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Brief Paper

An iterative learning algorithm for boundary control of a stretched moving string ☆

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Abstract

In this paper, iterative learning control technique is applied for the first time to a class of flexible systems. Specifically, learning control of a material transport system is considered. The system consists of a stretched string and a transporter. The motion of transporter is subject to such external disturbances as imperfect wheels and can cause string vibrations. The control objective is to damp out any string oscillation during transportation using iterative control applied at the boundaries. The control is designed using both discrete and continuous time Lyapunov functions. The proposed result is new and significant as it demonstrates that iterative learning control methodology is an effective technique for controlling distributed parameter systems.

Keywords: Boundary control; Iterative control; Learning control; Lyapunov direct method; Repetitive control; Lyapunov stability

1. Introduction

Boundary control of string systems is an interesting control problem as strings can be used to model dynamic behavior of many continuous time flexible systems, for example, telephone wires, cables, conveyor belts, and even human DNA (Lee, 1957; Mote, 1966; Abrate, 1992; Baicu Rahn, & Nibali, 1996; Morgul, Rao, & Conrad, 1994) Most of the existing results are concerned with boundary control of string model itself, often based on linear models and perfect knowledge. In other words, control in the presence of such exogenous signal as disturbance received little attention.

In this paper, a nonlinear string system extracted from device manufacturing and process automation is considered. The system, as sketched in Fig. 1, consists of a stretched string, sliding/control mechanism, and a supporting transporter which moves from one processing station to another. It is assumed that, due to the large difference between their masses, motions of the string and the control assemblies have little effect on that of the transporter. The motion of transporter is characterized by a constant cruising speed plus a small variation which is caused by disturbances and can be viewed as an exogenous signal to the overall system of string and control mechanism. The control problem studied in the paper is to design boundary controls capable of compensating for the unknown exogenous signal (through attenuation).

Iterative learning control is a control methodology that improves system performance over repeated trials, and it has been shown to be very effective in such applications as robotics and automation as these applications often involve tasks of repetitive nature (Hara, Yamamoto, Omata, & Nakano, 1988). While adaptive control is primarily limited to identifying unknown constants in the system dynamics (Narendra & Annaswamy, 1989) (for example, the result in Qu, 1999), iterative learning control can be used to estimate an unknown time varying function provided that it is periodic. Recent results on learning control and comparison between adaptive control and learning control can be found in Moore (1993), Qu and Dawson (1996), Arimoto (1996), Qu (1998) and the references cited therein.

In this paper, the idea of iterative learning is applied for the first time to the above-mentioned system whose model is described by a partial differential equation. Specifically, a string model with nonlinear tension and with an

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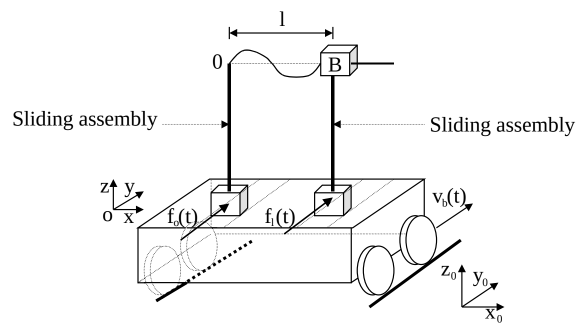

Fig. 1. A stretched string on a transporter.

exogenous signal is considered, and learning control is designed to attenuate the effects of exogenous signal $v_b(t)$ (representing the motion of the transporter) and its change. It is the periodic nature of the exogenous signal makes learning control (or repetitive control) an ideal choice.

The paper is organized into four sections. In Section 2, a string model of the system is presented. In Section 3, the learning control problem is defined, followed by the designs of conventional boundary controls and boundary learning controls. Stability and performance of the closed-loop system are analyzed. Conclusions are given in Section 4.

