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Robust Control of Nonlinear Systems by Estimating Time Variant Uncertainties

Z. Qu

***Abstract*—In this note, robust control design is considered for nonlinear systems with time variant uncertainties. Instead of assuming that bounding function on uncertainties is either known or parameterizable in terms of unknown constants, uncertainties or their bounding functions are estimated. It is shown that bounded uncertainties from a known or partially known exo-system can be estimated as a part of a globally stabilizing robust control. The proposed method extends the existing results of adaptive robust control, and it makes robust control more applicable by requiring less information on uncertainties.**

***Index Terms*—Bounding function, estimation, Lyapunov direct method, robust control, uncertainty.**

I. Introduction

Robust control of nonlinear uncertain systems have attracted a lot of attention. Much of the interests stem from the fact that nonlinear and uncertain dynamics are common in many applications and that robust control is the design method to guarantee stability and performance. Robust stabilizability in terms of structure properties of uncertain systems, robust stability and performance, properties of robust controls, their design procedures, and robust optimality are among of the subjects studied in [6], [2], [1], [3], [16], [17], [8], [10], [11], [4], [23], [21], [18]. Although exact knowledge of the plant is not required, robust control designs have to be done according to the extent of information that is known. Information required in most of the existing results include bounding functions on and structural properties of the uncertainties in the system under consideration. It is adequate to assume in analytical analysis that uncertainties be bounded in a certain sense, but getting information about the size of uncertainties could be very difficult in many applications. Since uncertainties are uncertain by nature, the less information about uncertainty we have to know in our control design, the more applicable the resulting robust control becomes.

In a standard robust control design result (for example, [2]), bounding function on nonlinear uncertainty is assumed to be known. In this case, robust control is designed to compensate for the worst uncertainty within the bounding function. In the case that bounding function can be parameterized in terms of unknown constants, adaptive robust control can be designed to adaptively estimate the bounding function [3]. While this result represents a major step forward in reducing required information on uncertainty, it is often restrictive and conservative as the unknowns have to be constants. In this note, we investigate how to design robust control for systems whose uncertainties or their bounding functions are parameterized in terms of unknown outputs from a known or partially known exo-system. Using the proposed method, time variant signals (not just unknown constants) will be estimated and a globally stabilizing robust control can be found.

The proposed result is also related to two other topics in systems and control. One of them is control design for systems with time-varying parameters. Standard adaptive control can be designed if system parameters are slowly time varying [7]. For systems with fast time varying and bounded parameters, robust control [19] or robust adaptive control [14] can be applied. In the case that time varying parameters are generated by a known quasilinear stable model, adaptive control can be designed by incorporating the dynamics and treating the parameters as state variables (see [13, Remark 4.5.2, p. 180]). The other is control design for feedback linearizable systems with internal dynamics [9], [5]. In terms of system structure and the resulting control (in the sense of being dynamic or static), treatment of internal dynamics and exo-systems is similar in a control design. In this note, the idea of identifying time varying unknowns is extended to bounding functions on nonlinear uncertainties, and the exo-system itself could be partially unknown.

The note is organized as follows. In Section II, conditions used in the literature and in this note are compared to illustrate the advantages of the proposed extensions. In Section III, robust control design is presented for the case that time variant unknowns are outputs of a known exo-system. In Section IV, direct estimation of nonlinear time variant uncertainties is studied. In Section V, robust control design is pursued for the case that exo-systems are only partially known. An illustrative example is given in Section VI.

II. Problem Statement

An uncertain system considered in the note is of form

$$\begin{aligned}\dot{x} = f(x,t) + \Delta f_u(x,\upsilon,t) + \Delta B_u(x,\upsilon,t)u \\ + B(x,t)[\Delta f_m(x,\upsilon,t) + \Delta B_m(x,\upsilon,t)u + u]\end{aligned} \quad (1)$$

where $x(t) \in \Re^n$ is the state, $u(t) \in \Re^m$ is the control to be designed, $\Omega \subset \Re^p$ is a bounded set, $\upsilon(t) \in \Omega$ denotes the time variant uncertainties, $f(x,t)$ and $B(x,t)$ are known parts of system dynamics, $\Delta f_u(x,\upsilon,t)$, $\Delta B_u(x,\upsilon,t)$, $\Delta f_m(x,\upsilon,t)$, and $\Delta B_m(x,\upsilon,t)$ are uncertainties, and the subscripts $_u$ and $_m$ in (1) denote the so-called unmatched and matched uncertainties [2] and [18], respectively.

The robust control problem is to design a control $u(x,t)$ such that the resulting closed loop system is stable (in the sense of either asymptotic stability or stability of uniform ultimate boundedness [2], [18]) for all possible values of bounded uncertain vector $\upsilon(t)$ in the prescribed set Ω. Robust control design requires several technical assumptions. Typically, robust control design is based upon stability or stabilizability

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The author is with the School of Electrical Engineering and Computer Science, University of Central Florida, Orlando, FL 32816 USA (e-mail: qu@pegasus.cc.ucf.edu).

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of known dynamics. Specifically, the system consisting of known dynamics

$$\dot{x} = f(x,t) + B(x,t)u \tag{2}$$

is referred to as the nominal system of system (1). The first assumption, given as follows, is on stability of the nominal system.

Assumption 1: The origin, $x = 0$, is globally asymptotically stable for the uncontrolled nominal system $\dot{x} = f(x,t)$. Therefore, there exists a C^1 function $V(x,t) : \Re^n \times \Re \to \Re^+$ such that

$$\gamma_1(\|x\|) \le V(x,t) \le \gamma_2(\|x\|),$$
$$\frac{\partial V(x,t)}{\partial t} + \nabla_x^T V(x,t) f(x,t) \le -\gamma(\|x\|) \tag{3}$$

where $\gamma_1, \gamma_2, \gamma : \Re^+ \to \Re^+$ are class $\mathcal{K}_\infty$ functions.

It is easy to show that, as will be in the illustrative example, Assumption 1 is equivalent to the nominal system being stabilizable under a known, nominal control. Assumption 1 is important as it provides Lyapunov function $V(x,t)$ used to synthesize robust control.

