the following boundary controls are robust and exponentially stabilizing everywhere [measured by $V_s(t)$ and in the region in which model (1) is valid] with respect to the equilibrium of the string:

$$\begin{aligned}f_0(t) &= -k_0Y_t(0,t) + 3c_f(0,t) + b_0\left[Y_t(0,t) - \delta_b(t)\right]\\ f_l(t) &= -k_l\big[Y_t(l,t)+\tfrac{3}{8}\alpha(l)y_x(l,t)\big] - 3c_f(l,t)\\ &\quad+ b_l\left[Y_t(l,t)-\delta_b(t)\right] - \tfrac{3}{8}M_l\alpha(l)y_{xt}(l,t) \qquad (22)\end{aligned}$$

where $k_0 \ge 0$ and $k_l \ge 2\alpha(l)\overline{m}$ are positive control gains, $c_f(0,t) = T(0,t)y_x(0,t)$ and $c_f(l,t) = T(l,t)y_x(l,t)$ are boundary contacting forces.

Proof: Choose a Lyapunov function candidate to be

$$V(t) = V_s + V_0 + V_l$$

where V_s is that in (9), and V_0 and V_l are defined to be the sub-Lyapunov functions for the control mechanism and as

$$\begin{aligned}V_0(t) &= \tfrac{1}{2}M_0\left[\dot{p}_0(t)+\delta_b(t)\right]^2,\\ V_l(t) &= \tfrac{1}{2}M_l\big[\dot{p}_l(t)+\delta_b(t)+\tfrac{3}{8}\alpha(l)y_x(l,t)\big]^2.\end{aligned}$$

It follows from dynamic equations (6) and (7) that, under boundary controls in (22)

$$\begin{aligned}\dot{V}_0 &= Y_t(0,t)\left[f_0(t) - T(0,t)y_x(0,t) - b_0 y_t(0,t)\right]\\ &= -\frac{2k_0}{M_0}V_0 + 2Y_t(0,t)T(0,t)y_x(0,t)\end{aligned}$$

and

$$\begin{aligned}\dot{V}_l =& \left[Y_t(l,t) + \tfrac{3}{8}\alpha(l)y_x(l,t)\right]\\ &\times\left[f_l(t) + T(l,t)y_x(l,t) - b_l y_t(l,t) + \tfrac{3}{8}M_l\alpha(l)y_{xt}(l,t)\right]\\ =& -\frac{2k_l}{M_l}V_l - 2\left[Y_t(l,t) + \frac{3}{8}\alpha(l)y_x(l,t)\right]T(l,t)y_x(l,t).\end{aligned}$$

Combining the above two results with (19) yields

$$\begin{aligned}\dot{V}(t) \le& -\frac{2k_0}{M_0}V_0 - \frac{2k_l}{M_l}V_l - \frac{1}{4}\alpha(l)T_0(l)y_x^2(l,t)\\ &+ \frac{1}{2}\alpha(l)m(l)Y_t^2(l,t) - \frac{\epsilon}{2l}\int_0^l\\ &\times\left\{Y_t^2(x,t) + y_x^2(x,t) + y_x^4(x,t)\right\}dx.\end{aligned}$$

It follows that, under a choice of $\alpha(l)$ satisfying (21) and under the choice of k_l

$$\begin{aligned}&-\tfrac{1}{2}k_l\left[Y_t(l,t) + \tfrac{3}{8}\alpha(l)y_x(l,t)\right]^2 - \tfrac{1}{4}\alpha(l)T_0(l)y_x^2(l,t)\\ &\quad+ \tfrac{1}{2}\alpha(l)m(l)Y_t^2(l,t)\\ &= -\tfrac{1}{2}\left[k_l - \alpha(l)m(l)\right]Y_t^2(l,t) - \left[\tfrac{9}{128}k_l\alpha^2(l) + \tfrac{1}{4}\alpha(l)T_0(l)\right]\\ &\quad\times y_x^2(l,t) - \tfrac{3}{8}k_l\alpha(l)Y_t(l,t)y_x(l,t)\\ &\le -\sqrt{k_l - \alpha(l)m(l)}\sqrt{\tfrac{9}{64}k_l\alpha^2(l) + \tfrac{1}{2}\alpha(l)T_0(l)}\\ &\quad\times\left|Y_t(l,t)y_x(l,t)\right| - \tfrac{3}{8}k_l\alpha(l)Y_t(l,t)y_x(l,t) \le 0.\end{aligned}$$

