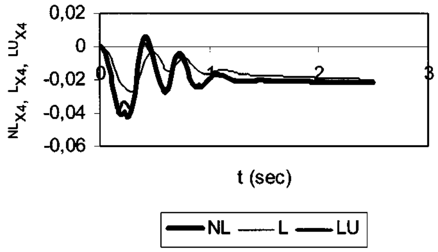

Fig. 4. x_4 (normalized variable which corresponds to PTT) time history comparison of nonlinear model, linear model, and linear model considering uncertainty.

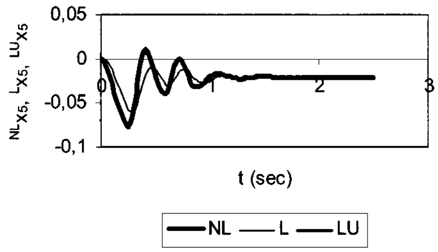

Fig. 5. x_5 (normalized variable which corresponds to TTT) time history comparison of nonlinear model, linear model, and linear model considering uncertainty.

$\lambda_2 = -5.305$, $\lambda_3 = -14.789 + j0.035$, $\lambda_4 = -14.789 - j0.035$, $\lambda_5 = -15.510$), and the linear model having uncertain eigenvalues ($\tilde{\lambda}_1^{\min} = -2.175$, $\tilde{\lambda}_1^{\max} = -0.055$, $\tilde{\lambda}_2^{\min} = -7.520$, $\tilde{\lambda}_2^{\max} = -2.186$, $\mathrm{Re}\{\tilde{\lambda}_3\}^{\min} = \mathrm{Re}\{\tilde{\lambda}_4\}^{\min} = -81.277$, $\mathrm{Re}\{\tilde{\lambda}_3\}^{\max} = \mathrm{Re}\{\tilde{\lambda}_4\}^{\max} = -8.565$, $\mathrm{Im}\{\tilde{\lambda}_3\}^{\min} = -\mathrm{Im}\{\tilde{\lambda}_4\}^{\max} = 0.042$, $\mathrm{Im}\{\tilde{\lambda}_3\}^{\max} = -\mathrm{Im}\{\tilde{\lambda}_4\}^{\min} = 0.068$, $\tilde{\lambda}_5^{\min} = -86.615$, $\tilde{\lambda}_5^{\min} = -47.632$) are compared. The linear model with uncertain eigenvalues is a perfect match for the nonlinear model. The modeling errors (1%–10%) meet current control requirements.

IV. Conclusion

A new method of multivariable linear model matrix parameter estimation was developed. The method enables obtaining the bounds for real and imaginary parts of uncertain matrix eigenvalues of a multivariable linear model from an aircraft turbofan engine detailed nonlinear model response in the time domain. This linear model only takes into account the difference between nonlinear and linear models and not the uncertainty in the nonlinear model itself. Such linear model and nonlinear model match perfectly. Its modeling errors meet current control requirements. The method uses nonlinear programming and can consequently consider any constraints for the estimated bounds for real and imaginary parts of uncertain matrix eigenvalues.

The results of this note may be applied to advanced turbofan engine and aircraft control systems, as well as to other dynamic systems, which may be described by multivariable linear models considering modeling uncertainty.

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Asymptotic Learning Control for a Class of Cascaded Nonlinear Uncertain Systems

Zhihua Qu and Jianxin Xu

***Abstract*—In this note, the problem of learning unknown functions in a class of cascaded nonlinear systems will be studied. The functions to be learned are those functions that are imbedded in the system dynamics and are of known period of time. In addition to the unknown periodic time functions, nonlinear uncertainties bounded by known functions of the state are also admissible. The objective of the note is to find an iterative learning control under which the class of nonlinear systems are globally stabilized (in the sense of being uniform bounded), their outputs are asymptotically convergent, and a combination of the time functions contained in system dynamics are asymptotically learned. To this end, a new type of differential-difference learning law is utilized to generate the proposed learning control that yields both asymptotic stability of the system output and asymptotic convergence of the learning error. The design is carried out by applying the Lyapunov direct method and the backward recursive design method.**

***Index Terms*—Learning control, Lyapunov design, periodic function, stability, uncertain system.**

I. Introduction

In a typical control design, one often begins with formulating the so-called tracking-error system to define the dynamics of the error between the plant output and a given output trajectory. There are many control applications in which the desired trajectory for the system output is repetitive. If so, the tracking-error system will contain time functions of known period. In case that there exist unknown dynamics (such as unknown parameters, unknown time functions, etc.) in the plant, periodic time functions in the resulting tracking-error system will be mixed with unknown, nonlinear dynamics of the plant. For such an uncertain system, three different design methodologies can be applied. One is the robust control theory which can be viewed as the worst-scenario control design method [20]. The second is the adaptive control which can be used to identify online nonlinear dynamics that are parameterized in terms of a set of unknown constants [12]. The

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third is the learning control theory using which an iterative learning control is designed to learn the unknown, periodic time functions and, hence, to establish stability and performance. In this note, new results will be presented to enrich the learning control theory so that it can be applied to more nonlinear systems and to achieve better performance.

Several model-based approaches have been proposed to design learning controls: linear learning design framework [2], [3], [4], [10] using functional norm; linear learning design based on internal model principle [21]; an approach parallel to adaptive control design [5], [9], [13]; generalized inversion of input matrix [7]; linear high-gain control [14], [17]; robustness analysis under disturbance [8]; removal of derivative measurement of the state [11], [18], [22]; and nonlinear design methods such as passivity design [1] and Lyapunov method [6], [18]. For the history of learning control and for other approaches of learning control that are not model based, the readers are referred to [15]. It is noted that these results aforementioned (as well as the proposed design in this note) are for systems with deterministic models and that there is also probability learning theory [23].

For nonlinear systems, learning control has to be designed using nonlinear techniques in order to achieve global stability. Although the learning control in [1] achieves asymptotic stability for torque-level, rigid-body mechanical systems, its extension to high-order system will call for derivative measurement of the state as it is based on a difference learning law. The result [6] overcomes this difficulty by employing a differential-difference learning law and by a judicious choice of Lyapunov function, but the resulting stability is a uniform bounded property. Convergence of learning error is not established in either of these two results. In this note, an improvement of the result in [6] is presented so that both asymptotic stability of the system output and asymptotic convergence of a composite learning error are obtained.

