OPTIMAL CONTROL APPLICATIONS AND METHODS
Optim. Control Appl. Meth., 2002; **23**:303–328 (DOI: 10.1002/oca.715)

A new suboptimal control design for cascaded non-linear systems

Zhihua Qu[1,*,†,‡] and James R. Cloutier[2,§]

[1] *School of Electrical Engineering and Computer Science, University of Central Florida, Orlando, FL 32816, U.S.A.*
[2] *Navigation and Control Branch at the Air Force Research Laboratory, Eglin AFB, FL 32542-6810, U.S.A.*

SUMMARY

A new suboptimal control design technique is proposed for a class of cascaded non-linear systems. The design is based on a forward recursive design rather than a backstepping design, and it utilizes a non-linear tracker derived using the state-dependent algebraic Riccati equation approach. The proposed design has two distinct features. First, it provides suboptimal performance with respect to a performance index that is defined in terms of the original state and control variables and thus can be prescribed. Second, the forward recursive procedure eliminates differentiation of fictitious controls (or their functions), which makes the design much simpler in applications. Due to the use of the non-linear tracker, the proposed design has the potential of producing less conservative results than non-linear servo results. Copyright © 2002 John Wiley & Sons, Ltd.

KEY WORDS: cascaded system; optimality, recursive design; Riccati equation; suboptimal control; tracking control

1. INTRODUCTION

For non-linear systems, there are several popular and successfully tested control laws such as adaptive control, robust control and L_2-gain optimal control [1–4]. Lyapunov's direct method is a method commonly used to design these controls. Recently, several recursive design procedures have been proposed to facilitate Lyapunov-based control design and stability analysis. Among them, the most notable is the backstepping design [5–8]; others include forward recursive design and recursive interlacing design [9]. On the other hand, optimal control is desired due to its performance guarantee [10, 11]. Since optimal controls have to be found by solving a vector partial differential equation, closed-form suboptimal controls are sought for the purpose of on-line implementation [12]. One promising technique to design suboptimal control is the

*Correspondence to: Zhihua Qu, School of Electrical Engineering and Computer Science, University of Central Florida, Orlando, FL 32816, U.S.A.
† E-mail: qu@pegasus.cc.ucf.edu
‡ Professor.
§ Principal Research Scientist.

Received 9 July 2001
Revised 25 July 2002
Accepted 8 August 2002

state-dependent algebraic Riccati equation (SDARE) method [13–15]. It has been shown therein that SDARE controls have performance very close to (and, in several cases, identical to) that of the optimal ones. The advantage of the SDARE method is that, if used appropriately, it can expand the normal LQ problem beyond the scope of the normal Hardy space (stable A matrices) and frozen-time controllable (and observable) systems. At the same time, the well-posed LQ problem can be shown to be a subset of the SDARE method as described in this paper.

Recursive designs have been shown to be effective particularly in (but not limited to) handling cascaded non-linear systems as the cascaded structure provides a unique avenue for developing a recursion. Many physical systems, especially such electrical–mechanical systems as robotic manipulators, satisfy the cascaded structure. And, the cascaded structure also ensures controllability of these systems. In a typical backstepping design, a sequence of state transformations involving fictitious controls are formed, their dynamics (or the rates of change of their corresponding sub-Lyapunov functions) are found by differentiation, and the differentiation operations generate numerous terms that must be compensated for by the actual control. This differentiation process makes the control derivation mathematically tedious and often leads to an overly compensating control as the designer tries to cancel a majority of the transformed dynamics. In the case that an optimal control is designed by backstepping, the performance index is inversely found in terms of transformed state, and its physical meaning is often unclear. To overcome these two shortcomings, a new suboptimal control design is proposed in this paper for a class of cascaded systems. The new method is based on a forward recursive design in which the SDARE technique is applied to generate fictitious control for each subsystem. Instead of using the SDARE regulators reported in Reference [15], an SDARE tracker is developed. By doing so, optimality is achieved for the individual subsystems, sub-optimality is achieved for the overall system, and recursive mapping of the fictitious controls into the actual control is accomplished in terms of algebraic equations rather than state transformation and differentiation.

The paper is organized as follows. In Section 2, optimality conditions and SDARE method are briefly reviewed. In Section 3, the new design methodology is proposed and compared to the existing methods. Second order systems are used to illustrate the new design procedure, and extension to high-order cascaded systems is guaranteed by its recursive nature. An illustrative example is presented in Section 4. Conclusions are given in Section 5.

2. NON-LINEAR OPTIMAL AND SUBOPTIMAL CONTROLS

Consider the following non-linear, affine system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{G}(\mathbf{x})\mathbf{u} \tag{1}$$

where $\mathbf{x} \in \Re^n$, $\mathbf{u} \in \Re^m$, and functions $\mathbf{f}(\cdot)$ and $\mathbf{G}(\cdot)$ are continuous. To study a more general class of LQ problems, one can rewrite system (1) as

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x})\mathbf{x} + \mathbf{B}(\mathbf{x})\mathbf{u} \tag{2}$$

where $\mathbf{B}(\mathbf{x}) = \mathbf{G}(\mathbf{x})$, and $\mathbf{A}(\mathbf{x})$ is a state-dependent parameterization of $\mathbf{f}(\mathbf{x})$ (namely, $\mathbf{f}(\mathbf{x}) = \mathbf{A}(\mathbf{x})\mathbf{x}$). The matrix $\mathbf{A}(\mathbf{x})$ is assumed to be well defined for all $\mathbf{x} \in \Re^n$.

The control objective studied in this paper is to devise a non-linear and continuous control

$$\mathbf{u} = \phi(\mathbf{x}) \tag{3}$$

such that the closed-loop, autonomous system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{G}(\mathbf{x})\phi(\mathbf{x}) \tag{4}$$

is asymptotically stable. This stabilization problem can be formulated into an optimal control problem as follows (or into a sub-optimal control problem to be stated later). Let the performance index be

$$J(\mathbf{x}(t_0), \mathbf{u}, t_0, t_f) = \frac{1}{2}\mathbf{x}^{\mathrm{T}}(t_f)\mathbf{S}\mathbf{x}(t_f) + \frac{1}{2}\int_{t_0}^{t_f} [\mathbf{x}^{\mathrm{T}}\mathbf{Q}(\mathbf{x})\mathbf{x} + \mathbf{u}^{\mathrm{T}}\mathbf{R}(\mathbf{x})\mathbf{u}]\,\mathrm{d}t \tag{5}$$

where $t_f \in [t_0, \infty]$ is the time interval of optimization, and $\mathbf{S}$ is a given constant positive definite matrix. Matrices $\mathbf{Q}$ and $\mathbf{R}$ are positive definite matrix functions of $\mathbf{x}$. The optimal control problem is to find the optimal control $\mathbf{u}^*$ that minimizes the performance index, that is, for all $\mathbf{u} \in \Re^m$

$$J^* \triangleq J(\mathbf{x}(t_0), \mathbf{u}^*, t_0, t_f) \leqslant J(\mathbf{x}(t_0), \mathbf{u}, t_0, t_f) \quad \text{and} \quad J(\mathbf{x}(t_0), \mathbf{u}^*, t_0, t_f) < \infty$$

2.1. Lagrangian method

The necessary conditions for optimality can be found using the calculus of variations. To this end, we form the Hamiltonian H as

$$H = \tfrac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{Q}(\mathbf{x})\mathbf{x} + \tfrac{1}{2}\mathbf{u}^{\mathrm{T}}\mathbf{R}(\mathbf{x})\mathbf{u} + \lambda^{\mathrm{T}}[\mathbf{f}(\mathbf{x}) + \mathbf{B}(\mathbf{x})\mathbf{u}] \tag{6}$$

where $\lambda \in \Re^n$ is the Lagrangian multiplier. Then, the necessary conditions for optimality are [11]:

$$\dot{\mathbf{x}} = \frac{\partial H}{\partial \lambda}, \quad \frac{\partial H}{\partial \mathbf{u}} = \mathbf{0} \quad \text{and} \quad \dot{\lambda} = -\frac{\partial H}{\partial \mathbf{x}} \tag{7}$$

Condition $\dot{\mathbf{x}} = \partial H/\partial\lambda$ is always satisfied. It follows from condition $\partial H/\partial \mathbf{u} = \mathbf{0}$ that a optimal control candidate in (3) should be of the form

$$\mathbf{u} = -\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}\mathbf{x} \tag{8}$$

provided that, for some matrix function $\mathbf{P}(\mathbf{x})$, the Lagrangian multiplier is chosen to be

$$\lambda = \mathbf{P}\mathbf{x} \tag{9}$$

Control (8) is optimal if matrix $\mathbf{P}(\mathbf{x})$ can be selected to satisfy the third and the last necessary condition $\dot{\lambda} = -\partial H/\partial \mathbf{x}$. By direct differentiation using parameterization (9), the third necessary condition of optimality can be rewritten (as did in Reference [14]) to be

$$\mathbf{0} = \dot{\mathbf{P}}\mathbf{x} + (\mathbf{PA} + \mathbf{A}^{\mathrm{T}}\mathbf{P} - \mathbf{PBR}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P} + \mathbf{Q})\mathbf{x} + \frac{1}{2}\mathrm{vec}\left\{\mathbf{x}^{\mathrm{T}}\frac{\partial \mathbf{Q}}{\partial x_i}\mathbf{x}\right\} + \frac{1}{2}\mathrm{vec}\left\{\mathbf{u}^{\mathrm{T}}\frac{\partial \mathbf{R}}{\partial x_i}\mathbf{u}\right\}$$
$$+ \ \mathrm{vec}\left\{\mathbf{x}^{\mathrm{T}}\frac{\partial \mathbf{A}^{\mathrm{T}}}{\partial x_i}\mathbf{Px} - \mathbf{x}^{\mathrm{T}}\mathbf{PBR}^{-1}\frac{\partial \mathbf{B}^{\mathrm{T}}}{\partial x_i}\mathbf{Px}\right\} \tag{10}$$

Since the first two necessary conditions in (7) have been satisfied, Equation (10) is the optimality condition. Substituting control (8) into (2) yields the optimal closed-loop system

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{BR}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P})\mathbf{x} \tag{11}$$

2.2. Hamilton–Jacobi theory

In the case that $t_f = \infty$, an optimal control can be derived by imbedding (5) into the performance index

$$V(t, \mathbf{x}) = \frac{1}{2}\int_t^{\infty} [\mathbf{x}^{\mathrm{T}}\mathbf{Q}(\mathbf{x})\mathbf{x} + \mathbf{u}^{\mathrm{T}}\mathbf{R}(\mathbf{x})\mathbf{u}]\,\mathrm{d}\tau$$

which can be optimized using dynamic programming. It can be shown using the principle of optimality that the necessary condition for optimality is given by the so-called Hamilton–Jacobi–Bellman equation. That is, if $V^*(t, \mathbf{x})$ is the optimal solution, it must be a solution to the partial differential equation

$$\frac{\partial V^*(t, \mathbf{x})}{\partial t} = -\min_{\mathbf{u}} H(\mathbf{x}, \mathbf{u}, \lambda)\bigg|_{\lambda = \frac{\partial V^*(t,\mathbf{x})}{\partial \mathbf{x}}} \tag{12}$$

where $H(\mathbf{x}, \mathbf{u}, \lambda)$ is the Hamiltonian in (6). Since system dynamics (1) and integrant $L(\mathbf{x}, \mathbf{u}) = 0.5\mathbf{x}^{\mathrm{T}}\mathbf{Q}(\mathbf{x})\mathbf{x} + 0.5\mathbf{u}^{\mathrm{T}}\mathbf{R}(\mathbf{x})\mathbf{u}$ do not explicitly depend on t and since the optimal control problem over the infinite horizon is being studied, $V(t, \mathbf{x}(t)) = V(\mathbf{x}(t))$ and consequently the left-hand side of the Hamilton–Jacobi equation (12) is zero. That is, if the closed-loop system is stable, the necessary and sufficient condition for optimality is

$$\min_{\mathbf{u}} H(\mathbf{x}, \mathbf{u}, \lambda)\bigg|_{\lambda = \frac{\partial V^*(\mathbf{x})}{\partial \mathbf{x}}} = \mathbf{0} \tag{13}$$

which remains a partial differential equation. It is obvious that the minimum of $H(\mathbf{x}, \mathbf{u}, \lambda)$ with respect to $\mathbf{u}$ is reached at

$$\mathbf{u}^* = -\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}^{\mathrm{T}}(\mathbf{x})\frac{\partial V^*(\mathbf{x})}{\partial \mathbf{x}}$$

which, identical to (8), is the optimal control law provided that the following non-linear parameterization (also equivalent to (9)) is employed: for a matrix function $\mathbf{P}(\mathbf{x})$,

$$\frac{\partial V^*(\mathbf{x})}{\partial \mathbf{x}} = \mathbf{P}(\mathbf{x})\mathbf{x} \tag{14}$$