Therefore, it is shown in the equation at the bottom of the page from which exponential stability can be concluded. □

Remark 3.1.3: Robust control (22) in the corollary is synthesized via the backward recursive design in [13]. In other words, its design is based on robust control (17), and their stability proofs are almost identical except that additional sub-Lyapunov functions are introduced to include state variables in the dynamics of control mechanism. Consequently, more feedback information is required in control (22), as both (22) and (17) belong to state feedback controls. ◇

Remark 3.1.4: If boundary values $m(l)$ and $T_0(l)$ are exactly known, condition (12) required in theorem 1 is no longer needed, condition (11) should be modified to be

$$11\alpha^2(l)m^2(l) \le 32T_0(l)$$

and admissible values for control gain $k_l(l)$ is given by the interval $[\underline{k}_l'(l), \overline{k}_l'(l)]$ where

$$\underline{k}_l'(l) = \frac{16T_0(l) - \sqrt{256T_0^2(l) - 88\alpha^2(l)m(l)T_0(l)}}{11\alpha(l)}$$

and

$$\overline{k}_l'(l) = \frac{16T_0(l) + \sqrt{256T_0^2(l) - 88\alpha^2(l)m(l)T_0(l)}}{11\alpha(l)}.$$

It is apparent that the control gain is a nonlinear function of boundary values of the system dynamics. ◇

Remark 3.1.5: In practice, boundary controls in (17) cannot have their gains exceed certain threshold values, which can be satisfied according to (18) by choosing a small $\alpha(l)$ as required also in remark 3.1.1. It is obvious that, if $b_0 = 0$, no control force is needed at $x = 0$ by setting $k_0 = 0$. That is, boundary $x = 0$ is free along its track, active control is only needed at $x = l$ to compensate for speed variations of the transporter. However, during the transient period that the transporter accelerates or decelerates, force must also be applied at $x = 0$. This is why it is better to implement control (17) with $k_0 > 0$ for all time. ◇

D. Robust and Adaptive Control Designs

In this section, the robust boundary control developed in Section 2-C is converted into an adaptive one in order to compensate for unknown motion of the transporter. For adaptive control design, consider the following Lyapunov function candidate

$$L(t) = \frac{1}{2k_a}\left[b_0 - \hat{b}_0(t)\right]^2 + \frac{1}{2k_a}\left[b_l - \hat{b}_l(t)\right]^2 + \frac{1}{2k_a}\sum_{i=1}^{4}\left[\xi_i - \hat{\xi}_i(t)\right]^2 \quad (23)$$

where

$k_a > 0$	adaptation gain;
ξ_1	$= b_0\eta_1\cos\eta_2$;
ξ_2	$= b_0\eta_1\sin\eta_2$;
ξ_3	$= b_l\eta_1\cos\eta_2$;
ξ_4	$= b_l\eta_1\sin\eta_2$;
$\hat{a}(t)$	estimate of a.

It follows from the certainty equivalence principle that, if $M_0 = M_l = 0$, robust and adaptive control should be chosen as

$$f_0(t) = k_0Y_t(0,t) + \hat{b}_0(t)Y_t(0,t) - \hat{\xi}_1\sin(w_bt) - \hat{\xi}_2\cos(w_bt) \quad (24)$$

and

$$f_l(t) = k_l(l)Y_t(l,t) + \hat{b}_l(t)Y_t(l,t) - \hat{\xi}_3\sin(w_bt) - \hat{\xi}_4\cos(w_bt) \quad (25)$$

where control gains k_0 and $k_l(l)$ are the same as those for controls in (17). Choices of adaptation laws and the resulting stability are given by the following theorem.