II. Problem Statement

In this note, the class of cascaded nonlinear systems is considered. Specifically, a system consisting of m sequentially connected subsystems is of the form

$$\begin{aligned} h_{i,j}(x_{1,j},\ldots,x_{i,j},t)\dot{x}_{i,j} \\ = F_{i,j}(x_{1,j},\ldots,x_{i,j},\zeta_i(t)) + x_{i+1,j} \end{aligned} \tag{1}$$

for $i = 1,\ldots,m-1$, and

$$\begin{aligned} h_{m,j}(x_{1,j},\ldots,x_{m,j},t)\dot{x}_{m,j} \\ = F_{m,j}(x_{1,j},\ldots,x_{m,j},\zeta_m(t)) + u_j \end{aligned} \tag{2}$$

where $t \in [0,T]$ is the time during a specific trial, T is the duration of all trials, $x_{i,j} \in \Re^n$ is the state of the ith subsystem during the jth trial, $\zeta_i(t)$ are unknown time functions of period T, and $u_j \in \Re^n$ is the control function (to be designed) during the jth trial. The design objective is to find a control law u_j (which will be a function of $x_{i,j}$ (for $i = 1,\ldots,m$) and u_{j-1} in a closed form) such that the closed-loop system is uniformly bounded, that the system output (given by $x_{1,j}$) is asymptotically convergent (as $j \to \infty$), and that combinations of unknown time functions are asymptotically learned. The design will be carried out using the backward recursive method. For the ease of presentation, we shall limit our derivation to the case of $m = 2$ so that the main idea of how to achieve asymptotic stability of the output and asymptotic convergence of the learning error can be emphasized.

Letting

$$\begin{aligned} \delta_i(t) =& F_{i,j}(0,\ldots,0,\zeta_i(t)) \text{ and } g_{i,j}(x_{1,j},\ldots,x_{i,j},t) \\ =& F_{i,j}(x_{1,j},\ldots,x_{i,j},\zeta_i(t)) - F_{i,j}(0,\ldots,0,\zeta_i(t)) \end{aligned}$$

we can rewrite system (1) and (2) with $m = 2$ as

$$h_{1,j}(x_{1,j},t)\dot{x}_{1,j} = \delta_1(t) + g_{1,j}(x_{1,j},t) + x_{2,j} \tag{3}$$

and

$$h_{2,j}(x_{1,j},x_{2,j},t)\dot{x}_{2,j} = \delta_2(t) + g_{2,j}(x_{1,j},x_{2,j},t) + u_j. \tag{4}$$

It is obvious that time functions $\delta_i(t)$ are periodic with respect to T.

In a typical application, repeated trials can be implemented in one of the following two ways: either resetting initial conditions (ICs) at the beginning of each trial or letting the system resume its operation (possibly after a pause but without resetting). Thus, it can be assumed without loss of any generality that IC of the jth trial is given by either $x_{i,j}(0) = \mathrm{IC}_{i0}$ or $x_{i,j}(0) = x_{i,j-1}(T)$ where IC_{i0} is a constant vector for all j (as, for the same repeated tasks, resetting the ICs at different values for different trials would not make much sense). If $\mathrm{IC}_{i0} \neq 0$, one can introduce the transformation $y_i = x_i - \mathrm{IC}_{i0}$ and rewrite system (3) and (4) as

$$h_{1,j}(y_{1,j} + \mathrm{IC}_{10},t)\dot{y}_{1,j} = \delta_1'(t) + g_{1,j}'(y_{1,j},t) + y_{2,j} \tag{5}$$

and

$$\begin{aligned} h_{2,j}(y_{1,j} + \mathrm{IC}_{10}, y_{2,j} + \mathrm{IC}_{20},t)\dot{y}_{2,j} \\ = \delta_2'(t) + g_{2,j}'(y_{1,j},y_{2,j},t) + u_j \end{aligned} \tag{6}$$

where

$$\begin{aligned} \delta_1'(t) =& \delta_1(t) + g_{1,j}'(\mathrm{IC}_{10},t) + \mathrm{IC}_{20} \\ g_{1,j}'(y_{1,j},t) =& g_{1,j}(y_{1,j} + \mathrm{IC}_{10},t) - g_{1,j}(\mathrm{IC}_{10},t) \\ \delta_2'(t) =& \delta_2(t) + g_{2,j}'(\mathrm{IC}_{10},\mathrm{IC}_{20},t) \end{aligned}$$

and

$$\begin{aligned} g_{2,j}'(y_{1,j},y_{1,2},t) = g_{2,j}(y_{1,j} + \mathrm{IC}_{10}, y_{2,j} + \mathrm{IC}_{20},t) \\ - g_{2,j}(\mathrm{IC}_{10},\mathrm{IC}_{20},t). \end{aligned}$$

Equations (5) and (6) have the same structure as the original equations (3) and (4) (which will be used in our control design for the case that $\mathrm{IC}_{i0} = 0$). Thus, $\mathrm{IC}_{i0} = 0$ will be assumed without loss of any generality in the subsequent discussion. In fact, in many applications, resetting of ICs is done by making the system return to its home position (which can be set as the origin of the state space).

For asymptotic stabilization of the system output and asymptotic convergence of learning error, dynamics of system (3) and (4) are required to satisfy the following assumptions.

Assumption 1: Time functions $\delta_i(t)$ are periodic with respect to T and are bounded in norm as, for all time

$$\begin{aligned} \|\delta_i(t)\| \le& \bar{\delta}_{i1} \quad \|\dot{\delta}_i(t)\| \le \bar{\delta}_{i2} \\ \|\ddot{\delta}_i(t)\| \le& \bar{\delta}_{i3}, \text{ and } \left\| \frac{d^3\delta_i(t)}{dt^3} \right\| \le \bar{\delta}_{i4}. \end{aligned}$$

Assumption 2: Matrix functions $h_{i,j}(\cdot)$ are known, periodic in t with respect to T, symmetric, and positive–definite as, for all $\{x_{1,j}, x_{2,j}, t\}$

$$0 < \underline{h}_i I \le h_{i,j} \le \overline{h}_i I$$

for constants $\underline{h}_i$ and $\overline{h}_i$. In addition, their time derivatives are bounded by known functions as

$$\|\dot{h}_{1,j}\| \le \rho_{h_1}(x_{1,j}) \text{ and } \|\dot{h}_{2,j}\| \le \rho_{h_2}(x_{1,j}, x_{2,j}).$$

Furthermore, matrix $h_{i,j}(\cdot)$ is diagonally dominant in the sense that, letting $w_{i,j} = h_{i,j}^{-1}$ and $w_{i,j}^{k,l}$ be the element of $w_{i,j}$ on the kth row and lth column, inequalities

$$w_{i,j}^{k,k} - \sum_{l=1,\, l\neq k}^{n} \left|w_{i,j}^{k,l}\right| \ge c_{w,k}$$

hold for $k = 1, \ldots, n$ and for constants $c_{w,k} \ge c_w > 0$.