Therefore, we can rewrite Hamilton–Jacobi–Bellman equation (13) as the so-called state-dependent algebraic Riccati equation for unsymmetrical solution (SDARE-US) $\mathbf{P(x)}$:

$$\mathbf{P}^{\mathrm{T}}(\mathbf{x})\mathbf{A}(\mathbf{x}) + \mathbf{A}^{\mathrm{T}}(\mathbf{x})\mathbf{P}(\mathbf{x}) + \mathbf{Q}(\mathbf{x}) - \mathbf{P}^{\mathrm{T}}(\mathbf{x})\mathbf{B}(\mathbf{x})\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}^{\mathrm{T}}(\mathbf{x})\mathbf{P}(\mathbf{x}) = \mathbf{0}, \quad \frac{n(n+1)}{2} \tag{15}$$

Since $V^*(\mathbf{x})$ is a scalar function, its Hessian matrix (second-order partial derivatives) must be symmetrical. In terms of non-linear parameterization (14), this symmetry condition becomes

$$P_{ij}(\mathbf{x}) + \sum_{k=1}^{n} \frac{\partial P_{ik}(\mathbf{x})}{\partial x_j} x_k = P_{ji}(\mathbf{x}) + \sum_{k=1}^{n} \frac{\partial P_{jk}(\mathbf{x})}{\partial x_i} x_k, \quad \frac{n(n-1)}{2} \tag{16}$$

The combination of Equations (15) and (16) is equivalent to the original Hamilton–Jacobi–Bellman equation (13).

The boundary condition for the Hamilton–Jacobi–Bellman equation is

$$V^*(\infty, \mathbf{x}(\infty)) = 0$$

which calls for stability of closed-loop system (11). It can be shown further that, if the closed-loop system is stable, then the Hamilton–Jacobi–Bellman equation is also a sufficient condition for optimality [16].

2.3. *Sufficient conditions for optimality*

Two sets of necessary conditions have been derived: optimality condition (10) from the minimum principle, and Hamilton–Jacobi–Bellman equation (15) in its matrix form and the corresponding symmetry condition (16). However, satisfying the necessary conditions do not necessarily ensure optimality and, even in certain cases, stability. Without imposing controllability, closed-loop stability and optimality can be obtained by requiring sufficient conditions.

Since $\partial^2 H(\mathbf{x}, \mathbf{u}, \lambda)/\partial \mathbf{u}^2 > 0$, control (8) is the so-called H-minimal control, and hence any bounded solution to (15) and partial differential equation (16) is optimal.

On the other hand, whether a solution to optimality condition (10) is optimal depends upon convexity of the performance index. In the general non-linear case, $\partial^2 H(\mathbf{x}, \mathbf{u}, \lambda)/\partial \mathbf{x}^2$ is too complicated to make general conclusions on convexity. Nonetheless, performance index (5) is locally convex around the origin, and the stationary point of $\mathbf{x} = \mathbf{0}$ is at least a local optimum. Thus, the positive definiteness of matrices $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{S}$ locally ensure the second-order conditions associated with the Hamiltonian.

A practical question is whether the optimal control, if exists, can be found and implemented on-line. If the answer is not (as will be shown in the subsequent section), we need then to investigate the options of devising suboptimal controls and how to make an appropriate choice among them. For the suboptimal controls to be introduced, stability is achieved by studying cascaded systems whose controllability is structurally guaranteed.

2.4. *Optimal control versus suboptimal control*

The optimal control problem is to find matrix $\mathbf{P}(\mathbf{x}(t_0), t_0, t_f)$, the solution to non-linear partial differential equation (10) or, if $t_p = \infty$, to algebraic and partial differential equations (15) and

(16). The solution to (10) is typically found numerically by backward and forward sweeps as it is a two-point boundary-value problem satisfying

$$\mathbf{x}(t_0) \text{ given}, \ \mathbf{P}(\mathbf{x}(t_0), t_0, t_f), = \mathbf{S} \quad \text{and} \quad 0 < \lim_{t_f \to \infty} \mathbf{P}(\mathbf{x}(t_0), t_0, t_f) < \infty$$

Thus, the optimal solution can only be found off-line.

To make real-time implementation possible, one has to avoid solving any two-point boundary-value problem (or partial differential equation) and hence resorts to sub-optimal control strategies. A promising method to achieve this goal is the sub-optimal design technique called SDARE method [13, 14]. The essential idea of the technique is to design a suboptimal control of form (8) by finding a symmetrical solution to the following SDARE:

$$\mathbf{PA} + \mathbf{A}^{\mathrm{T}}\mathbf{P} - \mathbf{PBR}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P} + \mathbf{Q} = \mathbf{0} \tag{17}$$

Such a solution avoids solving optimality condition (10) or partial differential equation (16) needed for unsymmetrical solution in SDARE-US (15). The resulting control can be implemented very efficiently through on-line numerical computation. Under additional conditions, the control (SDARE regulator) has been shown in Reference [17] to be globally asymptotically stable. Furthermore, it will be shown in this paper that SDARE control has many characteristics of the optimal control. In what follows, we shall study how to develop a recursive, SDARE-based design for a class of cascaded non-linear systems by first investigating SDARE control design for first-order systems.

2.5. SDARE control of scalar systems

Consider the scalar system:

$$\dot{x} = a(x)x + b(x)u \tag{18}$$

where $b(x) \neq 0$. Its associated SDARE is, for any $q(x) \geqslant \underline{q} > 0$ and $r(x) \geqslant \underline{r} > 0$,

$$2a(x)p(x) - b^2(x)r^{-1}(x)p^2(x) + q(x) = 0 \tag{19}$$

and the SDARE control is

$$u = -b(x)r^{-1}(x)p(x)x$$

It follows that positive solution to (19) is

$$p(x) = r(x)b^{-2}(x)[a(x) + \sqrt{a^2(x) + q(x)b^2(x)r^{-1}(x)}]$$

and that, by direct computation,

$$\begin{aligned}
\dot{p} &= \frac{r}{b^2}\left[1+\frac{a}{\sqrt{a^2+qb^2r^{-1}}}\right]\dot{a}+\frac{1}{2\sqrt{a^2+qb^2r^{-1}}}\dot{q}+\frac{1}{b^2}\left(a+\frac{2a^2r+qb^2}{2\sqrt{a^2r^2+qb^2r}}\right)\dot{r}\\
&\quad+\frac{r}{b^3}\left(-2a-2\sqrt{a^2+qb^2r^{-1}}+\frac{qb^2r^{-1}}{\sqrt{a^2+qb^2r^{-1}}}\right)\dot{b}\\
&= -\frac{r}{b^2}\left[\sqrt{a^2+qb^2r^{-1}}+a\right]x\frac{\partial a}{\partial x}+\frac{r}{b^3}\left[a+\sqrt{a^2+qb^2r^{-1}}\right]^2 x\frac{\partial b}{\partial x}-\frac{1}{2}x\frac{\partial q}{\partial x}\\
&\quad-\frac{1}{2b^2}\left[a+\sqrt{a^2+qb^2r^{-1}}\right]^2 x\frac{\partial r}{\partial x}\\
&= -\ px\frac{\partial a}{\partial x}+\frac{1}{r}p^2bx\frac{\partial b}{\partial x}-\frac{1}{2}x\frac{\partial q}{\partial x}-\frac{p^2b^2}{2r^2}x\frac{\partial r}{\partial x}
\end{aligned}$$

which, together with SDARE (19), is the scalar version of optimality condition (10).

Substituting solution $p(x)$ and the resulting control into system (18) yields the closed-loop optimal system

$$\dot{x} = -\sqrt{a^2(x)+q(x)b^2(x)r^{-1}(x)}x$$

It follows from the Lyapunov function[¶]

$$V(x) = \int_0^x p(\tau)\tau\,\mathrm{d}\tau$$

that

$$\dot{V} = x\,p(x)\dot{x} = -r(x)b^{-2}(x)\left[a(x)+\sqrt{a^2(x)+q(x)b^2(x)r^{-1}(x)}\right]\ \sqrt{a^2(x)+q(x)b^2(x)r^{-1}(x)}x^2$$

is strictly negative and therefore the system is globally asymptotically stable. Hence, the following lemma can be concluded.

Lemma 1

For scalar systems, the SDARE method always yields the optimal control (or, if $t_f \neq \infty$, inversely optimal with respect to some scalar value of S in performance index (5)) and the optimal control is globally stabilizing.

It should be mentioned that the solution $p(x)$ is not an explicit function of time. Thus, while the SDARE control is always optimal for regulating scalar systems over the infinite horizon,[‖] it is only suboptimal with respect to performance index (5) if the weighting S is given. This is

[¶] A simpler Lyapunov function is $V(x) = x^2$.

[‖] This result of optimality is also obvious from HJB equation (15) as symmetry property is not needed for scalar systems.

because, while solution $p(x)$ satisfies optimality condition (10), it may not satisfy the boundary condition $p(x(t_f)) = S$. The optimal solution $p(x(t_0), t_0, t_f)$ is generally a function of both x and t.

3. SDARE CONTROL FOR CASCADED SYSTEMS

In this section, we shall study the ways to design suboptimal control for cascaded non-linear systems. It has been shown that various controls such as adaptive control and robust control can be easily designed for cascaded systems using the backstepping method [7], a backward recursive design. In an application of the method, a fictitious control is designed first for each first-order (vector and square) subsystem, and the collection of the fictitious controls form a recursive mapping from which the actual control can be determined. In principle, SDARE method could be combined straightforwardly into the backstepping design so that fictitious controls are made to be suboptimal or even optimal. In what follows, we shall study this combination and motivate the alternative of using SDARE and a forward design.

3.1. Combinations of SDARE design and recursive designs

In terms of such features as optimality, stability and real-time implementability, Lemma 1 on SDARE control of scalar systems is the best result that one can hope for. While its extension to high-order systems is possible as shown by previous work [13, 14, 17], optimality (or sub-optimality) and global stability can only be guaranteed under several conditions. Incidentally, recursive designs (including backstepping, or forward recursive or interlacing design) are also based on design and stability results for scalar systems. For cascaded systems, the system structure makes it possible for the designer to choose a fictitious control and to study its impact on stability and performance, subsystem by subsystem. Combining a recursive design and the SDARE design would allow the designer to design a control for higher-order systems with guaranteed stability and performance (measured by certain optimality criteria).