The second assumption, originally defined in [6], [2] and given below, is the standard matching condition which ensures robust stabilizability. It has been shown in [18] and in the references therein that, in several cases, robust stability can be achieved without the matching condition. Nonetheless, the assumption is employed here so we can focus our attention on estimating uncertainties.

Assumption 2: Uncertain system (1) satisfies the matching conditions (MC's). That is, there exists a positive constant ϵ_b such that, for all $(x, \upsilon, t) \in \Re^n \times S \times \Re^+$, $\Delta f_u(x, \upsilon, t) = 0$, $\Delta B_u(x, \upsilon, t) = 0$, and $\|\Delta B_m(x, \upsilon, t)\| \le 1 - \epsilon_b$. To make mathematical derivations simpler, it is also assumed that $\Delta B_m(x, \upsilon, t) = 0$.

The third assumption will be on the size of uncertainty $\Delta f_m(x,t)$ because, as stated in the definition of robust control problem, uncertain variable vector $\upsilon(t)$ is bounded. In principle, the assumption is necessary as a successful robust control has to compensate for any and potentially destabilizing uncertainty and to be bounded itself. In other words, control in the presence of unbounded uncertainty is not only mathematically impossible but also physically unrealistic. However, there are several ways by which the assumption on uncertainty size can be made, and the choices will have major impact on whether and how robust control can be successfully applied.

Typically, uncertainties are handled and compensated for by defining or estimating their size bounding functions. The first option in making the third assumption, originally described in [6], [2] and restated below, is to assume that size information on uncertainty is known.

Assumption 3A: The uncertainty is bounded in Euclidean norm by a known nonlinear function as, for all $(x, \upsilon, t) \in \Re^n \times \Omega \times \Re^+$, $\|\Delta f_m(x, \upsilon, t)\| \le \rho_m(x,t)$, where $\rho_m(x,t)$ is Caratheodory, uniformly bounded with respect to t, and locally uniformly bounded with respect to x.

Assumption 3A states that, although uncertain vector $\upsilon(t)$ is unknown, its range of variation is known pointwise in the state space, and its contribution in $\Delta f_m(\cdot)$ and $\Delta B_m(\cdot)$ can be quantified so that known bounding functions can be found. Although this assumption is reasonable in many cases, uncertainties are unknown by nature and, in other cases, finding bounding functions may become the major obstacle to applying robust control. To overcome this difficulty, it was proposed in [3] that an adaptive version of robust control could be developed. Similar to the standard adaptive control results [12], [15], [20], [22], [11], adaptive robust control is applicable if bounding function $\rho_m(x,t)$ is parameterizable in terms of a set unknown but constant parameters as described by the following assumption.

Assumption 3B: The uncertainty is bounded in Euclidean norm as follows: for all $(x, \upsilon, t) \in \Re^n \times \Omega \times \Re^+$, $\|\Delta f_m(x, \upsilon, t)\| \le \rho_m(x,t)$, and

$$\rho_m(x,t) = W^T(x,t)\phi \tag{4}$$

where vector $\phi \in \Re^l$ contains all multiplicative and additive, unknown constant parameters, and $W(x,t)$ is a vector consisting of know functions that are Caratheodory, uniformly bounded with respect to t, and locally uniformly bounded with respect to x.

Since uncertain vector $\upsilon(t)$ is bounded, choosing constant vector ϕ as its magnitude vector can always be done. However, as will be shown in the subsequent section, such a treatment implies tradeoff. First, demanding a constant upper bound introduces conservatism for any time variant uncertainty. Second, adaptive robust control under Assumption 3B may not be one that is both continuous and asymptotically stabilizing. These limitations prompt us to study better designs of robust control. The approach we take in the note is to properly estimate time-varying uncertainties. To this end, we introduce the following assumptions as the new options of defining bounding functions on uncertainties. Compared with Assumptions 3A and 3B, bounding functions are now allowed to be in terms of unknown time varying parameters or to be simply the output of a known model.

Assumption 3C: The uncertainty is bounded in Euclidean norm as follows: for all $(x, \upsilon, t) \in \Re^n \times \Omega \times \Re^+$

$$\|\Delta f_m(x, \upsilon, t)\| \le W_1^T(x,t)\phi_1(t) \tag{5}$$

where unknown vector $\phi_1(t) \in \Re^{l_1}$ is assumed to be the bounded output of a quasilinear system

$$\dot{\phi}_1 = g_1(x,t)\phi_1 + g_2(x,t) \tag{6}$$

vector $W_1(x,t)$ and functions $g_i(x,t)$ are known, they are uniformly bounded with respect to t and locally uniformly bounded with respect to x, and there exists a constant, positive–definite matrix P_1 such that, for all $x \in \Re^n$ and for t, matrix

$$P_1 g_1(x,t) + g_1^T(x,t) P_1 \le 0 \tag{7}$$

is negative–semidefinite.

Assumption 3D: The uncertainty is bounded in Euclidean norm as follows: for all $(x, \upsilon, t) \in \Re^n \times \Omega \times \Re^+$, $\|\Delta f_m(x, \upsilon, t)\| \le W_2^T(x,t)\phi_2(t)$, and unknown vector $\phi_2(t) \in \Re^{l_2}$ is the bounded output of a nonlinear exosystem

$$\dot{\phi}_2 = h_1(x, \phi_2, t) + h_2(x,t) \tag{8}$$

where vector $W_2(x,t)$ and functions $h_i(\cdot)$ are known, they are uniformly bounded with respect to t and locally uniformly bounded with respect to x, and vector function $h_1(\cdot)$ has the property that, for all bounded x and for all z, w, t

$$(z-w)^T P_2 \left[h_1(x,z,t) - h_1(x,w,t)\right] \le 0 \tag{9}$$

is negative–semidefinite for a known positive–definite matrix P_2.

It is obviously that, if $g_i(x,t) = 0$, Assumption 3C reduces to Assumption 3B and that Assumption 3D includes Assumption 3C as the special case that $h_1(x, \phi_2, t)$ is linear with respect to ϕ_2. Note that a bound on $\phi_1(t)$ or $\phi_2(t)$ can only be developed by solving analytically the corresponding differential equation (which may be nonlinear and whose initial condition is unknown but bounded), and that the resulting bound would be in general a function of state x. Therefore, although $\phi_1(t)$ and $\phi_2(t)$ are assumed to be bounded (if x is bounded) and the model of exo-system is known, finding function $W(x,t)$ in (4) from either (5) or (8) could be quite difficult. In other words, the extensions from Assumption 3B to 3C and 3D are not trivial.