Assumption 3: Vector functions $g_{i,j}(\cdot)$ are bounded by known functions as

$$\|g_{1,j}(x_{1,j}, t)\| \le \rho_{g_1}(x_{1,j})\|x_{1,j}\| \tag{7}$$

and

$$\begin{aligned}\|g_{2,j}(x_{1,j}, x_{2,j})\| \le{}& \rho_{g_{21}}(x_{1,j}, x_{2,j})\|x_{1,j}\| \\ &+ \rho_{g_{22}}(x_{1,j}, x_{2,j})\|x_{2,j}\| \triangleq \rho_{g_2}(x_{1,j}, x_{2,j}).\end{aligned} \tag{8}$$

In essence, the assumptions imply that dynamics of the system are bounded by nonlinear functions of the state. Many physical systems meet these assumptions. For example, consider the dynamics of a rigid-body robot driven by dc motors [16]

$$\begin{aligned} M(q)\ddot{q} &= N(q,\dot{q}) + \tau \\ \tau &= K_t I \\ L_a \dot{I} &= -RI - E(\dot{q}) + v \end{aligned} \tag{9}$$

where q is the generalized coordinator, v is the voltage input, $E(\cdot)$ is the back electromotive force (EMF), and τ is the torque. Given a desired periodic trajectory $q^d(t)$, we can write dynamics of the tracking system as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ M(x_1 + q^d(t))\dot{x}_2 &= \delta_1(t) \\ &\quad + \left\{ \left[N(q,\dot{q}) - N(q^d, \dot{q}^d)\right] \right. \\ &\quad \left. + \left[M(q^d) - M(x_1 + q^d)\right]\ddot{q}^d \right\} + x_3 \\ L_a K_t^{-1}\dot{x}_3 &= \delta_2(t) - RK_t^{-1}x_3 - [E(\dot{q}) - E(\dot{q}^d)] + v \end{aligned}$$

where $x_1 = q - q^d$, $x_3 = \tau$

$$\delta_1(t) = N(q^d, \dot{q}^d) - M(q^d)\ddot{q}^d \text{ and } \delta_2(t) = -E(\dot{q}^d).$$

Due to the unknowns in the system (for example, friction in $N(q,\dot{q})$), functions $\delta_1(t)$ and $\delta_2(t)$ are unknown but periodic. The first subsystem has no unknowns or dynamics that need to be compensated for. It is easy to verify that subsystems 2 and 3 satisfy all the assumptions.

It is worth mentioning that, under the same assumptions listed above, robust controls have been proposed in [19] and [20] to ensure stability of being uniform and ultimate bounded. By taking advantage of periodicity, the proposed learning control can achieve asymptotic convergence for system output and for learning error, which is the main difference between the proposed result and the existing results.

III. Learning Control Design

Learning control will be designed for system (3) and (4) recursively. To this end, consider the following fictitious system:

$$h_{1,j}(x_{1,j}, t)\dot{x}_{1,j} = \delta_1(t) + g_{1,j}(x_{1,j}, t) + v_{1,j} \tag{10}$$

where $v_{1,j}$ is the fictitious control during the jth trial. An iterative learning control ensuring asymptotic stability for the first-order vector system (10) is provided by the following lemma.

Lemma 1: Consider system (10) under learning control

$$\begin{aligned} v_{1,j} = -&\left[\eta_1 x_{1,j} + \beta_1\rho_{g_1}(x_{1,j})x_{1,j} \right. \\ &\left. + \frac{1}{2}\rho_{h_1}(x_{1,j})x_{1,j} + \Delta_{1,j}\right] \end{aligned} \tag{11}$$

$$\begin{aligned} \gamma_1\dot{\Delta}_{1,j} = -&\Delta_{1,j} + (1-\gamma_1)\Delta_{1,j-1} \\ &+ \alpha_1 x_{1,j} + \gamma_1\lambda_1 h_{1,j}^{-1}\mathrm{sign}[x_{1,j}] \end{aligned} \tag{12}$$

where $\eta_1 > 0$ is a control gain, β_1 is another control gain satisfying inequality

$$\beta_1 > \max\left\{1, \frac{n}{c_w \underline{h}_1}\right\}$$

$0 < \gamma_1 < 1$ is the time constant of the differential-difference learning law, $\alpha_1 > 0$ is a learning gain, and λ_1 is another learning gain satisfying inequality

$$\begin{aligned}\lambda_1 \ge \max\Bigg\{ &\frac{(\beta_1+1)(\overline{\delta}_{11}+\overline{\delta}_{12})}{\beta_1 c_w - n\underline{h}_1^{-1}},\ \frac{3(\overline{\delta}_{11}+\overline{\delta}_{12})}{c_w}, \\ &\frac{\overline{\delta}_{11}+\overline{\delta}_{12}+\eta_1^{-1}\overline{h}_1(\overline{\delta}_{12}+\overline{\delta}_{13})}{c_w},\ \overline{h}_1(\overline{\delta}_{12}+\overline{\delta}_{11})\Bigg\}\end{aligned}$$

$\mathrm{sign}[\cdot]$ represents the vector sign function defined element by element, and $\Delta_{1,j}$ defined by (12) should be solved under IC $\Delta_{1,j}(0) = \Delta_{1,j-1}(T)$ with $\Delta_{1,-1}$ arbitrarily chosen. Then, under either a fixed IC (without loss of any generality, $x_{1,j}(0) = 0$ is used[1]) or IC resetting $(x_{1,j}(0) = x_{1,j-1}(T))$, Lyapunov function

$$\begin{aligned} V_{1,j} = \frac{1}{2}(1-\gamma_1)\int_0^T \|\delta_1(\tau) - \Delta_{1,j}(\tau)\|^2 d\tau \\ + \frac{1}{2}\gamma_1\|\delta_1(T) - \Delta_{1,j}(T)\|^2 \end{aligned} \tag{13}$$

has the property that, for a constant c_1 (independent of j)

$$\begin{aligned} V_{1,j} \le \sum_{k=1}^{j}\Bigg[-\gamma_1\int_0^T \|\delta_1 - \Delta_{1,k}\|^2 d\tau \\ -\eta_1\alpha_1\int_0^T \|x_{1,k}\|^2 d\tau\Bigg] - \frac{1}{2}\underline{h}_1\|x_{1,j}(T)\| + c_1. \end{aligned} \tag{14}$$

Furthermore, the system is asymptotically stable and its learning error converges to zero.

Proof: It follows from the choice of IC of learning law (12) and from periodicity of function $\delta_1(t)$ that the difference of Lyapunov function between two successive trials $\delta V_{1,j} = V_{1,j} - V_{1,j-1}$ can be rewritten as

$$\begin{aligned} \delta V_{1,j} = \frac{1}{2}(1-\gamma_1)\int_0^T \left[\|\delta_1 - \Delta_{1,j}\|^2 - \|\delta_1 - \Delta_{1,j-1}\|^2\right] d\tau \\ + \int_0^T [\delta_1 - \Delta_{1,j}]^T \left[\gamma_1\dot{\delta}_1 - \gamma_1\dot{\Delta}_{1,j}\right]^T d\tau. \end{aligned}$$

It follows from (12) that (15), as shown at the bottom of the next page, holds true.

[1] if $x_{1,j}(0) = \mathrm{IC}_{10} \neq 0$, a state transformation should be applied as discussed prior to system (10) before applying this lemma.