It is straightforward to combine the SDARE method and the backstepping design. For example, consider the second-order system

$$\dot{x}_1 = a_1(x_1)x_1 + b_1(x_1)x_2, \quad \dot{x}_2 = a_2(x_2)x_2 + b_2(x_2)u \tag{20}$$

where $b_i(\cdot)$ do not assume the value of zero. To design a control recursively, one rewrites the first subsystem as

$$\dot{x}_1 = a_1(x_1)x_1 + b_1(x_1)v_1 + b_1(x_1)(x_2 - v_1) \overset{\triangle}{=} a_1(x_1)x_1 + b_1(x_1)v_1 + b_1(x_1)z_2$$

Now, design v_1 for the fictitious system

$$\dot{x}_1 = a_1(x_1)x_1 + b_1(x_1)v_1 \tag{21}$$

in which case the SDARE method can readily be applied to optimize the performance index

$$I_1 = \frac{1}{2}s_1x_1^2(t_f) + \frac{1}{2}\int_{t_0}^{t_f} [q_1(x_1)x_1^2 + r_1(x_1)v_1^2]\,\mathrm{d}t \tag{22}$$

It follows from the result in Section 2.5 that the SDARE control is

$$v_1 = -b_1(x_1)r_1^{-1}(x_1)p_1(x_1)x_1 \tag{23}$$

where

$$p_1(x_1) = r_1(x_1)b_1^{-2}(x_1)\left[a_1(x_1) + \sqrt{a_1^2(x_1) + q_1(x_1)b_1^2(x_1)r_1^{-1}(x_1)}\right]$$

Based on fictitious control v_1, one can derive a dynamic equation for z_2, i.e.

$$\dot{z}_2 = a_2(x_2)x_2 - \frac{\partial v_1}{\partial x_1}a_1(x_1)x_1 - \frac{\partial v_1}{\partial x_1}b_1(x_1)x_2 + b_2(x_2)u$$

Now, letting

$$u = \frac{b_2(z_2)}{b_2(x_2)}v_2 + \frac{1}{b_2(x_2)}\left[a_2(z_2)z_2 - a_2(x_2)x_2 + \frac{\partial v_1}{\partial x_1}a_1(x_1)x_1 + \frac{\partial v_1}{\partial x_1}b_1(x_1)x_2\right] - \frac{b_1(x_1)p_1(x_1)}{b_2(x_2)p_2(z_2)}x_1 \tag{24}$$

where $p_2(z_2)$ will be defined shortly, we can rewrite the dynamics of the second subsystem as

$$\dot{z}_2 = a_2(z_2)z_2 + b_2(z_2)v_2 - b_1(x_1)p_1(x_1)p_2^{-1}(z_2)x_1$$

Again, v_2 can be designed to optimize performance index

$$I_2 = \frac{1}{2}s_2z_2^2(t_f) + \frac{1}{2}\int_{t_0}^{t_f}[q_2(z_2)z_2^2 + r_2(z_2)v_2^2]\,\mathrm{d}t \tag{25}$$

for the fictitious system

$$\dot{z}_2 = a_2(z_2)z_2 + b_2(z_2)v_2 \tag{26}$$

That is, the SDARE control that optimizes I_2 is

$$v_2 = -b_2(z_2)r_2^{-1}(z_2)p_2(z_2)z_2 \tag{27}$$

where

$$p_2(z_2) = r_2(z_2)b_2^{-2}(z_2)\left[a_2(z_1) + \sqrt{a_2^2(z_2) + q_2(z_2)b_2^2(z_2)r_2^{-1}(z_2)}\right]$$

By combining the SDARE design into the backstepping method in the above manner, we have the following result on stability and performance.

Lemma 2
Consider system (20) under control (24). Then, the closed-loop system has the following stability properties:

(i) Measured by performance indices (22) and (25), fictitious controls v_1 and v_2 in (23) and (27) are individually optimal (inversely with respect to some values of s_1 and s_2) for fictitious systems (21) and (26), respectively.

(ii) The actual control u defined by (23), (27), and (24) is globally stabilizing.
(iii) The control u is also optimal with respect to performance index $I_1 + I_2$ with $t_f = \infty$.

Proof

Statement (i) follows from Lemma 1. To verify statement (ii), consider Lyapunov function

$$V(x_1, z_2) = \int_0^{x_1} \tau_1 p_1(\tau_1)\,\mathrm{d}\tau_1 + \int_0^{z_2} \tau_2 p_2(\tau_2)\,\mathrm{d}\tau_2 \tag{28}$$

It follows that $V(\cdot)$ is a positive definite function of x_1 and z_2 and that, by the dynamics of x_1 and z_2,

$$\begin{aligned}\dot{V} =& x_1 p_1(x_1)[a_1(x_1)x_1 + b_1(x_1)v_1] + z_2 p_2(z_2)[a_2(z_2)z_2 + b_2(z_2)v_2] \\ =& -r_1(x_1)b_1^{-2}(x_1)\left[a_1(x_1) + \sqrt{a_1^2(x_1) + q_1(x_1)b_1^2(x_1)r_1^{-1}(x_1)}\right]\sqrt{a_1^2(x_1) + q_1(x)b_1^2(x_1)r_1^{-1}(x_1)}x_1^2 \\ & - r_1(z_2)b_2^{-2}(z_2)\left[a_2(z_2) + \sqrt{a_2^2(z_2) + q_2(z_2)b_2^2(z_2)r_2^{-1}(z_2)}\right]\sqrt{a_2^2(z_2) + q_2(z_2)b_2^2(z_2)r_2^{-1}(z_2)}z_2^2\end{aligned}$$

which is negative definite.

To verify statement (iii), consider again the value function (28). It follows that symmetry condition holds for $V(x_1, z_2)$ as

$$\frac{\partial^2 V(x_1, z_2)}{\partial x_1 \partial z_2} = \frac{\partial^2 V(x_1, z_2)}{\partial z_2 \partial x_1} = 0$$

With respect to performance index $I = I_1 + I_2$ with $t_f = \infty$, HJB equation (13) reduces to

$$\begin{aligned}0 =& \left[\frac{\partial V}{\partial x_1}\frac{\partial V}{\partial z_2}\right]\begin{bmatrix} a_1(x_1)x_1 + b_1(x_1)v_1 + b_1(x_1)z_2 \\ a_2(z_2)z_2 - b_1(x_1)p_1(x_1)p_2^{-1}(z_2)x_1 \end{bmatrix} \\ & - \frac{1}{r_2(z_2)}\left[\frac{\partial V}{\partial x_1}\frac{\partial V}{\partial z_2}\right]\begin{bmatrix} 0 \\ b_2(z_2) \end{bmatrix}[0 \;\; b_2(z_2)]\begin{bmatrix} \frac{\partial V}{\partial x_1} \\ \frac{\partial V}{\partial z_2} \end{bmatrix} + \frac{1}{2}q_1 x_1^2 + \frac{1}{2}r_1 v_1^2 + \frac{1}{2}q_2 z_2^2 + \frac{1}{2}r_2 v_2^2\end{aligned}$$

Performing vector products in the above equation yields

$$p_1 x_1(a_1 x_1 + b_1 v_1) + p_2 z_2^2 a_2 - \frac{b_2^2}{r_2} p_2^2 z_2^2 + \frac{1}{2}q_1 x_1^2 + \frac{1}{2}r_1 v_1^2 + \frac{1}{2}q_2 z_2^2 + \frac{1}{2}r_2 v_2^2 = 0$$

Substituting the expressions of v_1 and v_2 into the above equation yields

$$\left[p_1 x_1^2 a_1 - \frac{b_1^2}{2r_1} p_1^2 x_1^2 + \frac{1}{2} q_1 x_1^2\right] + \left[p_2 z_2^2 a_2 - \frac{b_2^2}{2r_2} p_2^2 z_2^2 + \frac{1}{2} q_2 z_2^2\right] = 0$$

which is obviously valid as the two brackets are the SDAREs for p_1 and p_2, respectively. □

Although Lemma 2 is stated and proven for second-order systems, its extension to high-order cascaded systems is obvious. It is also worth mentioning that feedback linearization is applicable to cascaded systems and, if applied, the system can be mapped into a linear one of the form

$$\dot{z}_1 = z_2 \quad \dot{z}_2 = v'$$

Then, one can easily design a linear optimal control for the above system to optimize quadratic performance index

$$\mathbf{z}^{\mathrm{T}}(t_f)\mathbf{S}\mathbf{z}(t_f) + \int_{t_0}^{t_f} [\mathbf{z}^{\mathrm{T}}\mathbf{Q}\mathbf{z} + v'Rv']\,\mathrm{d}t$$

in which case Lemma 2 reduces to a linear result.

In the above backstepping design, differentiation of fictitious control v_1 is performed in the backstepping step. Equivalently, differentiation of a sub-Lyapunov function of form $V_2(x_1, x_2, v_1)$ can be done in the design, which is beneficial for the case that the fictitious control itself is not differentiable [18]. Such operations produces many additional terms in the transformed dynamics. In fact, the higher the order of the system, the more terms one must consider in control design, which makes the design more involved and less accessible to application engineers.

The main feature of Lemma 2 is that performance index I is inversely determined through a backstepping design rather than prescribed, and most available results are along this line. One worth mentioning is the result reported in Reference [19]. It was shown in that paper that, for a special class of cascade systems, backstepping design can produce a control that is optimal with respect to a non-linear, inversely determined performance index and is also locally optimal with respect to a prescribed linear quadratic performance index.

There are three unresolved issues in the above non-linear optimal (or suboptimal) control design. First, in an optimal or suboptimal control design, can the designer use a quadratic-type non-linear performance index that is in terms of the original state and original control variables? Second, is there a recursive design that works for cascaded systems but does not require any differentiation operation? Finally, can tracking performance be considered in the design? The new recursive suboptimal design procedure proposed in the paper provide positive answers to these questions. Specifically, the new design method has the following distinct features:

- The performance index is defined to be quadratic-like and in terms of original state and control variables.
- The tracking formulation is used to define the control problem.
- The mapping of the fictitious controls into the actual control consists of a sequence of successive algebraic substitutions so that differentiation of fictitious controls (or their functions) is completely avoided.

The proposed technique is based on a forward recursive design and on the so-called SDARE tracker. To this end, the optimal tracker and the SDARE tracker will be developed first in the next section.

3.2. Non-linear optimal tracker and SDARE tracker

Consider the following non-linear, affine system:

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x})\mathbf{x} + \mathbf{B}(\mathbf{x})\mathbf{u}, \quad \mathbf{y} = \mathbf{C}(\mathbf{x})\mathbf{x} \tag{29}$$

where $\mathbf{x} \in \Re^n$, $\mathbf{u} \in \Re^m$, $\mathbf{y} \in \Re^p$ and functions $\mathbf{A}(\cdot)$, $\mathbf{B}(\cdot)$ and $\mathbf{C}(\cdot)$ are continuous. The objective is to devise a non-linear, continuous control so that the output of the system tracks its desired output $\mathbf{y}^{\mathrm{d}}(t)$, where $\mathbf{y}^{\mathrm{d}}$ is a smooth time function. This tracking problem can again be recast as an optimal control problem by introducing the performance index

$$J(\mathbf{x}(t_0), \mathbf{u}, \mathbf{y}^{\mathrm{d}}, t_0, t_f) = \frac{1}{2}[\mathbf{y}(t_f) - \mathbf{y}^{\mathrm{d}}]^{\mathrm{T}}\mathbf{S}[\mathbf{y}(t_f) - \mathbf{y}^{\mathrm{d}}] + \frac{1}{2}\int_{t_0}^{t_f} \{[\mathbf{y} - \mathbf{y}^{\mathrm{d}}]^{\mathrm{T}}\mathbf{Q}[\mathbf{y} - \mathbf{y}^{\mathrm{d}}] + \mathbf{u}^{\mathrm{T}}\mathbf{R}\mathbf{u}\}\,\mathrm{d}t \tag{30}$$

where $t_f \in [t_0, \infty]$ is the time interval of optimization, and matrices $\mathbf{S}$, $\mathbf{Q}$ and $\mathbf{R}$ are defined as before. Formulating the Hamiltonian H as