Assumption 3C and 3D imply that bounding function on uncertainty $\Delta f_m(x, \upsilon(t), t)$ is generated by a known exosystem, either linear or nonlinear. Various conditions in the assumptions are to ensure boundedness of the bounding function. Since $\upsilon(t)$ is the source of time variant uncertainty and since time variant uncertainty generated from a known exosystem can be estimated, robust control design may be pursued without the operation of developing a bounding function, which the subject of Section IV. The issue of further relaxing Assumption 3C (or 3D) to admit an uncertain model for exogenous signal $\upsilon(t)$ will be studied in Section V.

III. Robust Control Designs

We begin with two of existing results on robust control designs. The first one is the standard result in [6] and [2].

Lemma 1: If system (1) satisfies Assumptions 1, 2 and 3A, the closed loop system is either uniformly ultimate bounded (if $\epsilon > 0$) or asymptotically stable (if $\epsilon = 0$) under control

$$u(x,t) = -\rho_m(x,t)\frac{\mu(x,t)}{\|\mu(x,t)\| + \epsilon} \tag{10}$$

where $\epsilon \geq 0$ is a design constant, and $\mu(x,t) \triangleq B^T(x,t)\nabla_x V(x,t)\rho_m(x,t)$.

Proof: It follows from Assumptions 2 and 3A that a system under our consideration can be rewritten as

$$\dot{x} = f(x,t) + B(x,t)[\Delta f_m(x,t) + u].$$

It follows from Assumptions 1 and 3A that, along the trajectory of the above system

$$\begin{aligned}\dot{V} &\leq -\gamma(\|x\|) + \nabla_x^T V(x,t)B(x,t)\Delta f_m(x,t)\\ &\quad + \nabla_x^T V(x,t)B(x,t)u(x,t)\\ &\leq -\gamma(\|x\|) + \|\nabla_x^T V(x,t)B(x,t)\|\rho_m(x,t)\\ &\quad + \nabla_x^T V(x,t)B(x,t)u(x,t) \qquad (11)\\ &\leq -\gamma(\|x\|) + \epsilon \qquad (12)\end{aligned}$$

where the last inequality is ensured under robust control (10). □

One can see from (11) to (12) that, if the bounding function is known, robust control (10) achieves its compensation for uncertainties through size domination. There are many robust controls equivalent to that in (10), and all of them satisfy an inequality similar to that from (11) to (12). In case the bounding function is unknown but parameterizable, an adaptive robust control can be designed using the certainty-equivalence principle as did in a standard adaptive control design. The main result in [3] can be restated as the following lemma.

Lemma 2: Suppose that system (1) satisfies Assumptions 1, 2, and 3B. Then, the closed-loop system is uniformly ultimate bounded under control

$$u(x,t) = -\hat{\rho}_m(x,t)\frac{\hat{\mu}(x,t)}{\|\hat{\mu}(x,t)\| + \epsilon} \tag{13}$$

where $\hat{\rho}_m(x,t) = W^T(x,t)\hat{\phi} \geq 0$, $\hat{\mu}(x,t) = B^T(x,t)\nabla_x V(x,t)\hat{\rho}_m(x,t)$, ϵ is a design parameter given by

$$\dot{\epsilon} = -k_\epsilon \epsilon \tag{14}$$

with $\epsilon(t_0) > 0$, $\hat{\phi}$ is the estimate of ϕ and is generated by adaptation law

$$\dot{\hat{\phi}} = W(x,t)\|B^T(x,t)\nabla_x V(x,t)\| - k_a\hat{\phi} \tag{15}$$

and $k_a \geq 0$ and $k_\epsilon \geq 0$ are gains. Furthermore, if $k_\epsilon > 0$ and $k_a = 0$, the original state $x(t)$ becomes asymptotically stable with respect to the origin of $x = 0$ in the original state space.

Proof: Consider the Lyapunov function $L(x,t,\phi,\hat{\phi}) = V(x,t) + 0.5\|\tilde{\phi}\|^2 + k_l\epsilon$, where $\tilde{\phi} = \phi - \hat{\phi}$ is the parameter estimation error, and $k_l = 0$ if $k_\epsilon = 0$ and $k_l = 1/k_\epsilon$ if otherwise. It follows from (10), (13), (12), (11), and (14) that

$$\begin{aligned}\dot{L} &\leq -\gamma(\|x\|) + \|\mu(x,t)\| + \epsilon - \|\hat{\mu}(x,t)\| + \tilde{\phi}^T\dot{\tilde{\phi}} + k_l\dot{\epsilon}\\ &\leq -\gamma(\|x\|) - \frac{k_a}{2}\|\tilde{\phi}\|^2 + \frac{k_a}{2}\|\phi\|^2 + (1 - k_l k_\epsilon)\epsilon \qquad (16)\end{aligned}$$

from which the claimed stability result can be concluded using stability theorems in [2], [18]. □

The above adaptive robust control scheme provides an avenue for us to apply robust control to the cases that uncertainties are bounded by a parameterizable nonlinear function. Since uncertainties are often generated by exo-systems, Assumption 3C or 3D is introduced in order to make robust control more applicable while reducing conservatism. Robust control designs under these two assumptions are given by the following two theorems.

Theorem 1: Assume that system (1) satisfy Assumptions 1, 2, and 3C. Then, the closed-loop system is uniformly ultimate bounded under control

$$u(x,t) = -\hat{\rho}_{m1}(x,t)\frac{\hat{\mu}_1(x,t)}{\|\hat{\mu}_1(x,t)\| + \epsilon} \tag{17}$$

where $\hat{\rho}_{m1}(x,t) = W_1^T(x,t)\hat{\phi}_1 \geq 0$, $\hat{\mu}_1(x,t) = B^T(x,t)\nabla_x V(x,t)\hat{\rho}_{m1}(x,t)$, ϵ is a design parameter given by (14), $\hat{\phi}_1$ is the estimate of $\phi_1(t)$ and is given by adaptation law

$$\begin{aligned}\dot{\hat{\phi}}_1 &= g_1(x,t)\hat{\phi}_1 + g_2(x,t) + P_1^{-1}W_1(x,t)\\ &\qquad \|B^T(x,t)\nabla_x V(x,t)\| - k_a P_1^{-1}\hat{\phi}_1 \qquad (18)\end{aligned}$$

and $k_a \geq 0$ and $k_\epsilon \geq 0$ are gains. Moreover, $x(t)$ converges asymptotically to zero if $k_\epsilon > 0$ and $k_a = 0$.