2. System model

It follows from the discussions in Benaroya (1998) and Qu (1999) that dynamic model for the system in Fig. 1 is given by

$$m y_{tt}(x,t) - \left[T_0 + \frac{3}{2}AE y_x^2(x,t)\right] y_{xx}(x,t) = -m\dot{v}_b(t), \quad (1)$$

where t is time, x is axial coordinate along the equilibrium of the string, $y(x,t)$ is transverse displacement of the string with respect to its equilibrium, A and ρ are cross-section area and linear density of the string, respectively, $m=\rho A$, E is elastic modulus, l is axial length between supports, and T_0 is string initial tension. Initial conditions for displacement and velocity of the string are

$$y(x,0) = c_1(x) \quad \text{and} \quad y_t(x,0) = c_2(x). \quad (2)$$

In the system, the string is supported and controlled along parallel tracks on a moving transporter and hence there is no distributed force applied to transverse motion of the string. Without control, sliding assemblies will move freely along the track, and the string will oscillate due to either speed variation of the transporter or non-zero initial conditions of the string. The two boundary control forces at points $x=0, l$, denoted by $f_0(t)$ and $f_l(t)$, are the control variables to be designed to attenuate possible oscillations. It is assumed that dynamics of the control mechanism themselves can be neglected.[1] Using the setting, the boundary conditions needed to solve partial differential equation (1) are

$$T(0,t)y_x(0,t) = f_0(t) \quad \text{and} \quad T(l,t)y_x(l,t) = -f_l(t), \quad (3)$$

where $T(x,t) = T_0 + 0.5AE y_x^2(x,t)$ is the nonlinear tension in the string.

3. Learning control and stability analysis

The learning control problem studied in this paper is to design iterative learning controls for attenuating vibrations under the following assumption. Compared with Qu (1999), the following assumption does not assume any explicit expression of the periodic speed variation.

Assumption. Motion profile of the transporter can be expressed as

$$v_b(t) = c_b + \zeta_b(t), \quad (4)$$

where c_b is a constant cruising speed, ζ_b represents a periodic speed variation of known period T_b. That is, if $t = jT_b + \tau$, then $\zeta_b(t) = \zeta_b(\tau)$, where $j = 0, 1, \ldots$ and $\tau \in [0, T_b]$.

3.1. Boundary control and stability analysis

The proposed boundary, iterative learning controls are: during the jth trial, $t = jT_b + \tau$ with $\tau \in [0, T_b]$ and

$$f_l(t) = f_{l,j}(\tau) = k_l[y_t(l,t) + \Delta_j(\tau)], \quad (5)$$

$$f_0(t) = f_{0,j}(\tau) = k_0[y_t(0,t) + \Delta_j(\tau)], \quad (6)$$

where j is the subscript indicating the trial number that relates the local time τ during each trial to time t, λ is a positive design parameter bounded from above as

$$0 < \lambda < \min\left\{\sqrt{\frac{T_0}{m}}, \sqrt{\frac{32T_0}{9m}}\right\}, \quad (7)$$

k_l is a positive control gain satisfying the inequality

$$\frac{\lambda m}{2 + 2\sqrt{1 - \frac{9\lambda^2 m}{32T_0}}} < k_l < \frac{16T_0}{9\lambda}\left(1 + \sqrt{1 - \frac{9\lambda^2 m}{32T_0}}\right), \quad (8)$$

and $k_0 > 0$ is a positive gain.

The introduction of Δ_j into controls (5) and (6) will make them iterative. It is this term that learns the unknown time

[1] If not, the backward recursive design technique in Qu (1998) can be combined into the proposed result to generate a learning control as did in Qu (2000) which is on robust and adaptive control and is based on the result in Qu (1999).

function $\zeta_b(t)$ and ensures stability. Boundary controls given by (5) and (6) are synthesized using Lyapunov's direct method (Hale, 1977). Specifically, the following continuous time Lyapunov function candidate is adopted as one of two Lyapunov functions used in stability analysis and control synthesis:

$$V_c(t)=\int_0^l m\left\{[y_t(x,t)+\zeta_b(t)]^2+\frac{T_0}{m}y_x^2(x,t)+\frac{AE}{4m}y_x^4(x,t)+\frac{\lambda x}{l}[y_t(x,t)+\zeta_b(t)]y_x(x,t)\right\}\mathrm{d}x, \tag{9}$$

where its initial condition can be computed using the initial conditions in (2), and $\lambda>0$ is given by (7).