$$H = \tfrac{1}{2}[\mathbf{C}(\mathbf{x})\mathbf{x} - \mathbf{y}^{\mathrm{d}}]^{\mathrm{T}}\mathbf{Q}[\mathbf{C}(\mathbf{x})\mathbf{x} - \mathbf{y}^{\mathrm{d}}] + \tfrac{1}{2}\mathbf{u}^{\mathrm{T}}\mathbf{R}\mathbf{u} + \lambda^{\mathrm{T}}[\mathbf{A}(\mathbf{x})\mathbf{x} + \mathbf{B}(\mathbf{x})\mathbf{u}] \tag{31}$$

we have

$$\begin{aligned}\frac{\partial H}{\partial \mathbf{x}} &= \mathbf{C}^{\mathrm{T}}(\mathbf{x})\mathbf{Q}[\mathbf{C}(\mathbf{x})\mathbf{x} - \mathbf{y}^{\mathrm{d}}] + \mathrm{vec}\left\{[\mathbf{C}(\mathbf{x})\mathbf{x} - \mathbf{y}^{\mathrm{d}}]^{\mathrm{T}}\mathbf{Q}\frac{\partial \mathbf{C}}{\partial x_i}\mathbf{x}\right\} \\ &\quad + \frac{1}{2}\mathrm{vec}\left\{[\mathbf{C}(\mathbf{x})\mathbf{x} - \mathbf{y}^{\mathrm{d}}]^{\mathrm{T}}\,\frac{\partial \mathbf{Q}(\mathbf{x})}{\partial x_i}[\mathbf{C}(\mathbf{x})\mathbf{x} - \mathbf{y}^{\mathrm{d}}]\right\} \\ &\quad + \frac{1}{2}\mathrm{vec}\left\{\mathbf{u}^{\mathrm{T}}\frac{\partial \mathbf{R}(\mathbf{x})}{\partial x_i}\mathbf{u}\right\} + \left(\frac{\partial[\mathbf{A}(\mathbf{x})\mathbf{x}]}{\partial \mathbf{x}}\right)^{\mathrm{T}}\lambda + \mathrm{vec}\left\{\mathbf{u}^{\mathrm{T}}\frac{\partial \mathbf{B}^{\mathrm{T}}}{\partial x_i}\lambda\right\}\end{aligned}$$

Since the tracking problem reduces to the regulation problem discussed in Section 2 if $\mathbf{y}^{\mathrm{d}} = 0$, the tracking control structure should contain the same feedback mechanism as before but also have an additive feedforward/feedback part. That is, we can parameterize the Lagrangian multiplier as

$$\lambda = \mathbf{P}\mathbf{x} + \mathbf{w}(t)$$

where $w(t)$ is the feedforward/feedback control part. If the system is linear and the performance index is quadratic, $w(t)$ does not depend on $\mathbf{x}$ and hence becomes feedforward only, and $w(t) = 0$ if $\mathbf{y}^{\mathrm{d}}(t) = 0$ in addition. Then, by applying the necessary conditions for optimality in (7), we can conclude that the optimal tracker is

$$\mathbf{u} = -\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}[\mathbf{P}\mathbf{x} + \mathbf{w}(t)] \tag{32}$$

where the auxiliary signal $\mathbf{w}(t)$ and the matrix $\mathbf{P}$ are the solution to the optimality condition $\dot{\lambda} = -\partial H/\partial \mathbf{x}$, i.e.

$$\begin{aligned}
\mathbf{0} = \Bigg\langle & \dot{\mathbf{P}}\mathbf{x} + (\mathbf{PA} + \mathbf{A}^{\mathrm{T}}\mathbf{P} - \mathbf{PBR}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P} + \mathbf{C}^{\mathrm{T}}\mathbf{QC})\mathbf{x} \\
& + \mathrm{vec}\left\{\mathbf{x}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}\mathbf{Q}\frac{\partial \mathbf{C}}{\partial x_i}\mathbf{x}\right\} + \frac{1}{2}\mathrm{vec}\left\{\mathbf{x}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}\frac{\partial \mathbf{Q}}{\partial x_i}\mathbf{Cx}\right\} \\
& + \frac{1}{2}\mathrm{vec}\left\{\mathbf{x}^{\mathrm{T}}\mathbf{PBR}^{-1}\frac{\partial \mathbf{R}}{\partial x_i}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{Px}\right\} + \mathrm{vec}\left\{\mathbf{x}^{\mathrm{T}}\frac{\partial \mathbf{A}^{\mathrm{T}}}{\partial x_i}\mathbf{Px} - \mathbf{x}^{\mathrm{T}}\mathbf{PBR}^{-1}\frac{\partial \mathbf{B}^{\mathrm{T}}}{\partial x_i}\mathbf{Px}\right\}\Bigg\rangle \\
& + \Bigg\langle \dot{\mathbf{w}}(t) + [\mathbf{A}^{\mathrm{T}} - \mathbf{PBR}^{-1}\mathbf{B}^{\mathrm{T}}]\mathbf{w}(t) + \mathrm{vec}\left\{\mathbf{x}^{\mathrm{T}}\frac{\partial \mathbf{A}^{\mathrm{T}}}{\partial x_i}\mathbf{w}(t)\right\} - \mathbf{C}^{\mathrm{T}}\mathbf{Qy}^{\mathrm{d}} + \mathrm{vec}\left\{[\mathbf{y}^{\mathrm{d}}]^{\mathrm{T}}\mathbf{Q}\frac{\partial \mathbf{C}^{\mathrm{T}}}{\partial x_i}\mathbf{x}\right\} \\
& - \mathrm{vec}\left\{\mathbf{x}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}\frac{\partial \mathbf{Q}}{\partial x_i}\mathbf{y}^{\mathrm{d}}\right\} + \frac{1}{2}\mathrm{vec}\left\{[\mathbf{y}^{\mathrm{d}}]^{\mathrm{T}}\frac{\partial \mathbf{Q}}{\partial x_i}\mathbf{y}^{\mathrm{d}}\right\} + \mathrm{vec}\left\{\mathbf{x}^{\mathrm{T}}\mathbf{PBR}^{-1}\frac{\partial \mathbf{R}}{\partial x_i}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{w}(t)\right\} \\
& + \frac{1}{2}\mathrm{vec}\left\{\mathbf{w}^{\mathrm{T}}(t)\mathbf{BR}^{-1}\frac{\partial \mathbf{R}}{\partial x_i}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{w}(t)\right\} - \mathrm{vec}\left\{\mathbf{x}^{\mathrm{T}}\mathbf{PBR}^{-1}\frac{\partial \mathbf{B}^{\mathrm{T}}}{\partial x_i}\mathbf{w}(t)\right\} \\
& - \mathrm{vec}\left\{\mathbf{w}^{\mathrm{T}}(t)\mathbf{BR}^{-1}\frac{\partial \mathbf{B}^{\mathrm{T}}}{\partial x_i}\mathbf{Px}\right\} - \mathrm{vec}\left\{\mathbf{w}^{\mathrm{T}}\mathbf{BR}^{-1}\frac{\partial \mathbf{B}^{\mathrm{T}}}{\partial x_i}\mathbf{w}(t)\right\}\Bigg\rangle
\end{aligned} \tag{33}$$

and where the boundary conditions for $\mathbf{w}(t)$ and $\mathbf{P}(t)$ are

$$\mathbf{x}(t_0) \text{ given}, \quad \mathbf{P}(\mathbf{x}(t_0), t_0, t_f) = \mathbf{C}^{\mathrm{T}}\mathbf{SC}, \quad \mathbf{w}(t_f) = \mathbf{C}^{\mathrm{T}}\mathbf{Sy}^{\mathrm{d}}(t_f) \quad \text{and} \quad 0 < \lim_{t_f \to \infty} \mathbf{P}(\mathbf{x}(t_0), t_0, t_f) < \infty$$

The optimal tracker has to be solved as a two-point boundary value problem as the optimality condition (33) should be integrated backwards. The optimality condition (33) is the sum of two parts (as grouped by $\langle \cdot \rangle$): the first part contains terms that are associated with the matrix $\mathbf{P}$ and the state $\mathbf{x}$, and the second part includes the terms associated with the command $\mathbf{y}^{\mathrm{d}}$ and the auxiliary signal $\mathbf{w}(t)$.

To avoid the two-point boundary value problem, an SDARE tracker can be formulated in a similar fashion as the SDARE regulator. The proposed SDARE tracker of form (32) is generated in three steps. First, matrix $\mathbf{P}$ is the symmetrical solution to the output state-dependent Riccati equation (OSDARE):

$$\mathbf{PA} + \mathbf{A}^{\mathrm{T}}\mathbf{P} - \mathbf{PBR}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P} + \mathbf{C}^{\mathrm{T}}\mathbf{QC} = \mathbf{0} \tag{34}$$

From (34), we know that the matrix $\mathbf{P}$ is a function of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{Q}$ and $\mathbf{R}$. Second, separate the auxiliary signal from the optimality condition (33) by setting its dynamics to be

$$
\begin{aligned}
\dot{\mathbf{w}}(t) = {} & [-\mathbf{A}^{\mathrm{T}} + \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}]\mathbf{w}(t) - \mathrm{vec}\left\{\mathbf{x}^{\mathrm{T}}\frac{\partial\mathbf{A}^{\mathrm{T}}}{\partial x_i}\mathbf{w}(t)\right\} + \mathbf{C}^{\mathrm{T}}\mathbf{Q}\mathbf{y}^{\mathrm{d}} + \mathrm{vec}\left\{[\mathbf{y}^{\mathrm{d}}]^{\mathrm{T}}\mathbf{Q}\frac{\partial\mathbf{C}^{\mathrm{T}}}{\partial x_i}\mathbf{x}\right\} \\
& + \mathrm{vec}\left\{\mathbf{x}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}\frac{\partial\mathbf{Q}}{\partial x_i}\mathbf{y}^{\mathrm{d}}\right\} - \frac{1}{2}\mathrm{vec}\left\{[\mathbf{y}^{\mathrm{d}}]^{\mathrm{T}}\frac{\partial\mathbf{Q}}{\partial x_i}\mathbf{y}^{\mathrm{d}}\right\} - \mathrm{vec}\left\{\mathbf{x}^{\mathrm{T}}\mathbf{P}\mathbf{B}\mathbf{R}^{-1}\frac{\partial\mathbf{R}}{\partial x_i}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{w}(t)\right\} \\
& - \frac{1}{2}\mathrm{vec}\left\{\mathbf{w}^{\mathrm{T}}(t)\mathbf{B}\mathbf{R}^{-1}\frac{\partial\mathbf{R}}{\partial x_i}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{w}(t)\right\} + \mathrm{vec}\left\{\mathbf{x}^{\mathrm{T}}\mathbf{P}\mathbf{B}\mathbf{R}^{-1}\frac{\partial\mathbf{B}^{\mathrm{T}}}{\partial x_i}\mathbf{w}(t)\right\} \\
& + \mathrm{vec}\left\{\mathbf{w}^{\mathrm{T}}(t)\mathbf{B}\mathbf{R}^{-1}\frac{\partial\mathbf{B}^{\mathrm{T}}}{\partial x_i}\mathbf{P}\mathbf{x}\right\} + \mathrm{vec}\left\{\mathbf{w}^{\mathrm{T}}\mathbf{B}\mathbf{R}^{-1}\frac{\partial\mathbf{B}^{\mathrm{T}}}{\partial x_i}\mathbf{w}(t)\right\} \\
& + \mathrm{vec}\left\{\sum_{j=1}^{n}\sum_{k=1}^{n}\mathbf{x}^{\mathrm{T}}\frac{\partial\mathbf{P}}{\partial A_{jk}}\frac{\partial A_{jk}}{\partial x_i}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{w}(t)\right\} + \mathrm{vec}\left\{\sum_{j=1}^{n}\sum_{k=1}^{m}\mathbf{x}^{\mathrm{T}}\frac{\partial\mathbf{P}}{\partial B_{jk}}\frac{\partial B_{jk}}{\partial x_i}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{w}(t)\right\} \\
& + \mathrm{vec}\left\{\sum_{j=1}^{p}\sum_{k=1}^{n}\mathbf{x}^{\mathrm{T}}\frac{\partial\mathbf{P}}{\partial C_{jk}}\frac{\partial C_{jk}}{\partial x_i}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{w}(t)\right\} + \mathrm{vec}\left\{\sum_{j=1}^{p}\sum_{k=1}^{p}\mathbf{x}^{\mathrm{T}}\frac{\partial\mathbf{P}}{\partial Q_{jk}}\frac{\partial Q_{jk}}{\partial x_i}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{w}(t)\right\} \\
& + \mathrm{vec}\left\{\sum_{j=1}^{m}\sum_{k=1}^{m}\mathbf{x}^{\mathrm{T}}\frac{\partial\mathbf{P}}{\partial R_{jk}}\frac{\partial R_{jk}}{\partial x_i}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{w}(t)\right\} \qquad (35)
\end{aligned}
$$