Applying control (11), together with learning term defined by (12), to (10) yields

$$h_{1,j}\dot{x}_{1,j} = \delta_1 + g_{1,j} - \eta_1 x_{1,j} - \beta_1 \rho_{g_1} x_{1,j} - \frac{1}{2}\rho_{h_1} x_{1,j} - \Delta_{1,j}$$

which renders

$$-[\delta_1 - \Delta_{1,j}] = -h_{1,j}\dot{x}_{1,j} - \frac{1}{2}\rho_{h_1} x_{1,j} - \eta_1 x_{1,j} - \beta_1 \rho_{g_1} x_{1,j} + g_{1,j}. \quad (16)$$

Then, substituting the previous solution of $-[\delta_1 - \Delta_{1,j}]$ into the second integral term in (15) yields the first equation shown at the bottom of the next page. It should be noted that, under either a fixed IC ($x_{1,j}(0) = 0$ is used here) or IC resetting (i.e., $x_{1,j}(0) = x_{1,j-1}(T)$), $nx_{1,k}^T(0)h_{1,k}(x_{1,k}(0), 0)x_{1,k}(0) - x_{1,k-1}^T(T)h_{1,k-1}(x_{1,k-1}(T), T)$ $x_{1,k-1}(T) \leq 0$. Hence

$$\begin{aligned} &-\sum_{k=1}^{j} \frac{1}{2} x_{1,k}^T h_{1,k} x_{1,k} \Big|_0^T \\ &\leq -\frac{1}{2} x_{1,j}^T(T) h_{1,j}(x_{i,j}(T), T) x_{1,j}(T) \\ &\quad + \frac{1}{2} x_{1,1}^T(0) h_{1,1}(x_{1,1}(0), 0) x_{1,1}(0) \\ &\leq -\frac{1}{2} \underline{h}_1 \|x_{1,j}(T)\|^2 + \frac{1}{2} \overline{h}_1 \|x_{1,1}(0)\| \end{aligned} \quad (17)$$

in which the positive term is independent of j.

On the other hand, it follows from (16) that the third integral term in (15) is

$$\begin{aligned} &\int_0^T [\delta_1 - \Delta_{1,j}]^T \left\{ [\dot{\delta}_1 + \delta_1] - \lambda_1 h_{1,j}^{-1} \mathrm{sign}[x_{1,j}] \right\} d\tau \\ &= \int_0^T \left\{ \lambda_1 h_{1,j}^{-1} \mathrm{sign}[x_{1,j}] - [\dot{\delta}_1 + \delta_1] \right\}^T \\ &\quad \times \left[-\eta_1 x_{1,j} - h_{1,j}\dot{x}_{1,j} - \frac{1}{2}\rho_{h_1} x_{1,j} \right] d\tau \\ &\quad + \int_0^T \left\{ \lambda_1 h_{1,j}^{-1} \mathrm{sign}[x_{1,j}] - [\dot{\delta}_1 + \delta_1] \right\}^T \\ &\quad \times [-\beta_1 \rho_{g_1} x_{1,j} + g_{1,j}]\, d\tau. \end{aligned} \quad (18)$$

It follows from Assumptions 2 and 3 that the second integral in (18) is bounded from above as shown in the second equation at the bottom of the next page. Similarly, it follows that the first integral in (18) is bounded from above as the third equation shown at the bottom of the next page, where $x_{1,j,k}$ denotes the kth element of vector $x_{1,j}(t)$. As in (17), one can see that, since $\lambda_1 > \overline{h}_1(\overline{\delta}_{12} + \overline{\delta}_{11})$

$$-\lambda_1 \sum_{k=1}^{n} |x_{1,j,k}| + [\dot{\delta}_1 + \delta_1]^T h_{1,k} x_{1,k} \leq 0$$

and, therefore

$$\begin{aligned} &\sum_{k=1}^{j} \left[-\lambda_1 \sum_{k=1}^{n} |x_{1,j,k}|\big|_0^T + [\dot{\delta}_1 + \delta_1]^T\; h_{1,k} x_{1,k}\big|_0^T \right] \\ &\leq -\lambda_1 \sum_{k=1}^{n} |x_{1,j,k}(T)| + [\dot{\delta}_1(T) + \delta_1(T)]^T \\ &\quad \times h_{1,j}(x_{1,j}(T), T) x_{1,j}(T) + \lambda_1 \sum_{k=1}^{n} |x_{1,j,k}(0)| \\ &\quad - [\dot{\delta}_1(0) + \delta_1(0)]^T h_{1,1}(x_{1,1}(0), 0) x_{1,1}(0) \\ &\leq -\left[\lambda_1 - \overline{h}_1(\overline{\delta}_{12} + \overline{\delta}_{11})\right] \|x_{1,j}(T)\| \\ &\quad + \left[\lambda_1 \sqrt{n} + \overline{h}_1(\overline{\delta}_{12} + \overline{\delta}_{11})\right] \|x_{1,1}(0)\| \\ &\leq \left[\lambda_1 \sqrt{n} + \overline{h}_1(\overline{\delta}_{12} + \overline{\delta}_{11})\right] \|x_{1,1}(0)\| \end{aligned}$$

which is again independent of j.

By noting that, for $j > 0$

$$V_{1,j} - V_{1,0} = \sum_{k=1}^{j} \delta V_{1,k}$$

and by combining the results on the last two integral terms in (15), we can conclude inequality (14) and, therefore, convergence in L_2 norm to zero for both the state and the learning error.

To establish asymptotic stability of the state and asymptotic convergence of the learning error, we note the following facts. First, by applying the argument of induction up to the $(j-1)$th trial, one can assume that ICs $\|x_{1,j}(0)\|$ and $\|\Delta_{1,j}(0) - \delta_1(0)\|$ are uniformly bounded. Second, one can show using (14) that $\|x_{1,j}(T)\|$ and $\|\Delta_{1,j}(T) - \delta_1(T)\|$ are uniformly bounded. Third, to show that