Boundary controls (5) and (6) are chosen so that Lyapunov function (9) has a dissipative property as defined in the following lemma.

Lemma 1. *If constant λ is chosen according to inequality (7), Lyapunov function $V_c(t)$ defined by (9) is positive definite with respect to $y_x(x,t)$ and $[y_t(x,t)+\zeta_b(t)]$. Furthermore, along every trajectory of system (1) with boundary conditions in (3) and under boundary controls (5) and (6), the time derivative of Lyapunov function $V_c(t)$ satisfies the inequality that*

$$\dot V_c\leqslant-\varepsilon_v V_c-2k_0[y_t(0,t)+\Delta_j(t-jT_b)]^2-\left[2k_l-\frac{1}{2}\lambda m-\frac{9k_l^2}{16T_0}\lambda\right][y_t(l,t)+\Delta_j(t-jT_b)]^2+[\zeta_b(t)-\Delta_j(t-jT_b)]\{H[y_t,\Delta_j]+c_v\zeta_b(t)\}, \tag{10}$$

where $k_0>0$, k_l is chosen according to (8), $c_v=0.5\lambda m$,

$$\varepsilon_v=\frac{1}{l}\min\left\{\frac{\lambda}{2+\lambda},\frac{\lambda T_0}{2T_0+\lambda m},\frac{3}{2}\lambda\right\}, \tag{11}$$

$H[y_t,\Delta_j]$ is the short-hand for $H[y_t(0,t),y_t(l,t),\Delta_j(\tau)]=-[2k_0y_t(0,t)+(2k_l-\lambda m)y_t(l,t)+(2k_l+2k_0-0.5\lambda m)\Delta_j(t-jT_b)]$, and $H[\cdot]$ will be used as the parameter and the feedback function in the learning control to be designed later in Section 2.3.

Proof. It follows from inequality $a^2+b^2\geqslant 2ab$ that, if $\lambda<\sqrt{T_0/m}$, Lyapunov function is positive definite with respect to $[y_t(x,t)+\zeta_b(t)]$ and $y_x(x,t)$ as

$$V_c(t)\geqslant\frac{1}{2}\{m,T_0,0.5AE\}V_o(t),$$

and

$$V_c(t)\leqslant\max\{m+0.5\lambda m,T_0+0.5\lambda m,0.25AE\}V_o(t), \tag{12}$$

where $V_o(t)=\int_0^l\{[y_t(x,t)+\zeta_b(t)]^2+y_x^2(x,t)+y_x^4(x,t)\}\,\mathrm{d}x$.

It follows from dynamic equation (1) that the time derivative of $V_c(t)$ is

$$\dot V_c(t)=\int_0^l\left\{2\frac{\partial\{T(x,t)y_x(x,t)[y_t(x,t)+\zeta_b(t)]\}}{\partial x}+\frac{1}{2l}\lambda mx\frac{\partial[y_t(x,t)+\zeta_b(t)]^2}{\partial x}+\frac{1}{2l}\lambda xT_0\frac{\partial y_x^2(x,t)}{\partial x}+\frac{3}{8l}\lambda xAE\frac{\partial y_x^4(x,t)}{\partial x}\right\}\mathrm{d}x. \tag{13}$$

Performing the operation of integration by part to the last three terms in (13) yields

$$\int_0^l\frac{1}{2l}\lambda mx\frac{\partial[y_t(x,t)+\zeta_b(t)]^2}{\partial x}\mathrm{d}x=\frac{1}{2}\lambda m[y_t(l,t)+\zeta_b(t)]^2-\frac{1}{2l}\lambda m\int_0^l[y_t(x,t)+\zeta_b(t)]^2\,\mathrm{d}x, \tag{14}$$

$$\int_0^l\frac{1}{2l}\lambda xT_0\frac{\partial y_x^2(x,t)}{\partial x}\mathrm{d}x=\frac{1}{2}\lambda T_0y_x^2(l,t)-\frac{1}{2l}\lambda T_0\int_0^l y_x^2(x,t)\,\mathrm{d}x,$$

and

$$\int_0^l\frac{3}{8l}\lambda xAE\frac{\partial y_x^4(x,t)}{\partial x}\mathrm{d}x=\frac{3}{8}\lambda AEy_x^4(l,t)-\frac{3}{8l}\lambda AE\int_0^l y_x^4(x,t)\,\mathrm{d}x.$$