Therefore, under (17) and (35), the optimality condition (33) reduces to

$$
\begin{aligned}
\mathbf{0} = {} & \dot{\mathbf{P}}\mathbf{x} + \mathrm{vec}\left\{\mathbf{x}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}\mathbf{Q}\frac{\partial\mathbf{C}}{\partial x_i}\mathbf{x}\right\} + \frac{1}{2}\mathrm{vec}\left\{\mathbf{x}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}\frac{\partial\mathbf{Q}}{\partial x_i}\mathbf{C}\mathbf{x}\right\} + \frac{1}{2}\mathrm{vec}\left\{\mathbf{x}^{\mathrm{T}}\mathbf{P}\mathbf{B}\mathbf{R}^{-1}\frac{\partial\mathbf{R}}{\partial x_i}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}\mathbf{x}\right\} \\
& + \mathrm{vec}\left\{\mathbf{x}^{\mathrm{T}}\frac{\partial\mathbf{A}^{\mathrm{T}}}{\partial x_i}\mathbf{P}\mathbf{x} - \mathbf{x}^{\mathrm{T}}\mathbf{P}\mathbf{B}\mathbf{R}^{-1}\frac{\partial\mathbf{B}^{\mathrm{T}}}{\partial x_i}\mathbf{P}\mathbf{x}\right\} + \mathrm{vec}\left\{\sum_{j=1}^{n}\sum_{k=1}^{n}\mathbf{x}^{\mathrm{T}}\frac{\partial\mathbf{P}}{\partial A_{jk}}\frac{\partial A_{jk}}{\partial x_i}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{w}(t)\right\} \\
& + \mathrm{vec}\left\{\sum_{j=1}^{n}\sum_{k=1}^{m}\mathbf{x}^{\mathrm{T}}\frac{\partial\mathbf{P}}{\partial B_{jk}}\frac{\partial B_{jk}}{\partial x_i}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{w}(t)\right\} + \mathrm{vec}\left\{\sum_{j=1}^{p}\sum_{k=1}^{n}\mathbf{x}^{\mathrm{T}}\frac{\partial\mathbf{P}}{\partial C_{jk}}\frac{\partial C_{jk}}{\partial x_i}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{w}(t)\right\} \\
& + \mathrm{vec}\left\{\sum_{j=1}^{p}\sum_{k=1}^{p}\mathbf{x}^{\mathrm{T}}\frac{\partial\mathbf{P}}{\partial Q_{jk}}\frac{\partial Q_{jk}}{\partial x_i}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{w}(t)\right\} + \mathrm{vec}\left\{\sum_{j=1}^{m}\sum_{k=1}^{m}\mathbf{x}^{\mathrm{T}}\frac{\partial\mathbf{P}}{\partial R_{jk}}\frac{\partial R_{jk}}{\partial x_i}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{w}(t)\right\} \quad (36)
\end{aligned}
$$

Third, to overcome the two-point boundary value problem, we will reverse the time in (35) so that the auxiliary signal is generated forward in time by

$$\begin{aligned}\dot{\mathbf{w}}(t) = {} & [\mathbf{A}^{\mathrm{T}} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}]\mathbf{w}(t) + \mathrm{vec}\left\{\mathbf{x}^{\mathrm{T}}\frac{\partial \mathbf{A}^{\mathrm{T}}}{\partial x_i}\mathbf{w}(t)\right\} - \mathbf{C}^{\mathrm{T}}\mathbf{Q}\mathbf{y}^{\mathrm{d}} - \mathrm{vec}\left\{[\mathbf{y}^{\mathrm{d}}]^{\mathrm{T}}\mathbf{Q}\frac{\partial \mathbf{C}^{\mathrm{T}}}{\partial x_i}\mathbf{x}\right\} \\
& - \mathrm{vec}\left\{\mathbf{x}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}\frac{\partial \mathbf{Q}}{\partial x_i}\mathbf{y}^{\mathrm{d}}\right\} + \frac{1}{2}\mathrm{vec}\left\{[\mathbf{y}^{\mathrm{d}}]^{\mathrm{T}}\frac{\partial \mathbf{Q}}{\partial x_i}\mathbf{y}^{\mathrm{d}}\right\} + \mathrm{vec}\left\{\mathbf{x}^{\mathrm{T}}\mathbf{P}\mathbf{B}\mathbf{R}^{-1}\frac{\partial \mathbf{R}}{\partial x_i}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{w}(t)\right\} \\
& + \frac{1}{2}\mathrm{vec}\left\{\mathbf{w}^{\mathrm{T}}(t)\mathbf{B}\mathbf{R}^{-1}\frac{\partial \mathbf{R}}{\partial x_i}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{w}(t)\right\} - \mathrm{vec}\left\{\mathbf{x}^{\mathrm{T}}\mathbf{P}\mathbf{B}\mathbf{R}^{-1}\frac{\partial \mathbf{B}^{\mathrm{T}}}{\partial x_i}\mathbf{w}(t)\right\} \\
& - \mathrm{vec}\left\{\mathbf{w}^{\mathrm{T}}(t)\mathbf{B}\mathbf{R}^{-1}\frac{\partial \mathbf{B}^{\mathrm{T}}}{\partial x_i}\mathbf{P}\mathbf{x}\right\} - \mathrm{vec}\left\{\mathbf{w}^{\mathrm{T}}\mathbf{B}\mathbf{R}^{-1}\frac{\partial \mathbf{B}^{\mathrm{T}}}{\partial x_i}\mathbf{w}(t)\right\} \\
& - \mathrm{vec}\left\{\sum_{j=1}^{n}\sum_{k=1}^{n}\mathbf{x}^{\mathrm{T}}\frac{\partial \mathbf{P}}{\partial A_{jk}}\frac{\partial A_{jk}}{\partial x_i}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{w}(t)\right\} - \mathrm{vec}\left\{\sum_{j=1}^{n}\sum_{k=1}^{m}\mathbf{x}^{\mathrm{T}}\frac{\partial \mathbf{P}}{\partial B_{jk}}\frac{\partial B_{jk}}{\partial x_i}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{w}(t)\right\} \\
& - \mathrm{vec}\left\{\sum_{j=1}^{p}\sum_{k=1}^{n}\mathbf{x}^{\mathrm{T}}\frac{\partial \mathbf{P}}{\partial C_{jk}}\frac{\partial C_{jk}}{\partial x_i}\mathbf{B}R^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{w}(t)\right\} - \mathrm{vec}\left\{\sum_{j=1}^{p}\sum_{k=1}^{p}\mathbf{x}^{\mathrm{T}}\frac{\partial \mathbf{P}}{\partial Q_{jk}}\frac{\partial Q_{jk}}{\partial x_i}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{w}(t)\right\} \\
& - \mathrm{vec}\left\{\sum_{j=1}^{m}\sum_{k=1}^{m}\mathbf{x}^{\mathrm{T}}\frac{\partial \mathbf{P}}{\partial R_{jk}}\frac{\partial R_{jk}}{\partial x_i}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{w}(t)\right\}\end{aligned} \tag{37}$$

with initial condition $\mathbf{w}(t_0)$ properly chosen (and $\mathbf{w}(t_0) = 0$ if $\mathbf{y}^{\mathrm{d}}(t) = 0$). The above derivation leads naturally to the following lemma.

Lemma 3
The non-linear tracker defined by (32) together with (34) and (37) yields the optimal control (provided that the initial condition $\mathbf{w}(t_0)$ is properly generated from the boundary conditions and that the value of $\mathbf{S}$ is inversely determined).

The SDARE tracker is only suboptimal since the optimality condition (36) is not guaranteed in general. Stability analysis of the closed-loop system under the non-linear tracker or SDARE tracker can be pursued in the similar fashion as that in Reference [17]. It is worth noting that, for internal stability, some multiple-input and multiple-output systems may not be able to track an arbitrary continuous function $\mathbf{y}^{\mathrm{d}}(t)$ asymptotically.

3.3. SDARE tracker for scalar systems

To implement the new SDARE design for cascaded systems, let us reconsider scalar system (18) with $y = c(x)x$, where $c(x) \geqslant \underline{c} > 0$. Its associated OSDARE is, for $q(x) \geqslant \underline{q} > 0$ and $r(x) \geqslant \underline{r} > 0$,

$$2a(x)p(x) - b^2(x)r^{-1}(x)p^2(x) + c^2(x)q(x) = 0 \tag{38}$$

and the resulting SDARE control is

$$u = -b(x)r^{-1}(x)\,p(x)x - b(x)r^{-1}(x)w \tag{39}$$

where $w(t)$ is generated by (37), i.e.

$$\begin{aligned}\dot{w} = {} & [a - pb^2r^{-1}]w + x\frac{\partial a}{\partial x}w - cq\,y^{\mathrm{d}} - y^{\mathrm{d}}q\frac{\partial c}{\partial x}x - xc\frac{\partial q}{\partial x}y^{\mathrm{d}} + \frac{1}{2}\frac{\partial q}{\partial x}[y^{\mathrm{d}}]^2 \\ & + xpb^2r^{-2}\frac{\partial r}{\partial x}w + \frac{1}{2}b^2r^{-2}\frac{\partial r}{\partial x}w^2 - 2xpbr^{-1}\frac{\partial b}{\partial x}w - br^{-1}\frac{\partial b}{\partial x}w^2 \\ & - x\left(\frac{\partial p}{\partial a}\frac{\partial a}{\partial x} + \frac{\partial p}{\partial b}\frac{\partial b}{\partial x} + \frac{\partial p}{\partial c}\frac{\partial c}{\partial x} + \frac{\partial p}{\partial q}\frac{\partial q}{\partial x} + \frac{\partial p}{\partial r}\frac{\partial r}{\partial x}\right)r^{-1}b^2w\end{aligned} \tag{40}$$

with initial condition $w(t_0)$ given. The positive solution to OSDARE (38) is

$$p(x) = r(x)b^{-2}(x)\left[a(x) + \sqrt{a^2(x) + c^2(x)q(x)b^2(x)r^{-1}(x)}\right]$$

under which the closed-loop system becomes

$$\dot{x} = -\sqrt{a^2(x) + c^2(x)q(x)b^2(x)r^{-1}(x)}\,x - b^2(x)r^{-1}(x)w \tag{41}$$

Stability and performance of the closed-loop system is summarized in the following lemma.

Lemma 4
The scalar system (18) under the SDARE control (39) has the following properties:

(i) The closed-loop system (41) is input-to-state stable with respect to $w(t)$ if

$$\frac{b^2}{\sqrt{a^2r^2 + c^2qb^2r}} \tag{42}$$

is radially bounded.