Proof: Consider the Lyapunov function $L_1(x,t,\phi_1,\hat{\phi}_1) = V(x,t) + 0.5\tilde{\phi}_1^T P_1\tilde{\phi}_1 + k_l\epsilon$, where $\tilde{\phi}_1 = \phi_1 - \hat{\phi}_1$ is the output estimation error, and $k_l = 0$ if $k_\epsilon = 0$ and $k_l = 1/k_\epsilon$ if otherwise. It follows from (18) and (5) that

$$\begin{aligned}\dot{\tilde{\phi}}_1 &= g_1(x,t)\tilde{\phi}_1 - P_1^{-1}W_1(x,t)\|B^T(x,t)\nabla_x V(x,t)\|\\ &\qquad - k_a P_1^{-1}\tilde{\phi}_1 + k_a P_1^{-1}\phi_1.\end{aligned}$$

Therefore, differentiating L_1 and applying control (17) and adaptation law (7) yield

$$\begin{aligned}\dot{L}_1 &\leq -\gamma(\|x\|) + \epsilon - \|\hat{\mu}_1(x,t)\| + \|\mu_1(x,t)\| + \tilde{\phi}_1^T P_1\dot{\tilde{\phi}}_1 + k_l\dot{\epsilon}\\ &\leq -\gamma(\|x\|) - \frac{k_a}{2}\|\tilde{\phi}_1\|^2 + \frac{k_a}{2}\|\phi_1\|^2 + (1 - k_l k_\epsilon)\epsilon \qquad (19)\end{aligned}$$

from which stability claims can be made using theorems in [2], [18]. □

Theorem 2: If system (1) satisfies Assumptions 1, 2, and 3D, the closed-loop system is ensured to be uniformly ultimate bounded under control

$$u(x,t) = -\hat{\rho}_{m2}(x,t)\frac{\hat{\mu}_2(x,t)}{\|\hat{\mu}_2(x,t)\| + \epsilon} \tag{20}$$

where $\hat{\rho}_{m2}(x,t) = W_2^T(x,t)\hat{\phi}_2 \geq 0$, $\hat{\mu}_2(x,t) = B^T(x,t)\nabla_x V(x,t)\hat{\rho}_{m2}(x,t)$, ϵ is a design parameter given by (14), $\hat{\phi}_2$ is the estimate of ϕ_2 and is given by adaptation law

$$\begin{aligned}\dot{\hat{\phi}}_2 &= h_1(x,\hat{\phi}_2,t) + h_2(x,t) + P_2^{-1}W_2(x,t)\\ &\qquad \|B^T(x,t)\nabla_x V(x,t)\| - k_a P_2^{-1}\hat{\phi}_2 \qquad (21)\end{aligned}$$

and $k_a \geq 0$ and $k_\epsilon \geq 0$ are gains. In addition, $x(t)$ is asymptotically convergent to zero if $k_\epsilon > 0$ and $k_a = 0$.

Proof: Consider the Lyapunov function $L_2(x, t, \phi_2, \hat{\phi}_2) = V(x,t) + \frac{1}{2}\tilde{\phi}_2^T P_2 \tilde{\phi}_2 + k_l \epsilon$, where $\tilde{\phi}_2 = \phi_2 - \hat{\phi}_2$ is the estimation error, and $k_l = 0$ if $k_\epsilon = 0$ and $k_l = 1/k_\epsilon$ if otherwise. It follows from (21) that:

$$\dot{\tilde{\phi}}_2 = [h_1(x, \phi_2, t) - h_1(x, \hat{\phi}_2, t)] - P_2^{-1} W_2(x,t) \|B^T(x,t)\nabla_x V(x,t)\| + k_a P_2^{-1}\hat{\phi}_2.$$

Under Assumption 3D, we know from (20) and (19) that

$$\dot{L}_2 \leq -\gamma(\|x\|) - \frac{k_a}{2}\|\tilde{\phi}_2\|^2 + \frac{k_a}{2}\|\phi_2\|^2 + (1 - k_l k_\epsilon)\epsilon \tag{22}$$

from which the closed loop stability result and asymptotic convergence of $x(t)$ can be concluded using theorems in [2] and [18]. □

IV. Direct Estimation of Uncertainty

If system uncertainty is bounded by a function parameterized in terms of outputs from a known exo-system (rather than constants only), we can use the results in Theorems 1 and 2 to reduce conservatism in developing bounding functions. In the theorems, global boundedness of the estimation error and asymptotic convergence of state x can be concluded only if, in robust controls (17) and (20), design parameter ϵ is set to an exponential decaying function as defined by (14). It has been shown in [18] that, if $\epsilon(t)$ is an L_1 time function, robust control of form (17) becomes discontinuous in the limit unless $W_i(0,t) = 0$. Thus, trade-off between asymptotic stability of state x and continuity of robust control is needed in an application of the theorems.

Possible discontinuity of robust control is due to the use of bounding function in its design. As defined in (4) and (5), uncertainty $\Delta f_m(x, \upsilon, t)$ is known to be bounded in norm by a function. In robust control design and stability analysis, every possibility of the uncertainty within the given bounding function must be considered. The worst uncertainty that is admissible by inequality (5) is one of those that change arbitrarily fast within the bound and may even become discrete or jump dynamics in the limit. Therefore, to compensate asymptotically for such uncertainty, robust control must be capable of becoming discontinuous itself.

The extension from Assumption 3B to 3C (or 3D) brings about another way of handling uncertainties in robust control. Since uncertainties are general time-varying (due to their own dynamics), Assumption 3A or 3B must be made if only unknown constants can be estimated in the design. By admitting unknown time variant output of a known exo-system, we can get rid of the process of developing bounding function (which is conservative by its nature) and focus upon directly estimating time variant uncertainties. The following corollary corresponds to Theorem 1, its proof is almost identical to that of Theorem 1, and its resulting control will be both continuous and asymptotically stabilizing (for $x(t)$). Theorem 2 can be restated in a similar fashion.