It follows from boundary conditions in (3) that

$$\tfrac{1}{2}\lambda T_0y_x^2(l,t)+\tfrac{3}{8}\lambda AEy_x^4(l,t)=-\tfrac{3}{4}\lambda y_x(l,t)f_l(t)-\tfrac{1}{4}\lambda T_0y_x^2(l,t),$$

and that

$$\int_0^l\frac{\partial\{T(x,t)y_x(x,t)[y_t(x,t)+\zeta_b(t)]\}}{\partial x}=-f_l(t)[y_t(l,t)+\zeta_b(t)]-f_0(t)[y_t(0,t)+\zeta_b(t)].$$

It follows from boundary control (5) that, during the ith trial,

$$-f_l(t)[y_t(l,t)+\zeta_b(t)]=-k_l[y_t(l,t)+\Delta_j(t-jT_b)]^2-k_l[\zeta_b(t)-\Delta_j(t-jT_b)][y_t(l,t)+\Delta_j(t-jT_b)].$$

Similarly, it follows from boundary control (6) that

$$\begin{aligned}-f_0(t)[y_t(0,t)+\zeta_b(t)]\\ &= -k_0[y_t(0,t)+\Delta_j(t-jT_b)]^2\\ &\quad -k_0[\zeta_b(t)-\Delta_j(t-jT_b)][y_t(0,t)+\Delta_j(t-jT_b)].\end{aligned} \tag{15}$$

Substituting all the expressions from (14) up to (15) into (13) yields

$$\begin{aligned}\dot V_c(t) &= -2k_0[y_t(0,t)+\Delta_j(t-jT_b)]^2\\ &\quad -2k_l[y_t(l,t)+\Delta_j(t-jT_b)]^2-\frac{3}{4}\lambda y_x(l,t)f_l(t)\\ &\quad -\frac{1}{4}\lambda T_0 y_x^2(l,t)+\frac{1}{2}\lambda m[y_t(l,t)+\zeta_b(t)]^2\\ &\quad -\frac{1}{2l}\lambda m\int_0^l [y_t(x,t)+\zeta_b(t)]^2\,dx\\ &\quad -\frac{1}{2l}\lambda T_0\int_0^l y_x^2(x,t)\,dx-\frac{3}{8l}\lambda AE\int_0^l y_x^4(l,t)\,dx\\ &\quad -[\zeta_b(t)-\Delta_j(t-jT_b)][2k_l y_t(l,t)\\ &\quad +2k_0 y_t(0,t)+2(k_l+k_0)\Delta_j(t-jT_b)].\end{aligned}$$

Inequality (10) can be concluded based on the above expression of $\dot V_c$ by noting the following inequalities and equality. First, it follows from (12) that

$$\begin{aligned}&-\frac{1}{2l}\lambda m\int_0^l [y_t(x,t)+\zeta_b(t)]^2\,dx-\frac{1}{2l}\lambda T_0\int_0^l y_x^2(x,t)\,dx\\ &\quad-\frac{3}{8l}\lambda AE\int_0^l y_x^4(l,t)\,dx \leqslant -\varepsilon_v V_c,\end{aligned}$$

where ε_v is that defined in (11). Second, it follows from $a^2+b^2\geqslant 2ab$ that

$$\begin{aligned}&-\frac{9\lambda k_l^2}{16T_0}[y_t(l,t)+\Delta_j(t-jT_b)]^2-\frac{3}{4}\lambda y_x(l,t)f_l(t)\\ &\quad-\frac{1}{4}\lambda T_0 y_x^2(l,t)\leqslant 0.\end{aligned}$$

Finally, it follows that

$$\begin{aligned}&\tfrac{1}{2}\lambda m[y_t(l,t)+\zeta_b(t)]^2\\ &\quad=\tfrac{1}{2}\lambda m[y_t(l,t)+\Delta_j(t-jT_b)]^2+\tfrac{1}{2}\lambda m[\zeta_b(t)\\ &\qquad-\Delta_j(t-jT_b)][2y_t(l,t)+\zeta_b(t)+\Delta_j(t-jT_b)].\end{aligned}$$ □