(ii) The SDARE tracker, defined by (39) and (40), yields the optimal control (provided that the initial condition $w(t_0)$ is properly generated from the boundary conditions and that the value of S is inversely determined).

(iii) The closed-loop system under the SDARE tracker is globally uniformly bounded if $b^2(x)/r(x)$ and $q(x)$ are constant and that

$$a^2 + c^2qb^2r^{-1} + \left(2a\frac{\partial a}{\partial x} + 2cqb^2r^{-1}\frac{\partial c}{\partial x}\right)x \geqslant 0 \tag{43}$$

If in addition $c(x)$ is a constant and y^{d} is uniformly continuous, $|y - y^{\mathrm{d}}|$ can be made arbitrarily small by increasing q.**

**It follows from the proof that the requirement of $q(x)$ being constant can be relaxed and that, if $b^2(x)/r(x)$ is not constant but its partial derivative has a small magnitude, local stability can be concluded.

Proof

To show input-to-state stability for system (41), consider the Lyapunov function $V(x) = 0.5x^2$. Then,

$$\dot{V}(x) = -\sqrt{a^2 + c^2 q b^2 r^{-1}}\, x^2 - b^2 r^{-1} w x$$

which is negative definite outside an interval around the origin for any bounded $w(t)$ if condition (42) holds.

It follows from solution $p(x)$ that

$$\begin{aligned}
&\frac{\partial p}{\partial a}\frac{\partial a}{\partial x} + \frac{\partial p}{\partial b}\frac{\partial b}{\partial x} + \frac{\partial p}{\partial c}\frac{\partial c}{\partial x} + \frac{\partial p}{\partial q}\frac{\partial q}{\partial x} + \frac{\partial p}{\partial r}\frac{\partial r}{\partial x} \\
&\quad = \frac{r}{b^2}\frac{\sqrt{a^2 + c^2 q b^2 r^{-1}} + a}{\sqrt{a^2 + c^2 q b^2 r^{-1}}}\frac{\partial a}{\partial x} - \frac{r}{b^3}\frac{\left[a + \sqrt{a^2 + c^2 q b^2 r^{-1}}\right]^2}{\sqrt{a^2 + c^2 q b^2 r^{-1}}}\frac{\partial b}{\partial x} \\
&\qquad + \frac{cq}{\sqrt{a^2 + c^2 q b^2 r^{-1}}}\frac{\partial c}{\partial x} + \frac{1}{2}\frac{c^2}{\sqrt{a^2 + c^2 q b^2 r^{-1}}}\frac{\partial q}{\partial x} + \frac{1}{2b^2}\frac{\left[a + \sqrt{a^2 + c^2 q b^2 r^{-1}}\right]^2}{\sqrt{a^2 + c^2 q b^2 r^{-1}}}\frac{\partial r}{\partial x}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\dot{p} &= \left[\frac{\partial p}{\partial a}\frac{\partial a}{\partial x} + \frac{\partial p}{\partial b}\frac{\partial b}{\partial x} + \frac{\partial p}{\partial c}\frac{\partial c}{\partial x} + \frac{\partial p}{\partial q}\frac{\partial q}{\partial x} + \frac{\partial p}{\partial r}\frac{\partial r}{\partial x}\right]\dot{x} \\
&= -\, px\frac{\partial a}{\partial x} + \frac{1}{r}p^2 b x\frac{\partial b}{\partial x} - cqx\frac{\partial c}{\partial x} - \frac{1}{2}x\frac{\partial q}{\partial x} - \frac{p^2 b^2}{2r^2}x\frac{\partial r}{\partial x} \\
&\quad - \left[\frac{\partial p}{\partial a}\frac{\partial a}{\partial x} + \frac{\partial p}{\partial b}\frac{\partial b}{\partial x} + \frac{\partial p}{\partial c}\frac{\partial c}{\partial x} + \frac{\partial p}{\partial q}\frac{\partial q}{\partial x} + \frac{\partial p}{\partial r}\frac{\partial r}{\partial x}\right]\frac{b^2}{r}w
\end{aligned}$$

which, together with OSDARE (38), is the scalar version of optimality condition (36). Therefore, statement (ii) can be concluded based on Lemma 3.

To show stability of global uniform boundedness, consider the Lyapunov function

$$V'(x, w) = \frac{1}{2}x^2 + \frac{4}{q\underline{c}^4}w^2$$

and note that the dynamic equation (40) can be rewritten as

$$\dot{w} = \frac{\partial \bar{f}(x)}{\partial x}w - \frac{1}{2}\frac{\partial[b^2(x)/r(x)]}{\partial x}w^2 - \frac{\partial[c(x)q(x)x]}{\partial x}y^{\mathrm{d}} - \frac{1}{2}\frac{\partial q(x)}{\partial x}[y^{\mathrm{d}}]^2 \tag{44}$$

where

$$\bar{f}(x) \triangleq a(x)x - p(x)b^2(x)x/r(x) = -\sqrt{a^2 + c^2 q b^2 r^{-1}}\,x$$

By making $b^2(x)/r(x)$ be a constant, the term w^2 will be removed from dynamic equation of $\dot{w}$, which makes global stability possible. It follows that, if inequality (43) holds

$$\begin{aligned}\frac{\partial \bar{f}(x)}{\partial x} &= -\sqrt{a^2 + c^2 q b^2 r^{-1}} - \frac{x}{2\sqrt{a^2 + c^2 q b^2 r^{-1}}}\left(2a\frac{\partial a}{\partial x} + 2cqb^2 r^{-1}\frac{\partial c}{\partial x} + c^2 b^2 r^{-1}\frac{\partial q}{\partial x}\right)\\ &\leqslant -\frac{1}{2}\sqrt{a^2 + c^2 q b^2 r^{-1}}\end{aligned}$$

Therefore, along every trajectory of system (41) under the control (39) and (40), the time derivative of the Lyapunov function is

$$\begin{aligned}\dot{V}'(x, w) &\leqslant -\sqrt{a^2 + c^2 q b^2 r^{-1}}\,x^2 - \sqrt{q}\,b^2 r^{-1} x \frac{w}{\sqrt{q}}\\ &\quad - \frac{2}{\underline{c}^4}\sqrt{a^2 + c^2 q b^2 r^{-1}}\,\frac{w^2}{q} - \frac{4\sqrt{q}}{\underline{c}^4}\frac{\partial[c(x)x]}{\partial x}\frac{w}{\sqrt{q}}\,y^{\mathrm{d}}\\ &\leqslant -\sqrt{q}\left[\frac{1}{2}\frac{\sqrt{a^2 + c^2 q b^2 r^{-1}}}{\sqrt{q}}x^2 + \frac{1}{2\underline{c}^4}\frac{\sqrt{a^2 + c^2 q b^2 r^{-1}}}{\sqrt{q}}\frac{w^2}{q}\right.\\ &\quad \left. - \frac{8}{\underline{c}^4}\frac{\sqrt{q}}{\sqrt{a^2 + c^2 q b^2 r^{-1}}}\left(\frac{\partial[c(x)x]}{\partial x}\right)^2 [y^{\mathrm{d}}]^2\right]\end{aligned}$$

which is negative definite outside a ball around the origin of the plane $\{x, w/\sqrt{q}\}$. In addition, the radius of the ball does not increase as q increases. Therefore, by Theorem 2.14 in Reference [9], the state variables x and $w/\sqrt{q}$ are uniformly bounded (with respect to both t and q) and uniformly continuous. It follows from (41) that

$$y + \sqrt{\frac{b^2(x)}{r(x)}}\frac{w}{\sqrt{q}} = -\frac{1}{\sqrt{q}}\sqrt{\frac{r(x)}{b^2(x)}}\left[\dot{x} + \frac{a^2 x}{\sqrt{a^2 + c^2 q b^2 r^{-1}} + \sqrt{c^2 q b^2 r^{-1}}}\right]$$

By uniform boundedness of x and $w/\sqrt{q}$, we know that, for any $[t_1, t_2] \subset [t_0, t_f]$,

$$\lim_{q\to\infty}\int_{t_1}^{t_2}\left[y + \sqrt{\frac{b^2(x)}{r(x)}}\frac{w}{\sqrt{q}}\right]\mathrm{d}t = 0$$

Similarly, we can show using Equation (44) that, if $c(x)$ is a constant,

$$\lim_{q\to\infty}\int_{t_1}^{t_2}\left[\sqrt{\frac{b^2(x)}{r(x)}}\frac{w}{\sqrt{q}} + y^{\mathrm{d}}\right]\mathrm{d}t = 0$$

Subtracting the above two equations yields

$$\lim_{q\to\infty} \int_{t_1}^{t_2} [y - y^{\mathrm{d}}]\,\mathrm{d}t = 0$$

Since both y and y^{d} are uniformly continuous, the above equation implies that $|y - y^{\mathrm{d}}|$ can be made arbitrarily small by increasing q. □

Conditions in Lemma 4 can be satisfied through choices of $q(x)$ and $r(x)$. In some cases (as cascaded design in the next section), $c(x)$ can also be selected. It is worth noting that inequality (43) implies that the function

$$[a^2(x) + c^2(x)qb^2(x)/r(x)]x$$

is non-decreasing with respect to x.

3.4. Forward recursive control design using SDARE tracker

The development of SDARE tracker makes it possible to design recursively a new control for cascaded systems while optimizing a performance index defined in terms of the original state variables and the original control. The recursive design will be based on a forward recursion rather than backstepping. To illustrate the basic idea, reconsider the second-order system in (20) with $y = c_1x_1$. The design begins with the second subsystem

$$\dot{x}_2 = a_2(x_2)x_2 + b_2(x_2)u$$

and its output equation can be chosen to be $y_2 = x_2$. No matter how y_2^{d} is chosen for y_2 to track, the SDARE tracker can be applied to the above system and to optimize performance index

$$I_2 = \frac{1}{2}s_2[x_2(t_f) - y_2^{\mathrm{d}}(t_f)]^2 + \frac{1}{2}\int_{t_0}^{t_f} [q_2(x_2 - y_2^{\mathrm{d}})^2 + r_2(x_2)u^2]\,\mathrm{d}t$$

By the discussions in the previous section, such an SDARE tracker is given by

$$u = -b_2(x_2)r_2^{-1}(x_2)\,p_2(x_2)x_2 - b_2(x_2)r_2^{-1}(x_2)w_2 \tag{45}$$

where

$$p_2(x_2) = r_2(x_2)b_2^{-2}(x_2)\left[a_2(x_2) + \sqrt{a_2^2(x_2) + q_2b_2^2(x_2)r_2^{-1}(x_2)}\right]$$

$$\dot{w}_2 = -\frac{\partial\left[\sqrt{a_2^2(x_2) + q_2b_2^2(x_2)r_2^{-1}(x_2)}x_2\right]}{\partial x_2}w_2 - q_2y_2^{\mathrm{d}} \tag{46}$$

and q_2 and $r_2(x_2)$ are chosen such that the ratio $b_2^2(x_2)/r_2(x_2)$ is a constant and that the quantity $[a_2^2(x_2) + q_2b_2^2(x_2)/r_2(x_2)]x_2$ is a non-decreasing function of x_2.