$$\begin{aligned} \delta V_{1,j} =& \frac{1}{2}(1-\gamma_1) \int_0^T \left[\|\delta_1 - \Delta_{1,j}\|^2 - \|\delta_1 - \Delta_{1,j-1}\|^2 \right] d\tau + \int_0^T [\delta_1 - \Delta_{1,j}]^T \\ &\times \Big\{ \gamma_1 [\dot{\delta}_1 + \delta_1] - \alpha_1 x_{1,j} - \gamma_1 \lambda_1 h_{1,j}^{-1} \mathrm{sign}[x_{1,j}] \\ &\quad -[\delta_1 - \Delta_{1,j}] + (1-\gamma_1)[\delta_1 - \Delta_{1,j-1}] \Big\} d\tau \\ =& -\frac{1}{2}(1+\gamma_1) \int_0^T \|\delta_1 - \Delta_{1,j}\|^2 d\tau - \frac{1}{2}(1-\gamma_1) \int_0^T \|\delta_1 - \Delta_{1,j-1}\|^2 d\tau \\ &+ (1-\gamma_1) \int_0^T [\delta_1 - \Delta_{1,j}]^T [\delta_1 - \Delta_{1,j-1}] d\tau - \alpha_1 \int_0^T [\delta_1 - \Delta_{1,j}]^T x_{1,j} d\tau \\ &+ \gamma_1 \int_0^T [\delta_1 - \Delta_{1,j}]^T \left\{ [\dot{\delta}_1 + \delta_1] - \lambda_1 h_{1,j}^{-1} \mathrm{sign}[x_{1,j}] \right\} d\tau \\ \leq& -\gamma_1 \int_0^T \|\delta_1 - \Delta_{1,j}\|^2 d\tau - \alpha_1 \int_0^T [\delta_1 - \Delta_{1,j}]^T x_{1,j} d\tau \\ &+ \gamma_1 \int_0^T [\delta_1 - \Delta_{1,j}]^T \left\{ [\dot{\delta}_1 + \delta_1] - \lambda_1 h_{1,j}^{-1} \mathrm{sign}[x_{1,j}] \right\} d\tau. \end{aligned} \quad (15)$$

$\|x_{1,j}(t)\|$ and $\|\Delta_{1,j}(t) - \delta_1(t)\|$ are bounded during the jth trial, consider the continuous-time Lyapunov candidate

$$L_{1,j}(x_{1,j}(t), \Delta_{1,j}(t)) = \frac{1}{2}x_{1,j}^T h_{1,j} x_{1,j} + \frac{\gamma_1}{2\alpha_1}\|\Delta_{1,j} - \delta_1\|^2, \quad t \in [0, T].$$

It follows (19), as shown at the bottom of the next page, holds. In an argument of induction with respect to trial index j, the term associated with $\Delta_{1,j-1}$ in (19) is considered to be constant. Therefore, $\dot{L}_{1,j}$ is negative once $\|x_{1,j}\|$ or $\|\Delta_{1,j}(t) - \delta_1(t)\|$ is larger than a threshold value. Using the uniform bounded theorem in [20], one can easily establish by induction that $\|x_{1,j}(t)\|$ and $\|\Delta_{1,j}(t) - \delta_1(t)\|$ are uniformly bounded for all $t \in [0, T]$ and for all j. This result in turn implies uniform boundedness of $\dot{x}_{1,j}$ and $\dot{\Delta}_{1,j}$. By Barbalet lemma [20], asymptotic stability of the state and asymptotic convergence of the learning error can be concluded from their L_2 stability and convergence. $\square$

$$\begin{aligned}
&-\int_0^T [\delta_1 - \Delta_{1,j}]^T x_{1,j}\, d\tau \\
&= -\int_0^T x_{1,j}^T \left[h_{1,j}\dot{x}_{1,j} + \frac{1}{2}\rho_{h_1} x_{1,j} + \eta_1 x_{1,j} + \beta_1 \rho_{g_1} x_{1,j} - g_{1,j}\right] d\tau \\
&\le -\frac{1}{2}x_{1,j}^T h_{1,j} x_{1,j}\Big|_0^T - \frac{1}{2}\int_0^T x_{1,j}^T[\rho_{h_1} I - \dot{h}_{1,j}]x_{1,j}\, d\tau \\
&\quad - \eta_1 \int_0^T \|x_{1,j}\|^2 d\tau - \int_0^T \left[\beta_1\rho_{g_1}\|x_{1,j}\|^2 - \|x_{1,j}\|\|g_{1,j}\|\right] d\tau \\
&\le -\frac{1}{2}x_{1,j}^T h_{1,j} x_{1,j}\Big|_0^T - \eta_1 \int_0^T \|x_{1,j}\|^2 d\tau.
\end{aligned}$$

$$\begin{aligned}
&\int_0^T \left\{\lambda_1 h_{1,j}^{-1}\mathrm{sign}[x_{1,j}] - [\dot{\delta}_1 + \delta_1]\right\}^T [-\beta_1\rho_{g_1} x_{1,j} + g_{1,j}]\, d\tau \\
&\le -\int_0^T \Big\{\lambda_1 \left[\beta_1\rho_{g_1}\mathrm{sign}^T[x_{1,j}]w_{1,j}x_{1,j} - n\underline{h}_1^{-1}\|g_{1,j}\|\right] \\
&\qquad -(\|\dot{\delta}_1\| + \|\delta_1\|)\left[\beta_1\rho_{g_1}\|x_{1,j}\| + \|g_{1,j}\|\right]\Big\} d\tau \\
&\le -\int_0^T \Bigg\{\lambda_1\rho_{g_1}\left[\beta_1 \sum_k^n \left(w_{1,j}^{k,k} - \sum_{l=1,\ l\ne k}^n \left|w_{1,j}^{k,l}\right|\right)|x_{1,j}(k)| - n\underline{h}_1^{-1}\|x_{1,j}\|\right] \\
&\qquad -(\|\dot{\delta}_1\| + \|\delta_1\|)(\beta_1 + 1)\rho_{g_1}\|x_{1,j}\|\Bigg\} d\tau \\
&\le -\int_0^T \left\{\lambda_1\rho_{g_1}(\beta_1 c_w - n\underline{h}_1^{-1})\|x_{1,j}\| - (\|\dot{\delta}_1\| + \|\delta_1\|)(\beta_1 + 1)\rho_{g_1}\|x_{1,j}\|\right\} d\tau \\
&\le 0.
\end{aligned}$$

$$\begin{aligned}
&\int_0^T \left\{\lambda_1 h_{1,j}^{-1}\mathrm{sign}[x_{1,j}] - [\dot{\delta}_1 + \delta_1]\right\}^T \left[-\eta_1 x_{1,j} - h_{1,j}\dot{x}_{1,j} - \frac{1}{2}\rho_{h_1} x_{1,j}\right] d\tau \\
&= -\eta_1 \int_0^T \left\{\lambda_1 \mathrm{sign}^T[x_{1,j}]h_{1,j}^{-1}x_{1,j} - [\dot{\delta}_1 + \delta_1]^T x_{1,j}\right\} d\tau - \lambda_1 \sum_{k=1}^n |x_{1,j,k}|\big|_0^T + [\dot{\delta}_1 + \delta_1]^T\ h_{1,j}x_{1,j}\big|_0^T \\
&\quad - \int_0^T [(\dot{\delta}_1 + \delta_1)^T \dot{h}_{1,j}x_{1,j} + (\ddot{\delta}_1 + \dot{\delta}_1)^T h_{i,j}x_{1,j}]d\tau - \int_0^T \frac{1}{2}\rho_{h_1}\left\{\lambda_1 \mathrm{sign}^T[x_{1,j}]h_{1,j}^{-1}x_{1,j} - [\dot{\delta}_1 + \delta_1]^T x_{1,j}\right\} d\tau \\
&\le -\eta_1 \int_0^T \left\{\lambda_1 c_w - (\|\dot{\delta}_1\| + \|\delta_1\|) - \eta_1^{-1}\overline{h}_1(\|\ddot{\delta}_1\| + \|\dot{\delta}_1\|)\right\}\|x_{1,j}\|d\tau - \lambda_1 \sum_{k=1}^n |x_{1,j,k}|\Big|_0^T + [\dot{\delta}_1 + \delta_1]^T h_{1,j}x_{1,j}\Big|_0^T \\
&\quad - \int_0^T \frac{1}{2}\rho_{h_1}\left\{\lambda_1 c_w - 3(\|\dot{\delta}_1\| + \|\delta_1\|)\right\}\|x_{1,j}\|d\tau \\
&\le -\lambda_1 \sum_{k=1}^n |x_{1,j,k}|\big|_0^T + [\dot{\delta}_1 + \delta_1]^T\ h_{1,j}x_{1,j}\big|_0^T
\end{aligned}$$