If $\zeta_b(t)$ were known, there would be no need to learn. In this case, one can set $\Delta_j(t-jT_b)$ in boundary controls (5) and (6) as $\Delta_j(t-jT_b)=\zeta_b(t)$ under which inequality (10) becomes

$$\dot V_c \leqslant -\varepsilon_v V_c,$$

in which $k_0>0$ and inequality (8) is applied. The solution to the above differential inequality is

$$V(t)\leqslant V(t_0)e^{-\varepsilon_v t},$$

which demonstrates global and exponential stability as $\lambda>0$ is used in the analysis. In short, if $\zeta_b(t)$ were known, controls (5) and (6) would be globally and exponentially stabilizing, which provides a solid foundation for learning control design.

3.2. Design and analysis of learning algorithm

Since $\zeta_b(t)$ is unknown, learning term Δ_j must be designed properly to learn the unknown time function and hence ensure stability. The learning term, Δ_j, is updated from trial to trial by the following learning law: for $\tau\in[0,T_b]$ and for all $j\geqslant 0$,

$$\gamma\frac{d\Delta_j}{d\tau}+(1+\beta)\Delta_j(\tau)=(1-\gamma)\Delta_{j-1}(\tau)+\alpha H[y_t,\Delta_j], \tag{16}$$

where $i=1,\dots,m$, $j=0,1,\dots$, $\Delta_{-1}=0$, $H(\cdot)$ is the function defined in Section 2.2, α and β are learning control gains, γ is a design parameter, and ranges of their values are

$$\alpha>0,\beta\geqslant 0,\quad \text{and}\quad 0\leqslant\gamma<1. \tag{17}$$

The choice of γ determines whether a difference or difference-differential learning law is selected. Whenever $\gamma>0$ is set, Δ_j defined by (16) should be solved under initial condition $\Delta_j(0)=\Delta_{j-1}(T_b)$, where T_b denotes the duration of all learning trials.

The above learning control is synthesized based on Lyapunov's direct method using the following discrete Lyapunov function: for all j,

$$\begin{aligned}L_j &= \frac{1}{2}(1-\gamma)\int_0^{T_b}|\zeta_b(\tau)-\Delta_j(\tau)|^2\,d\tau\\ &\quad+\frac{1}{2}\gamma|\zeta_b(T_b)-\Delta_j(T_b)|^2,\end{aligned} \tag{18}$$

which consists of Euclidean norm and $\mathscr{L}_2$ norm (Khalil, 1992) of learning error $[\zeta_b(t)-\Delta_j]$. Property of learning control algorithm (16) is given by the following lemma.

Lemma 2. *Under iterative learning law* (16), *the incremental change of Lyapunov function* (18) *with respect to trials,* $\delta L_j \triangleq L_j-L_{j-1}$, *satisfies the inequality that*

$$\begin{aligned}\delta L_j &\leqslant \int_0^{T_b}|d(\tau)|\,d\tau-\beta\int_0^{T_b}|\zeta_b(\tau)-\Delta_j(\tau)|^2\,d\tau\\ &\quad-\alpha\int_{(j-1)T_b}^{jT_b}(\zeta_b(t)-\Delta_j(t-jT_b))\{H[y_t,\Delta_j]\\ &\quad+c_v\zeta_b(t)\}\,dt,\end{aligned} \tag{19}$$

where c_v is the constant defined in Section 2.2, and

$$d(t) = \frac{1}{4\gamma}[\gamma\dot{\zeta}_b + (\gamma + \beta + \alpha c_v)\zeta_b]^2$$

is an exogenous signal.