Obviously, the choice of $y_2^{\mathrm{d}}(t)$ determines the trajectory of state variable x_2. By the virtue of recursive design, it is natural to choose y_2^{d} to be the fictitious control for the following fictitious system that corresponds to the first subsystem, i.e.

$$\dot{x}_1 = a_1(x_1)x_1 + b_1(x_1)y_2^{\mathrm{d}}$$

It follows that, for any uniformly continuous function $y_1^{\mathrm{d}}(t)$ and for any $c_1 > 0$, fictitious control y_2^{d} that optimizes performance index

$$I_1 = \frac{1}{2}s_1[c_1x_1(t_f) - y_1^{\mathrm{d}}(t_f)]^2 + \frac{1}{2}\int_{t_0}^{t_f} [q_1(c_1x_1 - y_1^{\mathrm{d}})^2 + r_1(x_1)(y_2^{\mathrm{d}})^2]\,\mathrm{d}t$$

is

$$y_2^{\mathrm{d}} = -b_1(x_1)r_1^{-1}(x_1)p_1(x_1)x_1 - b_1(x_1)r_1^{-1}(x_1)w_1 \tag{47}$$

where q_1 and $r_1(x)$ are chosen such that $b_1^2(x_1)/r_1(x_1)$ is a constant and that $[a_1^2(x_1)+c_1^2q_1b_1^2(x_1)/r_1(x_1)]x_1$ is a non-decreasing function of x_1, with

$$p_1(x_1) = r_1(x_1)b_1^{-2}(x_1)\left[a_1(x_1) + \sqrt{a_1^2(x_1) + c_1^2q_1b_1^2(x_1)r_1^{-1}(x_1)}\right]$$

and

$$\dot{w}_1 = -\frac{\partial\left[\sqrt{a_1^2(x_1) + c_1^2q_1b_1^2(x_1)r_1^{-1}(x_1)}x_1\right]}{\partial x_1}w_1 - c_1q_1y_1^{\mathrm{d}} \tag{48}$$

The two controls, y_2^{d} and u, are two successive algebraic equations from which u is defined, their designs do not involve any operation of differentiation, and the overall control u can easily be calculated for real-time implementation by integrating differential equations of $\dot{w}_i$ and by algebraic substitution of y_2^{d}. The performance index intended to be optimized by u is

$$I = I_1 + I_2 = 0.5\int_{t_0}^{\infty} [q_1(c_1x_1 - y_1^{\mathrm{d}})^2 + r_1(y_2^{\mathrm{d}})^2 + q_2(x_2 - y_2^{\mathrm{d}})^2 + r_2u^2]\,\mathrm{d}t \tag{49}$$

and the following result on stability and performance can be concluded.

Theorem

For second-order systems of form (20), the SDARE tracker defined by (48), (47), (46) and (45) is suboptimal with respect to performance index (49). Furthermore, there exist (sufficiently large) values of q_1 and q_2 such that the closed-loop system is semi-globally stable in the sense of uniform boundedness and that tracking errors $|x_i - y_i^{\mathrm{d}}|$ are made (sufficiently) small.

Proof

The SDARE tracker is designed based on the SDARE method and on a forward recursion. According to statement (ii) of Lemma 4, the choice of u optimizes I_2. On the other hand, dynamics of the first subsystem can be rewritten as

$$\dot{x}_1 = a_1(x_1)x_1 + b_1(x_1)y_2^{\mathrm{d}} + b_1(x_1)[x_2 - y_2^{\mathrm{d}}]$$

The choice of y_2^{d} optimizes I_1 under the condition that the term $b_1(x_1)[x_2 - y_2^{\mathrm{d}}]$ is ignored, thus the overall control is merely suboptimal with respect to I.

Note that y_2^{d} is locally uniformly continuous, therefore by statement (iii) of Lemma 4, there exists q_2 such that x_2 is locally uniformly bounded and that $|x_2 - y_2^{\mathrm{d}}|$ is small. On the other hand, according to statements (i) and (iii) of Lemma 4, the first subsystem is input-to-state stable with respect to the bias term $b_1(x_1)[x_2 - y_2^{\mathrm{d}}]$ and, without the bias term, y_2^{d} is globally stabilizing and a large value of q_1 makes $c_1 x_1$ track y_1^{d}. Hence, semi-global stability and tracking performance can be concluded. □

As in the case of Lemma 2, extension of the theorem to higher-order cascaded systems is obvious. Unlike the optimal control obtained via the backstepping design and stated in Lemma 2, the forward recursive design always yields a suboptimal control as the coupling term such as $b_i(x_i)[x_{i+1} - y_{i+1}^{\mathrm{d}}]$ is considered not in optimization but only in the analysis of stability and tracking accuracy.

In case that $y_1^{\mathrm{d}} = 0$ and $t_f = \infty$, choosing $w(t_0) = 0$ implies $w_1(t) = 0$, the ratio y_2^{d}/x_1 is always well defined, and the overall performance index (49) can then be rewritten as

$$I = 0.5 \int_{t_0}^{\infty} \left\{ \mathbf{x}^{\mathrm{T}} \begin{bmatrix} c_1^2 q_1 + (r_1 + q_2)\left(\dfrac{y_2^{\mathrm{d}}}{x_1}\right)^2 & -q_2 \dfrac{y_2^{\mathrm{d}}}{x_1} \\ -q_2 \dfrac{y_2^{\mathrm{d}}}{x_1} & q_2 \end{bmatrix} \mathbf{x} + r_2 u^2 \right\} \mathrm{d}t$$

which is in the standard form and in terms of the original state and control variables. As a result, one can prescribe the performance index and see how it affects stability and tracking performance.

4. ILLUSTRATIVE EXAMPLE

Consider the second-order system

$$\dot{x}_1 = x_1 - x_1^3 + x_2, \quad \dot{x}_2 = x_2^4 + x_2 + u$$

Its non-linear parameterization is

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x})\mathbf{x} + \mathbf{B}u$$

where

$$\mathbf{A}(\mathbf{x}) = \begin{bmatrix} 1 - x_1^2 & 1 \\ 0 & x_2^3 + 1 \end{bmatrix} \triangleq \begin{bmatrix} a_1(x_1) & b_1 \\ 0 & a_2(x_2) \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \triangleq \begin{bmatrix} 0 \\ b_2 \end{bmatrix}$$

It is obvious that the pair $\{\mathbf{A}, \mathbf{B}\}$ is controllable.

To optimize performance index

$$J = \frac{1}{2}\mathbf{x}^{\mathrm{T}}(t_f)\mathbf{S}\mathbf{x}(t_f) + \frac{1}{2}\int_0^{t_f} [\mathbf{x}^{\mathrm{T}}\mathbf{Q}(\mathbf{x})\mathbf{x} + uR(\mathbf{x})u]\,\mathrm{d}\tau$$

the optimal control is

$$u = -R^{-1}\mathbf{B}^{\mathrm{T}}\lambda$$

where

$$\dot{\lambda} = -\frac{\partial H}{\partial \mathbf{x}}, \quad H = \frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} + \frac{1}{2}uRu + \lambda^{\mathrm{T}}[\mathbf{f}(\mathbf{x}) + \mathbf{B}(\mathbf{x})u]$$

and boundary conditions are $\mathbf{x}(0)$ and $\lambda(t_f) = \mathbf{S}\mathbf{x}(t_f)$. When t_f is chosen to be sufficiently large, the optimal control with $\mathbf{S} = \mathbf{P}$ reduces to the SDARE control given by

$$u = -R^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}\mathbf{x}$$

where $\mathbf{P}\mathbf{A} + \mathbf{A}^{\mathrm{T}}\mathbf{P} - \mathbf{P}\mathbf{B}R^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P} + \mathbf{Q} = \mathbf{0}$.

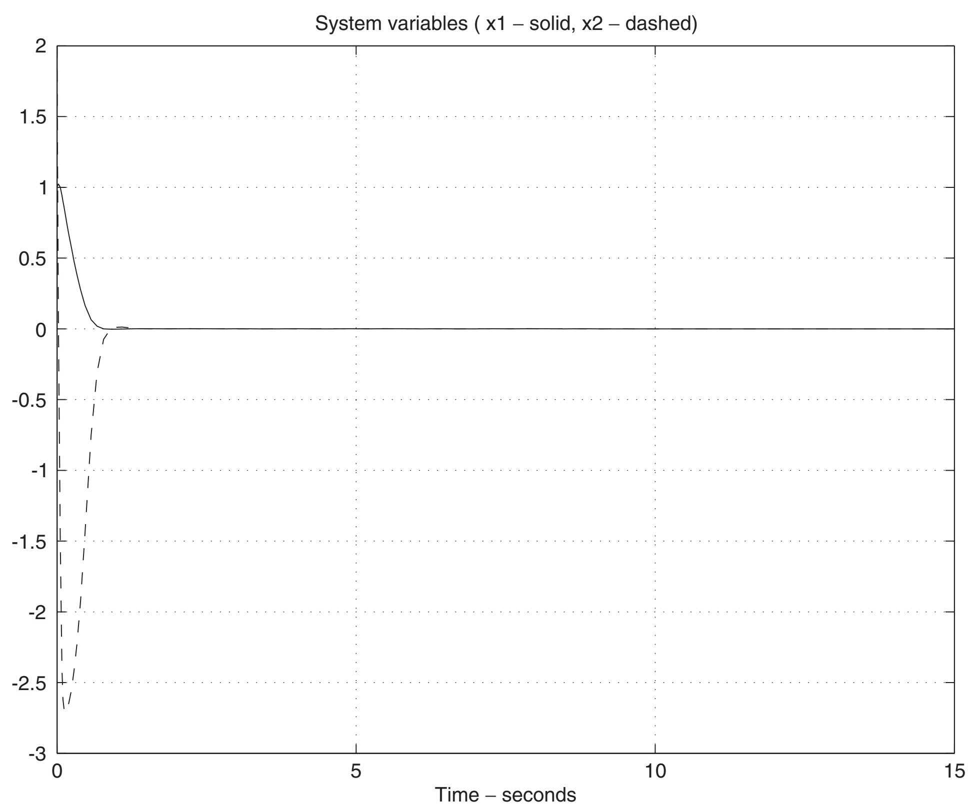

Figure 1. System response under the optimal control.

On the other hand, an SDARE tracker can be designed as proposed for subsystem 2 and then for subsystem 1. It follows that SDARE tracking control for subsystem 2 is

$$u = -\frac{1}{r_2}(p_2 x_2 + w_2)\dot{w}_2 = -\left[\sqrt{a_2^2(x_2) + q_2/r_2} + \frac{a_2(x_2)x_2}{\sqrt{a_2^2(x_2) + q_2/r_2}}\frac{\partial a_2(x_2)}{\partial x_2}\right] w_2 - q_2 y_2^{\mathrm{d}}$$

and that SDARE tracking control for subsystem 1 is

$$y_2^{\mathrm{d}} = -\frac{1}{r_1}(p_1 x_1 + w_1)\dot{w}_1 = -\left[\sqrt{a_1^2(x_1) + q_1/r_1} + \frac{a_1(x_1)x_1}{\sqrt{a_1^2(x_1) + q_1/r_1}}\frac{\partial a_1(x_1)}{\partial x_1}\right] w_1 - q_1 y_1^{\mathrm{d}}$$

where

$$p_2(x_2) = r_2\left[a_2(x_2) + \sqrt{a_2^2(x_2) + q_2/r_2}\right] \quad \text{and} \quad p_1(x_1) = r_1\left[a_1(x_1) + \sqrt{a_1^2(x_1) + q_1/r_1}\right]$$

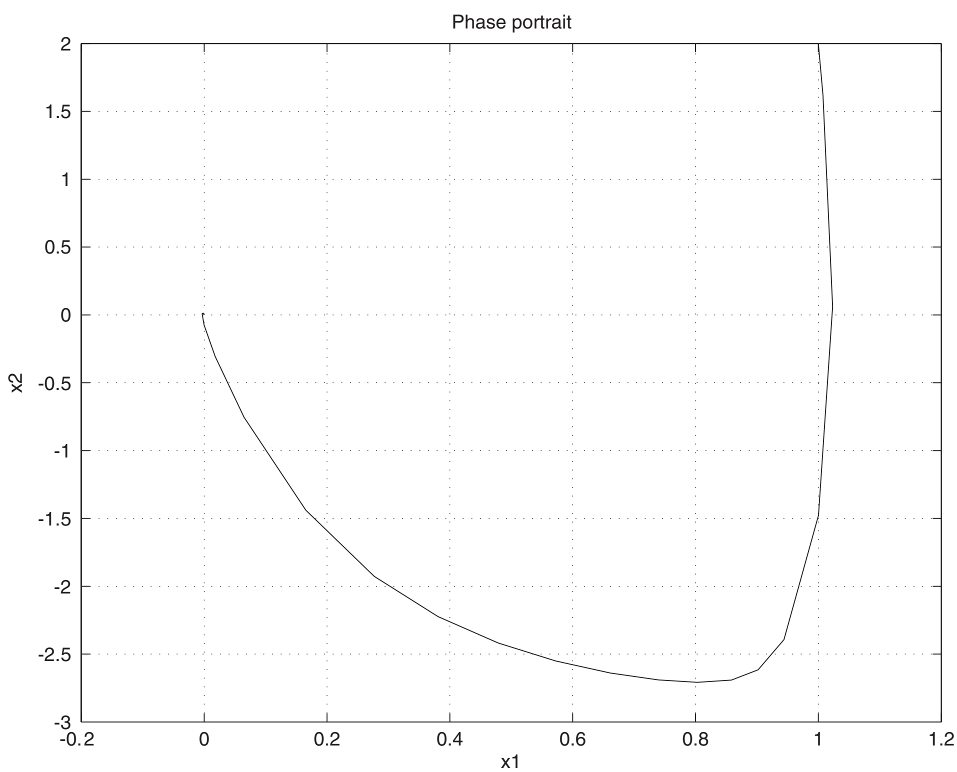

Figure 2. Phase portrait of the closed-loop, optimal system.