It should be noted that, for first-order vector systems in the form of (10), constant γ_1 can be set to be zero in which case the learning law (12) reduces to the standard iterative form, i.e., a difference learning law. Such a learning can ensure asymptotic stability of the state. However, for high-order vector systems, a differential-difference learning law (such as the one in (12) with $\gamma_1 > 0$) must be used in order not to require derivative measurement of the state in the implementation of the actual control.

Comparing system (3) (with $i = 1$) with fictitious system (10), we can rewrite dynamics of the first subsystem as

$$h_{1,j}(x_{1,j}, t)\dot{x}_{1,j} = \delta_1(t) + g_{1,j}(x_{1,j}, t) + v_{1,j} + z_{2,j} \tag{20}$$

where $z_{2,j} = x_{2,j} - v_{1,j}$. The following result can be concluded by mimicking the analysis in Lemma 1.

Lemma 2: Consider (20) under learning control (11) and (12). Then, under either a fixed IC ($x_{i,j}(0) = 0$ is used without loss of any generality) or IC resetting $(x_{i,j}(0) = x_{1,j-1}(T))$, the Lyapunov function (13) has the property that, for a constant c_1 (independent of j)

$$\begin{aligned} V_{1,j} \le \sum_{k=1}^{j} \Bigg[&-\gamma_1 \int_0^T \|\delta_1 - \Delta_{1,k}\|^2 d\tau - \eta_1\alpha_1 \int_0^T \|x_{1,k}\|^2 d\tau \\ &+ \alpha_1 \int_0^T x_{1,j}^T z_{2,j} - \gamma_1 \int_0^T \Big([\dot{\delta}_1 + \delta_1] \\ &- \lambda_1 h_{1,j}^{-1} \mathrm{sign}[x_{1,j}] \Big)^T z_{2,j} \Bigg] + c_1. \end{aligned} \tag{21}$$

Fictitious control design of $v_{1,j} = v_{1,j}(x_{1,j}, \Delta_{1,j})$ provides the avenue by which the actual control u_j can be found recursively. This is done by the partial state transformation $z_{2,j} = x_{2,j} - v_{1,j}$. Specifically, we want to find the equation governing dynamics of z_2 based on which u_j can be found in the same way as that for a first-order vector system [as did for fictitious system (10)]. It follows that:

$$\begin{aligned} &h_{2,j}(x_{1,j}, x_{2,j}, t)\dot{z}_{2,j} \\ &\quad = h_{2,j}(x_{1,j}, x_{2,j}, t)\dot{x}_{2,j} - h_{2,j}(x_{1,j}, x_{2,j}, t)\dot{v}_{1,j} \\ &\quad = \delta_2(t) + g_{2,j}(x_{1,j}, x_{2,j}, t) + u_j - h_{2,j}(x_{1,j}, x_{2,j}, t) \\ &\qquad \times \frac{\partial v_{1,j}}{\partial x_{1,j}} h_{1,j}^{-1}(x_{1,j}, t)[\delta_1(t) + g_{1,j}(x_{1,j}, t) + x_{2,j}] \\ &\qquad - h_{2,j}(x_{1,j}, x_{2,j}, t) \frac{\partial v_{1,j}}{\partial \Delta_{1,j}} \dot{\Delta}_{1,j} \\ &\quad = -\alpha_1 x_{1,j} + \gamma_1 \left\{ [\dot{\delta}_1 + \delta_1] - \lambda_1 h_{1,j}^{-1} \mathrm{sign}[x_{1,j}] \right\} + \delta_2'(t) \\ &\qquad + G_{2,j}(x_{1,j}, x_{2,j}, t) + G_{2,j}'(x_{1,j}, x_{2,j}, t) + u_j \end{aligned} \tag{22}$$

where $\dot{\Delta}_{1,j}$ is defined by (12), $\delta_2'(t)$ and $G_{2,j}(x_{1,j}, x_{2,j}, t)$ are those shown in the first equation at the bottom of the next page, and

$$\begin{aligned} G_{2,j}'(x_{1,j}, x_{2,j}, t) \triangleq\, &\alpha_1 x_{1,j} + \gamma_1\lambda_1 h_{1,j}^{-1}(x_{1,j}, t)\mathrm{sign}[x_{1,j}] \\ &- h_{2,j}(x_{1,j}, x_{2,j}, t) \frac{\partial v_{1,j}}{\partial x_{1,j}} h_{1,j}^{-1}(x_{1,j}, t) x_{2,j} \\ &- h_{2,j}(x_{1,j}, x_{2,j}, t) \frac{\partial v_{1,j}}{\partial \Delta_{1,j}} \dot{\Delta}_{1,j}. \end{aligned}$$

The first two terms in the right-hand side of (22) are to cancel the corresponding terms in (21) in a Lyapunov argument. The time function $\delta_2'(t)$ (to be learned) is chosen such that, if system dynamics are linear, $v_{1,j}$ is linear and $G_{2,j}(\cdot)$ contains only uncertainties that vanish at the origin. In other words, $\delta_2'(t)$ is the periodic time function that needs to be learned. The lumped uncertainty in system (22) can be bounded as shown in the second equation at the bottom of the next page. It is easy to verify that

$$\left\| \frac{d^{k-1}\delta_2'(t)}{dt^{k-1}} \right\| \le \overline{\delta}_{2k}', \qquad k = 1, 2, 3$$

where (23)–(24), as shown at the bottom of the next page, hold true.

Applying the same argument in Lemma 1 to system (22), and then combining the result with Lemma 2, we can reach the following conclusion.