Proof. It follows from the choice of initial condition of learning law (16) and from periodicity of $\zeta_b(\cdot)$ that the difference of Lyapunov function between two successive trials, $\delta L_j = L_j - L_{j-1}$, can be rewritten as

$$\delta L_j = \frac{1}{2}(1-\gamma)\int_0^{T_b} [|\zeta_b - \Delta_j|^2 - |\zeta_b - \Delta_{j-1}|^2]\,d\tau + \int_0^{T_b} (\zeta_b - \Delta_j)(\gamma\dot{\zeta}_b - \gamma\dot{\Delta}_j)\,d\tau.$$

It follows from (16) that

$$\begin{aligned}\delta L_j = &-\int_0^{T_b} \left|\frac{1}{2\sqrt{\gamma}}[\gamma\dot{\zeta}_b + (\beta + 0.5\alpha c_v - \gamma)\zeta_b] + \sqrt{\gamma}\Delta_j\right|^2 d\tau - \frac{1}{2}(1-\gamma)\int_0^{T_b} |\Delta_j - \Delta_{j-1}|^2\,d\tau \\ &- \beta\int_0^{T_b} |\zeta_b - \Delta_j|^2\,d\tau \\ &+ \frac{1}{4\gamma}\int_0^{T_b} |\gamma\dot{\zeta}_b + (\gamma + \beta + \alpha c_v)\zeta_b|^2\,d\tau \\ &- \alpha\int_0^{T_b} (\zeta_b - \Delta_j)\{H[y_t, \Delta_j] + c_v\zeta_b\}\,d\tau,\end{aligned}$$

from which inequality (19) can be obtained. □

3.3. Stability of the overall system

Stability of the overall system is guaranteed, as stated in the following theorem.

Theorem 1. *Consider system* (1) *with boundary conditions in* (3) *and under boundary controls* (5) *and* (6) *together with iterative learning law* (16). *If the control gains and design parameters are chosen according to* (7), (8), $k_0 > 0$, *and* (17), *the closed loop system is stable in the sense that* $y_x(x,t)$, $[y_t(x,t) + \zeta_b(t)]$, *and* $[y_t(x,t) + \Delta_j(\tau)]$ *are all uniformly continuous and uniformly bounded.*

Proof. Consider Lyapunov functional

$$\mathscr{E}(t) = \frac{1}{2}(1-\gamma)\int_t^{t-T_b} |\zeta_b(s) - \Delta(s)|^2\,ds + \frac{1}{2}\gamma|\zeta_b(t) - \Delta(t)|^2,$$

where $\Delta(w) = \Delta_j(w - jT_b)$ for an appropriate value of j and for $w \in [jT_b, jT_b + T_b]$. It is obvious that $\mathscr{E}(t)$ is positive definite with respect to $[\zeta_b(t) - \Delta(t)]$. It follows that

$$\begin{aligned}\dot{\mathscr{E}}(t) = &\tfrac{1}{2}(1-\gamma)[\zeta_b(t) - \Delta(t)]^2 - \tfrac{1}{2}(1-\gamma)[\zeta_b(t - T_b) \\ &- \Delta(t - T_b)]^2 + \gamma[\zeta_b(t) - \Delta(t)][\dot{\zeta}_b(t) - \dot{\Delta}(t)].\end{aligned}$$

On the other hand, learning law (16) can be rewritten as

$$\gamma\dot{\Delta}(t) + (1+\beta)\Delta(t) = (1-\gamma)\Delta(t - T_b) + \alpha H[y_t, \Delta].$$

Substituting the expression of $\dot{\Delta}(t)$ into that of $\dot{\mathscr{E}}(t)$ yields

$$\begin{aligned}\dot{\mathscr{E}}(t) \leqslant &-\beta[\zeta_b(t) - \Delta(t)]^2 + \frac{1}{4\gamma}\,d(t) \\ &- \alpha[\zeta_b(t) - \Delta(t)]\{H[y_t, \Delta] + c_v\zeta_b\},\end{aligned}$$

where $d(t)$ and c_v are those in Lemma 2. It follows from Lemma 1 that

$$\begin{aligned}\dot{V}_c + \dot{\mathscr{E}}(t) \leqslant &-\varepsilon_v V_c - 2k_0[y_t(0,t) + \zeta_b(t)]^2 \\ &- \left[2k_l - \frac{1}{2}\lambda m - \frac{9k_l^2}{16T_0}\lambda\right][y_t(l,t) + \zeta_b(t)]^2 \\ &- \beta[\zeta_b(t) - \Delta(t)]^2 + \frac{1}{4\gamma}d(t),\end{aligned}$$

from which uniform continuity and uniform boundedness can be concluded by Theorem 2.21 in Qu (1998). □

3.4. Performance of the overall system

Performance of the closed-loop, learning, boundary-control system can be established by combining the analysis in Sections 2.2 and 2.3 and by applying the following lemma which can easily be concluded using Barbalat lemma (Narendra & Annaswamy, 1989).