Theorem 2: Consider system (2) with boundary conditions in (5). If Assumptions 1–3 hold, if function $\alpha(x)$ is chosen according to inequalities (10)–(15), and if $M_0 = M_l = 0$, adaptive boundary controls (24) and (25) are asymptotically stabilizing everywhere [measured by $V_s(t)$ and in the region where model (1) is valid] with respect to the equilibrium of the string provided that the adaptation laws are set to be

$$\begin{aligned}\frac{d\hat{b}_0(t)}{dt} &= 2k_aY_t^2(0,t)\\ \frac{d\hat{\xi}_1(t)}{dt} &= -2k_aY_t(0,t)\sin(w_bt)\\ \frac{d\hat{\xi}_2(t)}{dt} &= -2k_aY_t(0,t)\cos(w_bt)\\ \frac{d\hat{b}_l(t)}{dt} &= 2k_a\left[Y_t(l,t) + \frac{3}{8}\alpha(l)y_x(l,t)\right]Y_t(l,t)\\ \frac{d\hat{\xi}_3(t)}{dt} &= -2k_a\left[Y_t(l,t) + \frac{3}{8}\alpha(l)y_x(l,t)\right]\sin(w_bt)\\ \frac{d\hat{\xi}_4(t)}{dt} &= -2k_a\left[Y_t(l,t) + \frac{3}{8}\alpha(l)y_x(l,t)\right]\cos(w_bt)\end{aligned} \quad (26)$$

where initial conditions can be selected by the designer.

$$\dot{V}(t) \le -\min\left\{\frac{2k_0}{m_0}, \frac{k_l}{m_l}, \frac{\epsilon}{2l\max\{\overline{m} + 0.5\max_{x\in[0,l]}\alpha^2(x)\overline{m}, \overline{c}_{T_0} + 0.5\overline{m}, 0.5\overline{w}\}}\right\}V \triangleq -\epsilon_vV.$$

Proof: It follows from (6) and (7), with $M_0 = M_l = 0$, that, under boundary controls given by (24) and (25)

$$T(0,t)y_x(0,t) = k_0 Y_t(0,t) - [b_0 - \hat{b}_0(t)]Y_t(0,t) + [\xi_1 - \hat{\xi}_1]\sin(w_b t) + [\xi_2 - \hat{\xi}_2]\cos(w_b t)$$

and

$$T(l,t)y_x(l,t) = -k_l(l)Y_t(l,t) + [b_l - \hat{b}_l(t)]Y_t(l,t) - [\xi_3 - \hat{\xi}_3]\sin(w_b t) - [\xi_4 - \hat{\xi}_4]\cos(w_b t).$$

Therefore, substituting the two expressions into (19), one can show

$$\begin{aligned}\dot{V}_s(t) \le& -2k_l(l)Y_t(l,t)\left[Y_t(l,t) + \tfrac{3}{8}\alpha(l)y_x(l,t)\right] \\ &- \tfrac{1}{4}\alpha(l)T_0(l)y_x^2(l,t) + \tfrac{1}{2}\alpha(l)m(l)Y_t^2(l,t) - 2Y_t(0,t) \\ &\times\Big\{-\left[b_0 - \hat{b}_0(t)\right]Y_t(0,t) + \left[\xi_1 - \hat{\xi}_1\right]\sin(w_b t) \\ &\quad + \left[\xi_2 - \hat{\xi}_2\right]\cos(w_b t)\Big\} + 2\left[Y_t(l,t) + \tfrac{3}{8}\alpha(l)y_x(l,t)\right] \\ &\times\Big\{\left[b_l - \hat{b}_l(t)\right]Y_t(l,t) - \left[\xi_3 - \hat{\xi}_3\right]\sin(w_b t) \\ &\quad - \left[\xi_4 - \hat{\xi}_4\right]\cos(w_b t)\Big\} - \frac{\epsilon}{2l}\int_0^l \\ &\times\left\{Y_t^2(x,t) + y_x^2(x,t) + y_x^4(x,t)\right\}dx.\end{aligned}$$

It follows that

$$\begin{aligned}&-2k_l(l)Y_t(l,t)\left[Y_t(l,t) + \tfrac{3}{8}\alpha(l)y_x(l,t)\right] \\ &\quad - \tfrac{1}{4}\alpha(l)T_0(l)y_x^2(l,t) + \tfrac{1}{2}\alpha(l)m(l)Y_t^2(l,t) \\ &\le -\left[2k_l(l) - \frac{1}{2}\alpha(l)m(l) - \frac{11}{16T_0(l)}\alpha(l)k_l^2(l)\right]Y_t^2(l,t) \\ &\quad - \tfrac{1}{22}\alpha(l)T_0(l)y_x^2(l,t) \\ &\le -\tfrac{1}{22}\alpha(l)T_0(l)y_x^2(l,t).\end{aligned}$$