Corollary 1: Assume that system (1) satisfy Assumptions 1 and 2. If the uncertainty is generated as: for all $(x, \upsilon, t) \in \Re^n \times \Omega \times \Re^+$, $\Delta f_m(x, \upsilon, t) = W_1(x,t)\phi_1(t)$, where unknown vector $\phi_1(t) \in \Re^{l_1}$ is the bounded output of exo-system (6), matrix $W_1(x,t) \in \Re^{m \times l_1}$ and functions $g_i(x,t)$ are known and continuous, they are uniformly bounded with respect to t and locally uniformly bounded with respect to x, and $g_1(x,t)$ satisfies (7) for a constant, positive definite matrix P_1. Then, the closed-loop system is uniformly ultimate bounded under control

$$u(x,t) = -W_1(x,t)\hat{\phi}_1 \tag{23}$$

where $\hat{\phi}_1$ is the estimate of ϕ_1 and is generated by the following adaptation law:

$$\dot{\hat{\phi}}_1 = g_1(x,t)\hat{\phi}_1 + g_2(x,t) + P_1^{-1}W_1(x,t) B^T(x,t)\nabla_x V(x,t) - k_a P_1^{-1}\hat{\phi}_1$$

for a constant scalar gain $k_a \geq 0$. In addition, $x(t)$ is asymptotically convergent to zero if $k_a = 0$.

Of course, Theorem 1 and Corollary 1 can be combined. Specifically, both uncertainty generated by an exosystem and uncertainty whose bounding function is generated by an exo-system can be estimated. Consider a system of form (1) in which $\Delta f_m(x,t) = W_{11}(x,t)\phi_{11}(t) + \Delta f'_m(x,t)$ and $\|\Delta f'_m(x,t)\| \leq W_{12}^T(x,t)\phi_{12}(t)$, where $\phi_{1i}(t)$ are outputs of known exo-systems. By Theorem 1 and its corollary, robust control should be chosen to be

$$u(x,t) = -W_{11}(x,t)\hat{\phi}_{11} - W_{12}^T(x,t)\hat{\phi}_{12} \frac{B^T(x,t)\nabla_x V(x,t) W_{12}^T(x,t)\hat{\phi}_{12}}{\|B^T(x,t)\nabla_x V(x,t)\| W_{12}^T(x,t)\hat{\phi}_{12} + \epsilon}$$

where $\hat{\phi}_{1i}$ are defined according to the expressions given in Theorem 1 and Corollary 1, respectively.

V. Estimation Based on Uncertain Model of Exo-System

As those in (1), dynamics of the exosystem (with output $\phi_i(t)$) may not be known exactly. If the model of the exosystem is merely a black box (completely unknown), the corresponding robust control problem is in general ill-defined. This is because, unlike a typical model identification problem or state observation problem, output of the exo-system is itself an uncertainty to be estimated. Thus, in this section, we shall investigate the problem of robust control design under the following assumption in order to extend the result of Theorem 2. In essence, the assumption implies that the exosystem is stable (for all bounded x), is partially known, and has a bounded output. Stability results in Theorem 1 and Corollary 1 can be extended in a similar fashion.

Assumption 3E: Uncertainty $\Delta f_m(x, \upsilon, t)$ is bounded by inequality (8) where unknown vector $\phi_2(t) \in \Re^{l_2}$ is assumed to be the bounded output of a nonlinear system

$$\dot{\phi}_2 = h_1(x, \phi_2, t) + \Delta h_1(x, \phi_2, t) + h_2(x,t) + \Delta h_2(x,t) \tag{24}$$

vector $W_2(x,t)$ and functions $h_1(x, \phi_2, t)$ and $h_2(x,t)$ are known, but functions $\Delta h_1(x, \phi_2, t)$ and $\Delta h_2(x,t)$ are uncertain. Furthermore, vector function $h_1(\cdot)$ has the property that, for a constant $\sigma > 1$, for a known positive definite matrix P_2, and for a nonnegative function $\beta(\cdot)$ and

$$(z - w)^T P_2 [h_1(x, z, t) - h_1(x, w, t)] \leq -\beta(\|x\|, \|z\|, \|w\|)\|z - w\|^\sigma$$

and uncertainties $\Delta h_1(x, \phi_2, t)$ and $\Delta h_2(x,t)$ are bounded as, for all x, for all t, and for a known scalar function $\alpha(\cdot)$, $\|\Delta h_1(x, \phi_2, t) + \Delta h_2(x,t)\| \leq \alpha(\|x\|)$.

Corollary 2: Consider that system (1) satisfies Assumptions 1, 2, and 3E. Then, the closed-loop system is globally, uniformly and ultimately bounded under robust control

$$u(x,t) = -\hat{\rho}_{m2}(x,t)\frac{\hat{\mu}'_2(x,t)}{\|\hat{\mu}'_2(x,t)\| + \epsilon} \tag{25}$$

where $\hat{\rho}_{m2}(x,t) = W_2^T(x,t)\hat{\phi}_2 \geq 0$, $\hat{\mu}'_2(x,t) = k_v V^b(x,t) B^T(x,t)\nabla_x V(x,t)\hat{\rho}_{m2}(x,t)$, ϵ

is a design parameter given by (14), $b > -1$ is another design parameter, $\hat{\phi}_2$ is the estimate of ϕ_2 and is given by adaptation law

$$\dot{\hat{\phi}}_2 = h_1(x, \hat{\phi}_2, t) + h_2(x,t) + k_v V^b(x,t) P_2^{-1} W_2(x,t) \|B^T(x,t)\nabla_x V(x,t)\| - k_a P_2^{-1}\hat{\phi}_2 \quad (26)$$

and gains $k_v > 0$, $k_a \geq 0$ and $k_\epsilon \geq 0$ and constant $b \geq 0$ are chosen such that

$$\lim_{s\to+\infty} \frac{k_v k_a \gamma_1^b(s)\gamma(s)}{\lambda_{\max}^2(P_2)\alpha^2(s)} > 1. \quad (27)$$