Definition. Consider a dynamic system whose input–output pair is $\{e(t), y(t)\}$. Then, *attenuation factor* η from the input to the output is defined as follows: if $e(t) \in L_2$, inequality

$$\int_0^t y^2(s)\,ds \leqslant c + \eta\int_0^t e^2(s)\,ds$$

holds for all t and for some constant c; or if $d(t) \notin L_2$, inequality

$$\lim_{\tau\to\infty}\sup_{t\geqslant\tau}\frac{\int_0^t y^2(s)\,ds}{\int_0^t e^2(s)\,ds} \leqslant \eta$$

holds for all t.

Lemma 3. *Consider the following inequality*:

$$w(t) + a\int_0^t w(s)\,ds \leqslant b + \int_0^t |d(s)|\,ds, \tag{20}$$

where $a, b > 0$ are constants, $w(t)$ is a nonnegative function, and $d(t)$ is a bounded exogenous signal. Then,

(i) *If $d(t) = 0$, $w(t)$ is uniformly bounded.*
(ii) *If $d(t) = 0$ or if $d(t) \in L_1$, $w(t)$ converges to zero as time approaches infinity provided that $w(t)$ is uniformly continuous.*

(iii) *The attenuation factor from* $\sqrt{|d(t)|}$ *to* $\sqrt{w(t)}$ *is no larger than* $1/a$.

Performance of the overall system will be concluded via the following three steps. First, integrating both sides of inequality (10) yields

$$\begin{aligned} V_c(t) + \varepsilon_v \int_0^t V_c(s)\,ds &\leqslant V_c(0) - 2k_0 \int_0^t [y_t(0,s) + \Delta(s)]^2\,d\tau \\ &- \left[2k_l - \frac{1}{2}\lambda m - \frac{9k_l^2}{16T_0}\lambda\right] \int_0^t [y_t(l,s) + \Delta(s)]^2\,ds \\ &+ \int_0^t [\zeta_b(s) - \Delta(s)]\{H[y_t,\Delta] + c_v\zeta_b(s)\}\,ds, \end{aligned}$$

where $\Delta(s) = \Delta_j(s - jT_b)$ for an appropriate value of j and for $s \in [jT_b, jT_b + T_b]$. Second, summing both sides of inequality (19) with respect to trials yields

$$\begin{aligned} L_j \leqslant L_0 &+ \int_0^{jT_b} |d(s)|\,ds - \beta \int_0^{jT_b} |\zeta_b(s) - \Delta(s)|^2\,ds \\ &- \alpha \int_0^{jT_b} [\zeta_b(s) - \Delta(s)]\{H[y_t,\Delta] + c_v\zeta_b(s)\}\,ds. \end{aligned}$$

In the third and final step, we add up both sides of the above two inequalities at $t = jT_b$ and have

$$\begin{aligned} &\alpha V_c(jT_b) + \varepsilon_v\alpha \int_0^{jT_b} V_c(s)\,ds + L_j + \beta \int_0^{jT_b} |\zeta_b(s) \\ &\quad -\Delta(s)|^2\,ds + 2k_0\alpha \int_0^{jT_b} [y_t(0,s) + \Delta(s)]^2\,ds \\ &\leqslant \alpha V_c(0) + L_0 + \int_0^{jT_b} |d(s)|\,ds - \alpha\left[2k_l - \frac{1}{2}\lambda m \right.\\ &\quad \left. - \frac{9k_l^2}{16T_0}\lambda\right] \int_0^{jT_b} [y_t(l,s) + \Delta(s)]^2\,ds, \end{aligned}$$

based on which the second main theorem of the paper can be concluded.