Therefore, under adaptation laws in (26), we have

$$\dot{V}_s + \dot{L} \le -\epsilon_{v_s}V_s - \tfrac{1}{22}\alpha(l)T_0(l)y_x^2(l,t) \le -\epsilon_{v_s}V_s \tag{27}$$

from which asymptotic stability can be concluded by invoking Lemma 3.6 in [12, p. 38]. □

Remark 3.2.1: Similar to other asymptotic adaptive control designs, the adaptive laws in (26) are chosen such that the time derivative of composite Lyapunov function $V_s + L$ is negative semi-definite. This is obvious by comparing (20) and (27). As in other combined estimation and control problems, adaptation gain k_a should be chosen to be larger than control gains k_0 and $k_l(l)$ so that estimates $\hat{\eta}_i(t)$ quickly converge and the boundary controls become effective. On the other hand, k_a being too large makes the closed-loop adaptive system sensitive to unmodeled dynamics. ⋄

Remark 3.2.2: In case that either M_0 or M_l is not zero (or sufficiently small), robust adaptive control can be synthesized based on theorem 2 and using the backstepping design in [7] (as did in the robust control design in Corollary 1 and based on Theorem 1). In this case, additional adaptation laws can be introduced to estimate online masses M_0 and M_l. ⋄

Compared to Theorem 1, adaptive control in Theorem 2 requires the measurement of boundary slope $y_x(l,t)$. In case that such a measurement is not available, but friction coefficient is known, the following corollary can be applied.

Corollary 2: Consider system (2) with boundary conditions in (5). If Assumptions 1–3 hold, if function $\alpha(x)$ is chosen according to inequalities (10)–(15), if $M_0 = M_l = 0$, and if b_l is known, adaptive boundary controls (24) and (25) (with $\hat{b}_l = b_l$) are uniformly and ultimately bounded with respect to the equilibrium of the string provided that the adaptation laws are chosen to be: there exists a constant $\underline{k}_r > 0$ such that, for any gain $k_r > 99\alpha(l)k_a/(8\underline{c}_{T_0})$,

$$\begin{aligned}\frac{d\hat{b}_0}{dt} &= -k_r\hat{b}_0 + 2k_aY_t^2(0,t) \\ \frac{d\hat{\xi}_1}{dt} &= -k_r\hat{\xi}_1 - 2k_aY_t(0,t)\sin(w_b t) \\ \frac{d\hat{\xi}_2}{dt} &= -k_r\hat{\xi}_2 - 2k_aY_t(0,t)\cos(w_b t) \\ \frac{d\hat{\xi}_3}{dt} &= -k_r\hat{\xi}_3 - 2k_aY_t(l,t)\sin(w_b t) \\ \frac{d\hat{\xi}_4}{dt} &= -k_r\hat{\xi}_4 - 2k_aY_t(l,t)\cos(w_b t)\end{aligned} \tag{28}$$

where initial conditions can be set by the designer.

Proof: It follows from the proof of Theorem 2 that, under boundary controls given by (24) and (25), but with adaptation laws in (28)

$$\begin{aligned}\dot{V}_s + \dot{L} \le& -\epsilon_{v_s}V_s - \tfrac{1}{22}\alpha(l)T_0(l)y_x^2(l,t) + \tfrac{3}{4}\alpha(l)y_x(l,t) \\ &\times\left\{-\left[\xi_3 - \hat{\xi}_3\right]\sin(w_b t) - \left[\xi_4 - \hat{\xi}_4\right]\cos(w_b t)\right\} \\ &+ \frac{k_r}{k_a}\left[b_0 - \hat{b}_0(t)\right]\hat{b}_0(t) + \frac{k_r}{k_a}\sum_{i=1}^4\left[\xi_i - \hat{\xi}_i(t)\right]\hat{\xi}_i(t) \\ \le& -\epsilon_{v_s}V_s + \frac{99\alpha(l)}{16T_0(l)}\left\{\left[\xi_3 - \hat{\xi}_3\right]^2 + \left[\xi_4 - \hat{\xi}_4\right]^2\right\} \\ &- \frac{k_r}{2k_a}\left[b_0 - \hat{b}_0(t)\right]^2 - \frac{k_r}{2k_a}\sum_{i=1}^4\left[\xi_i - \hat{\xi}_i(t)\right]^2 \\ &+ \frac{k_r}{2k_a}b_0^2 + \frac{k_r}{2k_a}\sum_{i=1}^4\xi_i^2 \\ \le& -\epsilon_{v_s}V_s - \left(k_r - \frac{99\alpha(l)k_a}{8T_0(l)}\right)L + \frac{k_r}{2k_a}b_0^2 + \frac{k_r}{2k_a}\sum_{i=1}^4\xi_i^2\end{aligned} \tag{29}$$