Furthermore, if gain $k_v > 0$ and constant $b > -1$ can be chosen such that, for all $s \geq 0$ and for all (x,t), $V^b(x,t)W_2^T(x,t)\|B^T(x,t)\nabla_x V(x,t)\|$ is well defined and

$$\inf_{\lambda_1,\lambda_2\geq 0} k_v \beta^{\frac{1}{\sigma-1}}(s,\lambda_1,\lambda_2)\gamma_1^b(s)\gamma(s) > (\sigma-1)2^{\frac{1}{\sigma-1}}\sigma^{-\frac{\sigma}{\sigma-1}}\lambda_{\max}^{\frac{\sigma}{\sigma-1}}(P_2)\alpha^{\frac{\sigma}{\sigma-1}}(s) \quad (28)$$

state vector $x(t)$ can be made globally and asymptotically convergent to zero by simply setting $k_a = 0$ and $k_\epsilon > 0$.

Proof: Consider the modified Lyapunov function

$$L_2'(x,t,\phi_2,\hat{\phi}_2) = \frac{k_v}{1+b}V^{b+1}(x,t) + \frac{1}{2}\tilde{\phi}_2^T P_2 \tilde{\phi}_2 + k_l \epsilon$$

where $\tilde{\phi}_2 = \phi_2 - \hat{\phi}_2$ is output estimation error, and $k_l = 0$ if $k_\epsilon = 0$ and $k_l = 1/k_\epsilon$ if otherwise. It follows from Assumption 3E and from (25), (26), and (22) that

$$\begin{aligned}\dot{L}_2' &\leq -k_v V^b(x,t)\gamma(\|x\|) + \epsilon - \|\hat{\mu}_2'(x,t)\| \\ &\quad + \|\mu_2'(x,t)\| + \tilde{\phi}_2^T P_2 \dot{\tilde{\phi}}_2 + k_l\dot{\epsilon} \\ &\leq -k_v\gamma_1^b(\|x\|)\gamma(\|x\|) - \beta\|\tilde{\phi}_2\|^\sigma - \frac{k_a}{2}\|\tilde{\phi}_2\|^2 \\ &\quad + \lambda_{\max}(P_2)\|\tilde{\phi}_2\|\alpha(\|x\|) + \frac{k_a}{2}\|\phi_2\|^2 + (1-k_l k_\epsilon)\epsilon.\end{aligned} \quad (29)$$

Depending upon the properties of functions $\beta(\cdot)$ and $\alpha(\cdot)$, two types of stability results can be concluded from (29) for $\|x\|$ (while $\|\tilde{\phi}_2\|$ is bounded). Consider first $\beta(s,\lambda_1,\lambda_2) > 0$ for all (s,λ_1,λ_2), in which case inequality

$$\begin{aligned}&-k_v\gamma_1^b(\|x\|)\gamma(\|x\|) - \beta\|\tilde{\phi}_2\|^\sigma + \lambda_{\max}(P_2)\|\tilde{\phi}_2\|\alpha(\|x\|) \\ &\quad\leq -k_v\gamma_1^b(\|x\|)\gamma(\|x\|) + (\sigma-1)2^{\frac{1}{\sigma-1}}\beta^{-\frac{1}{\sigma-1}}\sigma^{-\frac{\sigma}{\sigma-1}} \\ &\quad\quad \lambda_{\max}^{\frac{\sigma}{\sigma-1}}(P_2)\alpha^{\frac{\sigma}{\sigma-1}}(\|x\|) - \frac{1}{2}\beta\|\tilde{\phi}_2\|^\sigma\end{aligned}$$

can be concluded from the Hölder inequality. Therefore, if k_v and b can be chosen to satisfy inequality (28), the right-hand side of inequality (29) is negative–semidefinite (after setting $k_a = 0$ and $k_\epsilon > 0$), and control (25) is asymptotically stabilizing for $x(t)$.

In the general case that inequality (28) cannot be satisfied for all $s \geq 0$, it follows that, by setting $k_a > 0$:

$$-\frac{k_a}{4}\|\tilde{\phi}_2\|^2 + \lambda_{\max}(P_2)\|\tilde{\phi}_2\|\alpha(\|x\|) \leq \frac{\lambda_{\max}^2(P_2)}{k_a}\alpha^2(\|x\|).$$

By the gain selections specified by inequality (27), we know that the right-hand side of inequality (29) is negative if $\|x\|$ or $\|\tilde{\phi}_2\|$ exceeds a certain threshold value. Consequently, stability of uniform and ultimate boundedness can be claimed using the boundedness theorem in [2] and [18]. □

It is worth noting that inequality (28) implies that, as $\gamma(0) = 0$, uncertainties $\Delta h_i(\cdot)$ (rather than $v(t)$ or $\phi_2(t)$ as would be required by the existing results) must be vanishing in order to achieve asymptotic stability. On the other hand, inequality (29) can always be satisfied by increasing b and k_v.

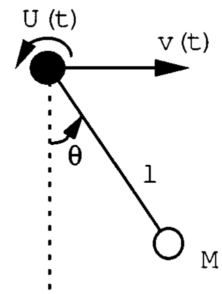

Fig. 1. A disturbed pendulum.

VI. Illustrative Example

The following example on a simple pendulum is used here, as did in [2], to illustrate the proposed robust control design based on estimating system uncertainties.