Theorem 2. *Consider system* (1) *with boundary conditions in* (3) *and under boundary controls* (5) *and* (6) *together with iterative learning law* (16). *If the control gains and design parameters are chosen according to* (7), (8), $k_0 > 0$, *and* (17), *the closed loop system has the following stability properties*:

(i) *The attenuation factors from* $\dot{\zeta}_b$ *and* ζ_b *to* $\sqrt{V_c(t)}$ (*which is a positive definite function of both* $y_x(x,t)$, $[y_t(x,t) + \zeta_b(t)]$) *are no larger than* $\gamma l \max\{2 + \lambda, 2 + \lambda m/T_0\}/(4\alpha\lambda)$ *and* $l(\gamma + \beta + 0.5\alpha\lambda m)^2 \max\{2 + \lambda, 2 + \lambda m/T_0\}/(4\alpha\lambda\gamma)$, *respectively*.

(ii) *The attenuation factors from* $\dot{\zeta}_b$ *and* ζ_b *to* $[\zeta_b(t) - \Delta(t)]$ *are no larger than* $\gamma/(4\beta)$ *and* $(\gamma + \beta + 0.5\alpha\lambda m)^2/(4\beta\gamma)$, *respectively*.

(iii) *The attenuation factors from* $\dot{\zeta}_b$ *and* ζ_b *to* $[y_t(0,s) - \Delta(t)]$ *are no larger than* $\gamma/(8k_0\alpha)$, $(\gamma + \beta + 0.5\alpha\lambda m)^2/(8k_0\alpha\gamma)$, *respectively*.

(iv) *The attenuation factors from* $\dot{\zeta}_b$ *and* ζ_b *to* $[y_t(l,s) - \Delta(t)]$ *are no larger than* $\gamma/(4\alpha k_l')$ *and* $(\gamma + \beta + 0.5\alpha\lambda m)^2/(4\alpha\gamma k_l')$, *respectively, where* $k_l' = 2k_l - 0.5\lambda m - 9k_l^2\lambda/(16T_0)$.

Proof. It follows from Lemma 2 that

$$|d(t)| \leqslant \frac{\gamma}{4}|\dot{\zeta}_b|^2 + \frac{1}{4\gamma}(\gamma + \beta + \alpha c_v)^2|\zeta_b|^2.$$

The attenuation gains can now be obtained using statement (iii) of Lemma 3. □

The expressions of attenuation factors enable the control designer to select control gains. For instance, let us consider the attenuation factors from $\dot{\zeta}_b$ and ζ_b to $[y_t(l,s) - \Delta(t)]$. These two attenuation factors can be made arbitrarily small by choosing control gains in one of the two ways: either $\lambda \ll 1$, $\lambda \sim \gamma \sim \beta$, $\alpha\lambda \sim 1$, and $k_l = 16T_0/(9\lambda)$; or $\lambda \ll 1$, $\lambda \sim \gamma \sim \beta$, $\alpha \sim 1$, and $k_l \sim 1$, where $a \sim b$ denotes that a and b are infinitesimal (or infinite) of the same order. Similarly, the attenuation factors from $\dot{\zeta}_b$ and ζ_b to $[y_t(0,s) - \Delta(t)]$ can also be made arbitrarily small.

On the other hand, expressions of other attenuation factors point to the need of an improved learning control design. Specifically, more emphasis should be placed on minimizing the attenuation factor from ζ_b to $\sqrt{V_c(t)}$. By doing so, vibration measured by $y_x(x,t)$ and $[y_t(x,t) - \zeta_b(t)]$ can be minimized. It is worth noting that the attenuation factor from ζ_b to $[\zeta_b(t) - \Delta(t)]$ is bounded from below by 1 and that the attenuation factor from ζ_b to $\sqrt{V_c(t)}$ is bounded from below by $0.5ml$. Further development is needed to minimize these two attenuation factors.

4. Conclusion

In this paper, learning control methodology is applied to generate boundary learning control for a string system that is modeled by a partial differential equation and is subject to an unknown but periodic exogenous signal. The proposed result represents the first attempt in extending learning control theory directly to partial differential equations, the original models for distributed-parameter systems. It is shown that, despite of nonlinear tension, boundary learning control can be successfully designed to ensure stability and performance and to attenuate the effect of exogenous signal. Future research is needed to develop asymptotically stabilizing learning control for flexible systems.

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