Theorem: Consider (3) and (4) with $m = 2$ under learning control

$$\begin{aligned} u_j = &- \Big[\eta_2 z_{2,j} + G_{2,j}' + \beta_2 \rho_{G_2} \mathrm{sign}[z_{2,j}] \\ &+ \frac{1}{2} \rho_{h_2} z_{2,j} + \Delta_{2,j} \Big] \end{aligned} \tag{25}$$

$$\begin{aligned} \gamma_2 \dot{\Delta}_{2,j} = &- \Delta_{2,j} + (1 - \gamma_2)\Delta_{2,j-1} + \alpha_2 z_{2,j} \\ &+ \gamma_2 \lambda_2 h_{2,j}^{-1} \mathrm{sign}[z_{2,j}] \end{aligned} \tag{26}$$

where $z_{2,j} = x_{2,j} - v_{1,j}$, $v_{1,j}$ is defined by (11) and (12), and $\Delta_{2,j}$ defined by (26) should be solved under IC $\Delta_{2,j}(0) = \Delta_{2,j-1}(T)$ with $\Delta_{2,-1}$ arbitrarily chosen, $\eta_2 > 0$ is a constant control gain, β_2 is another control gain satisfying inequality

$$\beta_2 > \max\left\{ 1, \frac{\overline{h}_2}{\underline{h}_2} \right]$$

$$\begin{aligned} \dot{L}_{1,j} = &x_{1,j}^T h_{1,j} \dot{x}_{1,j} + \frac{1}{2} x_{1,j}^T \dot{h}_{1,j} x_{1,j} + \frac{\gamma_1}{\alpha_1} [\Delta_{1,j} - \delta_1]^T [\dot{\Delta}_{1,j} - \dot{\delta}_1] \\ \le &- \eta_1 \|x_{1,j}\|^2 - (\beta_1 \rho_{g_1} \|x_{1,j}\| - \|g_{1,j}\|)\|x_{1,j}\| - \frac{1}{2}(\rho_{h_1} - \|\dot{h}_{1,j}\|)\|x_{1,j}\|^2 - x_{1,j}^T [\Delta_{1,j} - \delta_1] \\ &+ \frac{1}{\alpha_1} [\Delta_{1,j} - \delta_1]^T \left\{ -\gamma_1 [\dot{\delta}_1 + \delta_1] + \alpha_1 x_{1,j} + \gamma_1 \lambda_1 h_{1,j}^{-1} \mathrm{sign}[x_{1,j}] + [\delta_1 - \Delta_{1,j}] - (1 - \gamma_1)[\Delta_{1,j-1} - \delta_1] \right\} \\ \le &- \eta_1 \|x_{1,j}\|^2 - \frac{1}{4\alpha_1} \|\Delta_{1,j} - \delta_1\|^2 + \frac{1}{\alpha_1} \gamma_1^2 (\dot{\delta}_1 + \delta_1)^2 + \frac{\gamma_1^2 \lambda_1^2 n^2}{\alpha_1 \underline{h}_1^2} + \frac{(1 - \gamma_1)^2}{\alpha_1} \|\delta_1 - \Delta_{1,j-1}\|^2 \\ \le &- \eta_1 \|x_{1,j}\|^2 - \frac{1}{4\alpha_1} \|\Delta_{1,j} - \delta_1\|^2 + \frac{1}{\alpha_1} \gamma_1^2 (\overline{\delta}_1 + \overline{\delta}_1')^2 + \frac{\gamma_1^2 \lambda_1^2 n^2}{\alpha_1 \underline{h}_1^2} + \frac{(1 - \gamma_1)^2}{\alpha_1} \max_{t \in [0,T]} \|\delta_1 - \Delta_{1,j-1}\|^2. \end{aligned} \tag{19}$$

$0 < \gamma_2 < 1$ is the time constant of the differential-difference learning law, $\alpha_2 > 0$ is a learning gain, λ_2 is another learning gain satisfying inequality

$$\lambda_2 \geq \max\left\{ \frac{(\beta_2+1)\left(\overline{\delta}'_{21}+\overline{\delta}'_{22}\right)}{n\beta_2\overline{h}_2^{-1}-n\underline{h}_2^{-1}},\ \frac{3\left(\overline{\delta}'_{21}+\overline{\delta}'_{22}\right)}{c_w},\ \frac{\overline{\delta}'_{21}+\overline{\delta}'_{22}+\eta_2^{-1}\overline{h}_2\left(\overline{\delta}'_{22}+\overline{\delta}'_{23}\right)}{c_w},\ \left(\overline{\delta}'_{21}+\overline{\delta}'_{22}\right)\overline{h}_2 \right\}$$

and $\overline{\delta}'_{2k}$ (with $k = 1, 2, 3$) are defined by (23) up to (24). Then, under either a fixed IC ($x_{i,j}(0) = 0$ is used without loss of any generality) or IC resetting $(x_{i,j}(0) = x_{1,j-1}(T))$, the Lyapunov function

$$V_j = \sum_{i=1}^{2}\left[\frac{1}{2}(1-\gamma_i)\int_0^T \|\delta'_i(\tau)-\Delta_{i,j}(\tau)\|^2 d\tau + \frac{1}{2}\gamma_i\|\delta'_i(T)-\Delta_{i,j}(T)\|^2\right] \quad (27)$$

with $\delta'_1(t) = \delta_1(t)$ has the property that, for a constant c (independent of j)

$$V_j \leq \sum_{i=1}^{2}\sum_{k=1}^{j}\left[-\gamma_i\int_0^T \|\delta'_i-\Delta_{i,k}\|^2 d\tau - \eta_i\alpha_i\int_0^T \|z_{i,k}\|^2 d\tau\right] + c \quad (28)$$

where $z_{1,k} = x_{1,k}$. Furthermore, the transformed system (of state variables $x_{1,j}$ and $z_{2,j}$) is asymptotically stable and its learning error (vector of elements in $[\Delta_{i,j} - \delta'_i]$, $i = 1, 2$) converges asymptotically to zero.

It is worth noting again that, if there are only two subsystems, constant γ_2 can be chosen to be zero. In general, as in the case of recursively designing a linear control for linear time-invariant systems, design constants γ_i should be chosen such that $0 < \gamma_i \leq \gamma_l$ for $i > l$. Similarly, gains η_i, β_i, α_i, and λ_i should be chosen to be positive and nondecreasing as the index of the subsystems increases. This ensures that the subsystems from the mth to the first converge sequentially

$$\begin{aligned}
\delta'_2(t) &\triangleq \delta_2(t) - \gamma_1[\dot{\delta}_1(t)+\delta_1(t)] - h_{2,j}(0,0,t)\frac{\partial v_{1,j}(0,0)}{\partial x_1}h_{1,j}^{-1}(0,t)\delta_1(t),\\
G_{2,j}(x_{1,j},x_{2,j},t) &\triangleq g_{2,j}(x_{1,j},x_{2,j},t)\\
&\quad - \left[h_{2,j}(x_{1,j},x_{2,j},t)\frac{\partial v_{1,j}(x_{1,j},\Delta_{1,j})}{\partial x_1}h_{1,j}^{-1}(x_{1,j},t) - h_{2,j}(0,0,t)\frac{\partial v_{1,j}(0,0)}{\partial x_1}h_{1,j}^{-1}(0,t)\right]\delta_1(t)\\
&\quad - h_{2,j}(x_{1,j},x_{2,j},t)\frac{\partial v_{1,j}(x_{1,j},\Delta_{1,j})}{\partial x_1}h_{1,j}^{-1}(x_{1,j},t)g_{1,j}(x_{1,j},t)
\end{aligned}$$