If $y_1^{\mathrm{d}} = 0$, one can choose $w_1(t) = 0$. Therefore, the overall performance index is

$$I = \frac{1}{2}\int_{t_0}^{\infty} [\mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} + uRu]\,\mathrm{d}t$$

where $R = r_2$, and

$$\mathbf{Q} = \begin{bmatrix} q_1 + (r_1+q_2)\left[a_1(x_1) + \sqrt{a_1^2(x_1) + q_1/r_1}\right]^2 & -q_2\left[a_1(x_1) + \sqrt{a_1^2(x_1) + q_1/r_1}\right] \\ -q_2\left[a_1(x_1) + \sqrt{a_1^2(x_1) + q_1/r_1}\right] & q_2 \end{bmatrix}$$

The optimal control and the newly proposed SDARE-tracker-based suboptimal control are simulated with the following choices:

$$t_f = 15, \quad r_1 = r_2 = 5.0, \quad q_1 = 200, \quad q_2 = 500, \quad \text{and } \mathbf{x}(0) = [1 \;\; 2]^{\mathrm{T}}$$

Results of the simulation are shown in Figures 1–4. Figures 1 and 2 shows that the second-order system is very well stabilized. Figures 3 and 4 illustrate how much performance is lost due to

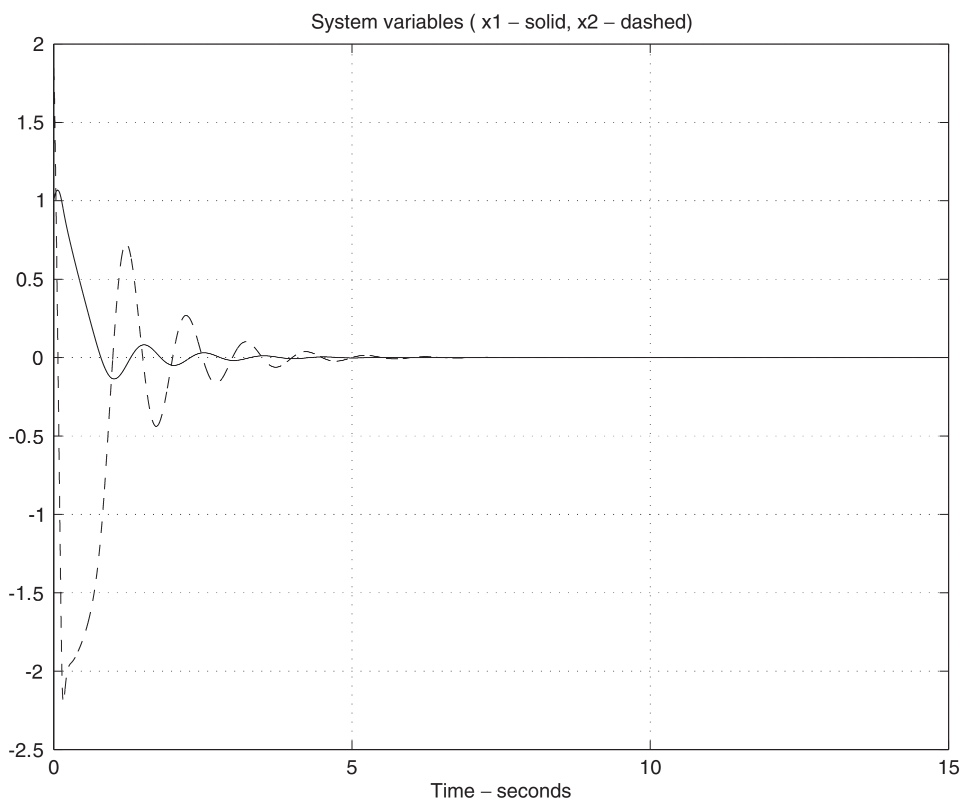

Figure 3. System response under the proposed suboptimal control.

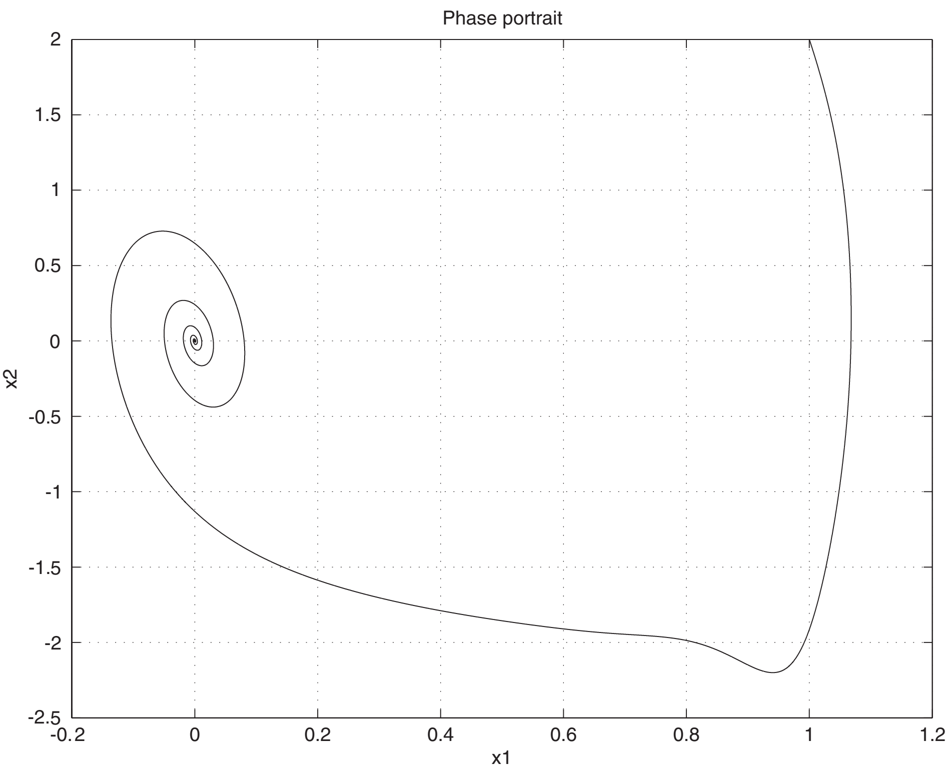

Figure 4. Phase portrait of the closed-loop, suboptimal system.

sub-optimality. In particular, the sub-optimal system takes longer to settle, and its damping is not sufficient to avoid oscillation. This conclusion can be observed by comparing either time responses in Figures 2 and 4 or phase portraits in Figures 1 and 3.

5. CONCLUSION

A non-linear (sub)optimal tracker is developed using the SDARE method. This SDARE tracker is then used as the seed controller to generate fictitious controls and the actual control in a new forward recursive design procedure for cascaded non-linear systems. State transformation and differentiation of fictitious controls are no longer needed; analysis and control design are done in terms of the original state and control variables; each fictitious control is designed to be optimal with respect to the dynamics of the associated subsystem; and the controls generated for each subsystem form a set of successive algebraic equations that are readily implementable as they are.

Semi-global stability and tracking performance are established for the overall, closed-loop system. The current analytical proof calls for large values of some of the control gains. Further research is needed to explicitly consider the coupling terms in the suboptimal design and hence to eliminate the need of using any large gain.

REFERENCES

1. Isidori A. *Nonlinear Control Systems* (3rd edn). Springer: Berlin, New York, 1995.
2. Basar T, Bernhard P. H^{∞}*-Optimal Control and Passivity Techniques in Nonlinear Control* (2nd edn). Birkhauser: London, 1995.
3. Khalil, H. *Nonlinear Systems* (3rd edn). Prentice-Hall: Upper Saddle River, 1996.
4. van der Schaft AJ. L_2*-Gain and Passivity Techniques in Nonlinear Control.* Springer: London, 1996.
5. Kanellakppoulos I, Kokotovic PV, Morse AS. Systematic design of adaptive controllers for feedback linearizable systems. *IEEE Transactions on Automatic Control* 1991; **36**:1241–1253.
6. Sepulchre R, Jankovic M, Kokotovic PV. *Constructive Nonlinear Control*. Springer: New York, 1997.
7. Krstic M, Kanellakppoulos I, Kokotovic PV. *Nonlinear and Adaptive Control Design*. Wiley: New York, 1995.
8. Freeman RA, Kokotovic PV. *Robust Nonlinear Control Design: State Space and Lyapunov Techniques*. Birkhauser: Boston, 1996.
9. Qu Z. *Robust Control of Nonlinear Uncertain Systems*. Wiley-Interscience: New York, 1998.
10. Athens M, Falb PL. *Optimal Control*. McGraw Hill: New York, 1966.
11. Bryson AE, Ho Y-C. *Applied Optimal Control* (2nd edn). Hemisphere Publishing Corporation: New York, 1975.
12. Byrnes CI. New methods for non-linear optimal control. *Proceedings of European Control Conference*, Grenoble, France, 1991.
13. Cloutier JR. State-dependent Riccati equation techniques: an overview. *1997 American Control Conference*, Albuquerque, MN, 1997; 932–936.
14. Cloutier JR, D'Souza CN, Mracek CP. Nonlinear regulation and nonlinear H_∞ control via the state-dependent Riccati equation technique: Part 1, Theory; Part 2, Examples'. *Proceedings of the First International Conference on Nonlinear Problems in Aviation and Aerospace*, 1996; 117–142. Available through University Press, Embry-Riddle Aeronautical University, Daytona Beach, FL, 32114, ISBN: 1-884099-06-8.
15. Cloutier JR *et al.* State-dependent Riccati equation techniques: theory and applications. *Workshop notes at 1998 American Control Conference*, 1998.
16. Jacobson DH. *Extensions of Linear Quadratic Control, Optimization and Matrix Theory*. Academic Press: New York, 1977.
17. Qu Z, Cloutier J, Mracek C. A new sub-optimal non-linear control design technique: state dependent algebraic Riccati equation method. *IFAC Congress*, **E**, San Francisco, CA, 1996; 365–370.
18. Qian C, Lin W. A continuous feedback approach to global strong stabilization of non-linear systems. *IEEE Transactions on Automatic Control* 2001; **46**:1061–1079.
19. Ezal K, Pan Z, Kokotovic PV. Locally optimal and robust backstepping design. *Proceedings of the 36th IEEE Conference on Decision and Control*, San Diego, 1997; 1767–1773.