Consider the simple pendulum shown in Fig. 1. It is of mass M and length l, and is subject to a control moment $U(t)$ and an unknown bounded disturbance $v(t)$ in the form of a horizontal acceleration of its point of support. It was shown in [2] and [18] that pendulum dynamics are described by

$$\dot{x}_1 = x_2 \quad \dot{x}_2 = -\frac{g}{l^2}\sin x_1 + u - \frac{v}{l}\cos x_1$$

where x_1 and x_2 are angular position and velocity of the rigid weightless link between the mass and the point of support, and $u(t) = (U(t)/Ml^2)$ is the control. To have a stable nominal system as required in Assumption 1, we choose the control as $u = -k_1x_1(t) - k_2x_2(t) + u_r$, where u_r is the robust control part to be selected, and gains $k_1, k_1', k_2 > 0$ are chosen such that $k_1 = k_1' + \sup_{x_1\in\Re}(-\sin x_1)g/(x_1 l^2)$. It is straightforward to verify using Lyapunov function (generated via the gradient technique)

$$V(x) = x^T P x + \frac{2g}{l}(1-\cos x_1) \quad \text{with} \quad P = \begin{bmatrix} k_1 + 0.5k_2^2 & 0.5k_2 \\ 0.5k_2 & 1\end{bmatrix}$$

that the nominal system under the nominal control is asymptotically stable and that, for Lyapunov function $V(x)$, $\gamma(\|x\|) = \min\{k_2, k_1'k_2\}$, $\gamma_1(\|x\|) = \lambda_{\min}(P)\|x\|^2$, and

$$\gamma_2(\|x\|) = \lambda_{\max}(P)\|x\|^2 + \begin{cases} 2g(1-\cos\|x\|)/l & \text{if } \|x\| \leq \pi \\ 4g/l & \text{if } \|x\| > \pi\end{cases}$$

where $\lambda_{\min}(P), \lambda_{\max}(P) = 0.5[1 + k_1 + 0.5k_2^2 \pm \sqrt{(1+k_1+0.5k_2^2)^2 - 4(k_1+0.25k_2^2)}]$.

It is assumed that the disturbance $v(t)$ is the bounded output of a partially known exosystem whose dynamics are described by

$$\begin{aligned}\dot{z}_1 &= -z_1 - z_1^3 + x_1^2 + \cos t, \\ \dot{z}_2 &= -z_2 + x_2^2 + \sin 2t + \Delta(x,t), \quad v = (1+x_2^2)z_1 + z_2\end{aligned}$$

where $\|\Delta(x,t)\| \leq \|x\|^2$. Therefore, by comparing Corollaries 1 and 2, we know that the robust control which directly estimates $v(t)$ should be

$$u_r = \frac{1}{l}\hat{v}\cos x_1 = \frac{1}{l}\cos x_1 \begin{bmatrix}1+x_2^2 & 1\end{bmatrix}\begin{bmatrix}\hat{z}_1 & \hat{z}_2\end{bmatrix}^T$$

where $\dot{\hat{z}}_1 = -\hat{z}_1 - \hat{z}_1^3 + x_1^2 + \cos t - k_vV^b(x)(1+x_2^2)(k_2x_1 + 2x_2) - k_a\hat{z}_1$, and $\dot{\hat{z}}_2 = -\hat{z}_2 + x_2^2 + \sin 2t - k_vV^b(x)(k_2x_1+2x_2) - k_a\hat{z}_2$. Computer simulation of the above control was performed using Matlab with the following choices: $l = \sqrt{g}$, $k_1 = 9$, $k_2 = 6$, $b = 1$, $k_v = 20$, $k_a = 0$, $x_1(0) = \pi/4$, $x_2(0) = 0$, $z_1(0) = z_2(0) = 0$, $\hat{z}_1(0) = \hat{z}_2(0) = 0$, $\Delta(x,t) = 5|x_1x_2|(1+\cos 3t)$. The simulation results, shown in Figs. 2 and 3, verify the theoretical analysis.

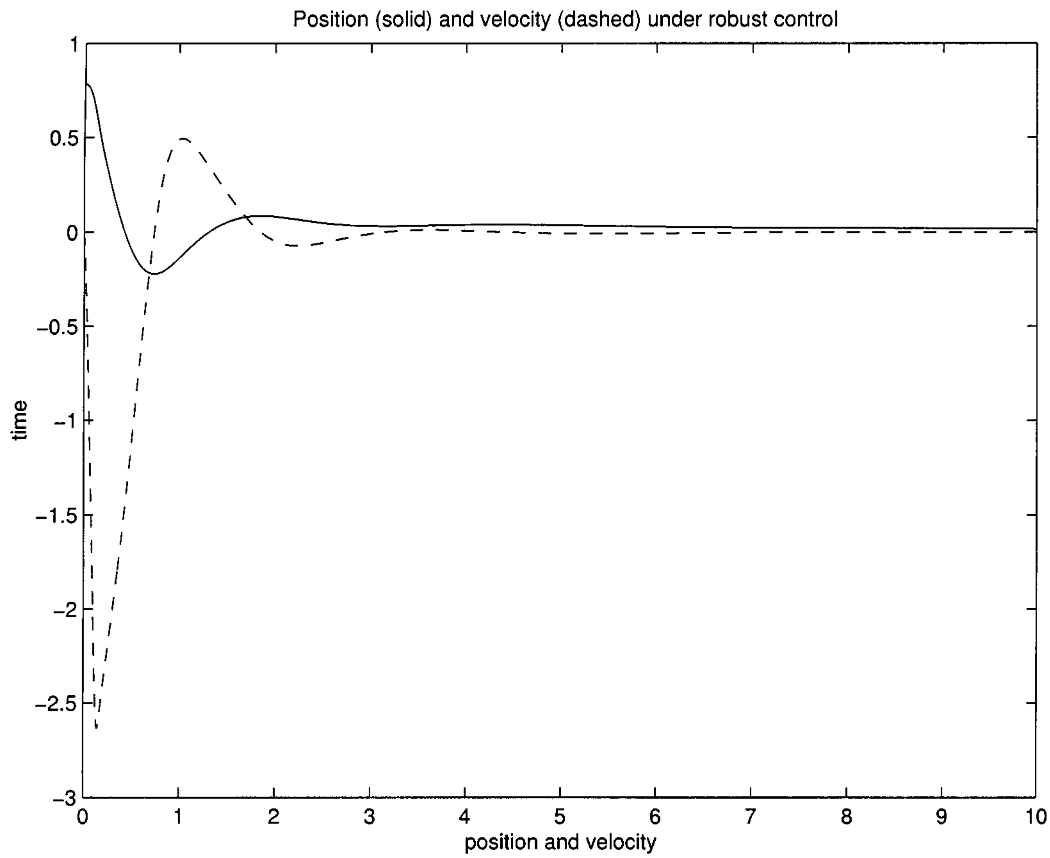

Fig. 2. State trajectory of the disturbed pendulum under robust control.