$$\begin{aligned}
\|G_{2,j}(x_{1,j},x_{2,j},t)\| &\leq \rho_{g_2}(x_{1,j},x_{2,j})\\
&\quad + \left\|h_{2,j}(x_{1,j},x_{2,j},t)\frac{\partial v_{1,j}(x_{1,j},\Delta_{1,j})}{\partial x_1}h_{1,j}^{-1}(x_{1,j},t) - h_{2,j}(0,0,t)\frac{\partial v_{1,j}(0,0)}{\partial x_1}h_{1,j}^{-1}(0,t)\right\|\overline{\delta}_{11}\\
&\quad + \left\|h_{2,j}(x_{1,j},x_{2,j},t)\frac{\partial v_{1,j}(x_{1,j},\Delta_{1,j})}{\partial x_1}h_{1,j}^{-1}(x_{1,j},t)\right\|\rho_{g_1}\|x_{1,j}\|\\
&\triangleq \rho_{G_2}(x_{1,j},x_{2,j}).
\end{aligned}$$

$$\begin{aligned}
\overline{\delta}'_{21} &\triangleq \overline{\delta}_{21} + \gamma_1(\overline{\delta}_{12}+\overline{\delta}_{11}) + \left\|h_{2,j}(0,0,t)\frac{\partial v_{1,j}(0,0)}{\partial x_1}h_{1,j}^{-1}(0,t)\right\|\overline{\delta}_{11},\\
\overline{\delta}'_{22} &\triangleq \overline{\delta}_{22} + \gamma_1(\overline{\delta}_{13}+\overline{\delta}_{12}) + \left\|h_{2,j}(0,0,t)\frac{\partial v_{1,j}(0,0)}{\partial x_1}h_{1,j}^{-1}(0,t)\right\|\overline{\delta}_{12} + \left\|\frac{\partial h_{2,j}(0,0,t)}{\partial t}\frac{\partial v_{1,j}(0,0)}{\partial x_1}h_{1,j}^{-1}(0,t)\right\|\overline{\delta}_{11}\\
&\quad + \left\|h_{2,j}(0,0,t)\frac{\partial v_{1,j}(0,0)}{\partial x_1}\frac{\partial h_{1,j}^{-1}(0,t)}{\partial t}\right\|\overline{\delta}_{11}
\end{aligned} \quad (23)$$

$$\begin{aligned}
\overline{\delta}'_{23} &\triangleq \overline{\delta}_{23} + \gamma_1(\overline{\delta}_{14}+\overline{\delta}_{13}) + \left\|h_{2,j}(0,0,t)\frac{\partial v_{1,j}(0,0)}{\partial x_1}h_{1,j}^{-1}(0,t)\right\|\overline{\delta}_{13} + 2\left\|\frac{\partial h_{2,j}(0,0,t)}{\partial t}\frac{\partial v_{1,j}(0,0)}{\partial x_1}h_{1,j}^{-1}(0,t)\right\|\overline{\delta}_{12}\\
&\quad + 2\left\|h_{2,j}(0,0,t)\frac{\partial v_{1,j}(0,0)}{\partial x_1}\frac{\partial h_{1,j}^{-1}(0,t)}{\partial t}\right\|\overline{\delta}_{12} + 2\left\|\frac{\partial h_{2,j}(0,0,t)}{\partial t}\frac{\partial v_{1,j}(0,0)}{\partial x_1}\frac{\partial h_{1,j}^{-1}(0,t)}{\partial t}\right\|\overline{\delta}_{11}\\
&\quad + \left\|\frac{\partial^2 h_{2,j}(0,0,t)}{\partial t^2}\frac{\partial v_{1,j}(0,0)}{\partial x_1}h_{1,j}^{-1}(0,t)\right\|\overline{\delta}_{11} + \left\|h_{2,j}(0,0,t)\frac{\partial v_{1,j}(0,0)}{\partial x_1}\frac{\partial^2 h_{1,j}^{-1}(0,t)}{\partial t^2}\right\|\overline{\delta}_{11}.
\end{aligned} \quad (24)$$

according to the physical interaction of cascaded systems so that the system output will converge smoothly.

It is obvious that the proposed design readily applies to system (9), as well as many other electrical–mechanical systems. The structure of cascaded subsystems ensures that unknown but periodic time functions can be compensated for by an iterative control law.

IV. Conclusion

In this note, a Lyapunov-based learning control design is presented for cascaded nonlinear systems. Compared with the result of uniform bounded stability in [6], the newly designed learning control is capable of achieving both asymptotic stability of the system output and asymptotic convergence of a composite vector of learning errors. The result is the first to show that periodic functions in a cascaded system can be learned using an iterative learning law while ensuring output asymptotic convergence. Because of the use of a robust control part, nonperiodic uncertainties are also admissible in the system dynamics.

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Repeatability of Inverse Kinematics Algorithms for Mobile Manipulators

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Abstract—We introduce and examine the property of repeatability of inverse kinematics algorithms for mobile manipulators. Similarly to stationary manipulators, repeatability of mobile manipulators is defined by requiring that a closed path in the taskspace should be transformed by the inverse kinematics algorithm into a closed path in the configuration space. In a simply connected, singularity-free region of the taskspace a necessary and sufficient condition for repeatability is derived as the integrability condition of a distribution associated with the inverse kinematics algorithm.

Index Terms—Distribution, integrability, inverse kinematics, mobile manipulator, repeatability.

I. Introduction

We shall be concerned with mobile manipulators composed of a nonholonomic mobile platform and a holonomic stationary manipulator mounted atop of the platform. An increasing interest in mobile manipulators observed recently in the literature has two sources: first, excellent performance characteristics of mobile manipulators, second, challenging motion planning and control problems [1]–[3]. For a comprehensive review of literature, the reader is directed to [4] and [5].

The kinematics of a stationary or mobile manipulator can be regarded as a map from a configuration space into a taskspace. The inverse kinematics, assigning a configuration to a prescribed point in the taskspace, are computed by inverse kinematics algorithms. An inverse kinematics algorithm is called repeatable, if it maps closed paths in the taskspace (cyclic sequences of tasks) to closed paths in the configuration space (cyclic sequences of configurations). It is well known that inverse kinematics algorithms based on optimization theory are repeatable by design. On the other hand, within the important class of Jacobian algorithms relying on a right pseudoinverse of analytic Jacobian of the kinematics, the repeatability issue becomes crucial. Repeatability of Jacobian algorithms has an appealing geometric interpretation. If the (regular) kinematics of a stationary or a mobile manipulator are treated

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