which is negative definite except for several constant bias terms. It follows from Lemma 3.4 in [12, p. 35] that the closed-loop system is robustly stable in the sense that all signals are uniformly and ultimately bounded. □

Remark 3.2.3: The adaptation laws in Corollary 2 belong to the class of leakage-like adaptation laws [11] or to the class of robust adaptive controls [13]. According to Theorem 1, design parameter $\alpha(l)$ should be chosen to be small. As a result, it follows from (29) that the leakage gain k_r can be made small as well. However, due to less feedback information required, performance ensured in Corollary 2 is weaker than that in Theorem 2. ⋄

III. CONCLUSION

In this note, the problem of designing a robust and adaptive boundary control for a string system is considered in the presence of both uncertain dynamics and unknown motion of its support. The system under consideration is modeled by a partial differential equation in which the tension may be an uncertain nonlinear function of both its transverse gradient and the position along its equilibrium. It is shown that, if the base motion is known (through feedback measurement), a robust boundary control can be designed to ensure exponential stability everywhere and that, if otherwise, the robust control can easily be converted into a robust and adaptive control to ensure either asymptotic stability or uniform and ultimate bounded stability. It is believed that the result is the first complete solution to the nonlinear robust boundary control problem of suppressing transverse oscillation for the string system.

This also represents an important step in extending nonlinear robust control theory to distributed-parameter systems.

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Identification in the Presence of Symmetry: Oscillator Networks

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***Abstract*—It is well known that the presence of symmetry in the equations of a dynamical system has a profound effect on the resulting behavior. This note examines how this effect is manifested in the corresponding parameter identification problem. Our work shows that standard ideas such as persistent excitation in a trajectory can be *explained* by symmetry. Moreover, by understanding how symmetry affects the dynamics, it may be possible to obtain sufficient information to achieve full identification even when typical trajectories are not persistently exciting. Alternately, our analysis shows how properly interpreting the output of the identification process can give useful information even if full identification is not possible.**

***Index Terms*—Coupled oscillators, identification, parameter estimation, symmetry.**

II. INTRODUCTION

Obtaining an accurate quantitative model for a given dynamical system is of central importance in many areas of systems theory, including control theory. In many applications the basic structural features of the system model can be determined from a consideration of the physical laws which govern the system behavior, so that what remains is *parametric system identification*, that is, determining the values of the parameters in the model using measurements of inputs and outputs. In this note, we focus in particular on *adaptive* parameter identification [1]–[3], since the inherently dynamical context lends itself easily to the standard methodology of equivariant dynamics, and also because this approach has proven to be compatible with a great many system theoretic objectives (e.g., model-based prediction and control). The results we obtain are a manifestation of a structural property in this particular context, but it is likely that similar restrictions will occur in the presence of symmetry regardless of the specific identification methodology considered.

The subject of adaptive identification (and control) has been studied extensively during the past few decades [1]–[3]. However, there has been very little attention devoted to parameter identification in systems for which the dynamics possesses a symmetry [4]. Symmetric systems have been the subject of a great deal work in the dynamics community, but have received limited attention in control theory despite the many parallels between the disciplines. One of the best known examples of the use of symmetry in engineering is based on the study of continuous symmetries of mechanical systems, which give rise to conservation laws [5]. Symmetries can also result from the geometry of spatial domains [6]. Also, observe that engineered devices might easily possess symmetric dynamics as a consequence of the way they are designed and constructed, for example because of the use of connection of identical components. An example of this kind of application is the analysis of gaits of locomotion systems [7]. The example we consider below is also interpretable as a system of N identical electrical oscillators with equal mutual coupling, see [8] for an application involving

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