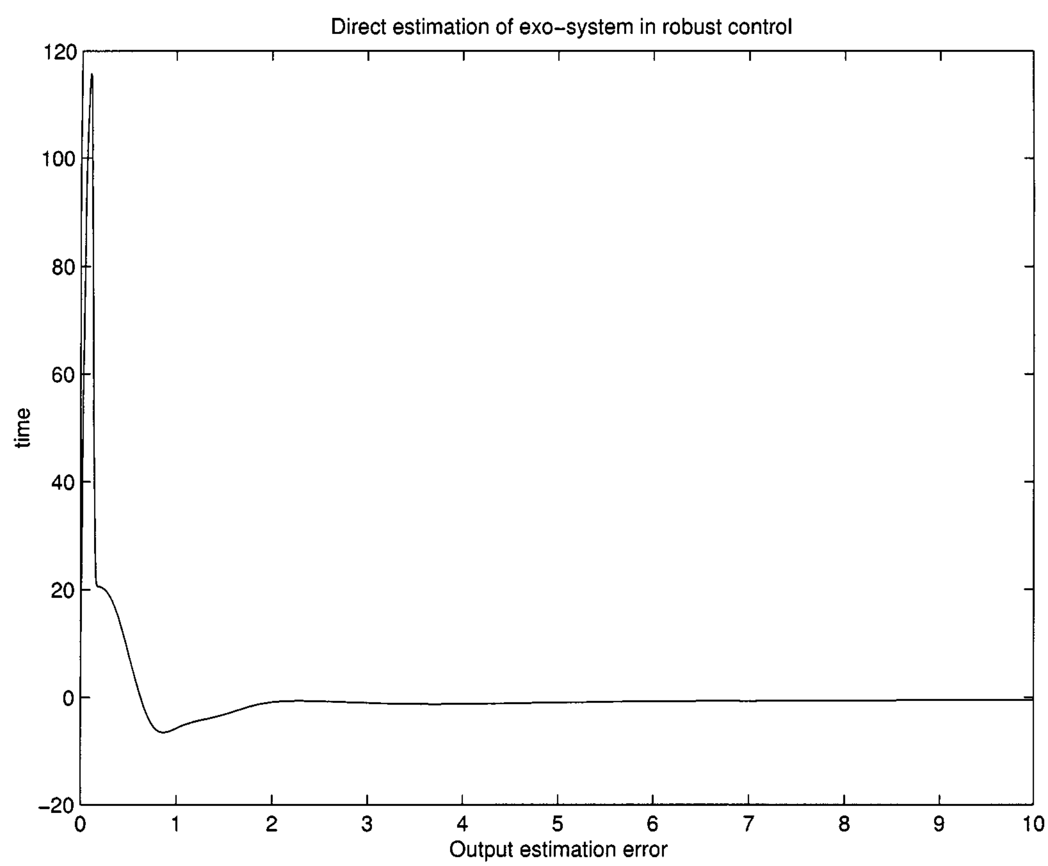

Fig. 3. Estimation error of time variant uncertainty.

VII. Conclusion

Robust control can be made more applicable by designing it in conjunction with estimation of nonlinear time variant uncertainties. It is proposed in the note that, despite of their nonlinearity and time variance, uncertainties or their bounding functions can be estimated as long as they are generated by exosystems whose models are either known or partially known. Technical conditions are found in the note under which estimation and stability can be achieved. Compared with the existing results on adaptive robust control, the proposed result represents a step forward in handling nonlinear time variant uncertainties.

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An Observer Design for Linear Time-Delay Systems

M. Hou, P. Zítek, and R. J. Patton

***Abstract*—An observer design is proposed for linear systems with time delay. The key of the design is to find a generalized coordinate change such that in the new coordinates all the time-delay terms are injected by the system's output. The existence of such a coordinate change is guaranteed by a rank condition on the observability matrix. Novelty of the proposed design is clearly reflected in the multiple-output case where a dimensional expansion in the coordinate change could become necessary and hence is allowed.**

***Index Terms*—Observability matrix, observer design, output injection, polynomial matrix, time-delay systems.**

I. Introduction

An observer asymptotically reconstructs the state of a dynamic system and has important applications in realization of feedback control, system supervision and fault diagnosis of dynamic processes. Time-delay systems describe a wide range of dynamic processes arising often in chemical, biological and economic applications.

There are various observer design methods for time-delay systems, see, e.g., [2], [4], [7], [9], [12], [13], and [17]. Most of these designs are based on the spectrum assignment [10], [11], which is considerably complicated in the multiple-input–multiple-output (MIMO) case in comparison with the conventional pole assignment. Recently, design of linear functional observers for time-delay systems was considered in [14] with a set of matrix equations for observers' coefficients to satisfy.

This paper presents a simple observer design for linear time-delay systems. The key in the design is to find a coordinate change such that in the new coordinates all the time-delay terms in the system description are associated with the output only. This method corresponds to the output injection approach in the nonlinear observer designs [1], [5], [6], [15]. The output injection approach can deal with a very restrictive class of nonlinear systems, while this method can cover all linear time-delay systems satisfying a rank condition on the observability matrix.

To illustrate the idea, consider the following example representing the dynamics of a heat exchanger with zero inlet temperature [17].

Example 1: A linear time-delay system with a single output is

$$\dot{x}_1(t) = -x_1(t) + x_2(t), \quad \dot{x}_2(t) = -x_1(t) - x_2(t-\tau),$$
$$y(t) = x_1(t).$$

By inspection, if choose a coordinate change as

$$\xi_1(t) = x_1(t), \quad \xi_2(t) = x_2(t) + x_1(t-\tau)$$

in the ξ-coordinates

$$\dot{\xi}_1(t) = \xi_2(t) - y(t-\tau) - y(t) \quad \dot{\xi}_2(t) = -y(t-\tau) - y(t),$$
$$y(t) = \xi_1(t).$$

Designing an observer for ξ and further for x then becomes trivial.

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M. Hou and R. J. Patton are with the Department of Engineering, University of Hull, Hull HU6 7RX, U.K. (e-mail: m.hou@eng.hull.ac.uk; r.j.patton@eng.hull.ac.uk).

P. Zítek is with the Department of Instrumentation & Control Engineering, Czech Technical University, 16607 Prague, Czech Republic (e-mail: zitek@fsid.cvut.cz).

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