VI. Conclusion

In this note, we have studied different concepts of nonlinear identifiability in the linear algebraic framework. Constructive procedures have been worked out for both geometric and algebraic identifiability of nonlinear systems. Relationships between different concepts have been completely characterized. As an application of the theory developed, we investigated the identifiability properties of a four dimensional model of HIV/AIDS. The questions answered in this study include the minimal number of measurement of the variables for a complete determination of all parameters and the best period of time to make such measurements. This information will be useful in formulating guidelines for the clinical practice.

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Robust Control of Nonlinear Systems in the Presence of Unknown Exogenous Dynamics

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***Abstract*—A robust control is designed for a class of uncertain systems, and it is distinct and novel that the proposed control does not require any information of a bounding function on nonlinear uncertainties in the system. Instead, the uncertainties to be compensated for are generated by an exogenous system whose dynamics are either completely unknown or partially unknown. The only requirements on the exogenous system are that its unknown dynamics are bounded by a known function and that its output is bounded. The proposed robust control is based on a nonlinear observer that estimates the uncertainties. It is shown that, under different sets of conditions, local, semiglobal, or global stability of uniform ultimate boundedness or asymptotic stability can be achieved.**

***Index Terms*—Bounding function, estimation, Lyapunov direct method, nonlinear uncertainty, observer, robust control.**

I. Introduction

Robustness is one of the essential concepts in control theory. Roughly speaking, a control system is robust if stability and performance can be maintained under a specific class of uncertainties which could be unknown functionals, parameter variations, unmodeled dynamics, disturbances, etc. Robust control of nonlinear uncertain systems has attracted a lot of attention. Classes of stabilizable uncertain systems have been found, and several robust control design procedures have been proposed [4]–[7], [9], [10], [12], [15], [19]–[21], [24], [26].

In most of the existing results, robust controls are designed to deal with significant but bounded uncertainties by assuming a known bounding function on the size of uncertainties. While uncertainties being bounded ensures that a stabilizing control (if found) will be of finite magnitude, determining a known bounding function of uncertainties is a nontrivial issue in many applications. Without knowledge of the bounding function, robust control must be designed to learn the size of uncertainties while compensating for them. To this end, progress has been made by combining robust and adaptive control designs. In [6], the robust control design problem is investigated under the assumption that the bounding function has a known functional expression and it is parameterized in terms of finite unknown constants. In this case, an adaptive robust control was proposed to adaptively estimate the unknown parameters in the bounding function. In [22], an extension is made so that the bounding function can be parameterized in terms of time varying parameters. Specifically,

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if the time varying parameters are outputs of a known or partially known exogenous system and if certain properties (such as having a known Lyapunov function) are met by the exogenous system, the bounding function can also be estimated using an adaptation law and a stabilizing robust control can be found. For existing results on nonlinear output feedback and observer designs, readers are referred to [1]–[3], [13], and [14].

In this note, robust control is sought so that stability and performance can be ensured under less priori information on the size of uncertainties. Specifically, the uncertainties in the system are assumed to be bounded (but their bounding function is not known or needed), and they are also the outputs of an exogenous system. Compared to [22], the proposed result in this note does not require any explicit stability property (other than boundedness) for the exogenous system. This improvement significantly reduces the knowledge needed for robust control design, and it is accomplished by using nonlinear observers (rather than adaptation laws in [22]). It is shown that, depending upon the location of uncertainties, a reduced-order or full-order nonlinear observer can be designed to estimate the uncertainties. It is also shown that local stability of uniform ultimate boundedness can be achieved under the proposed robust control. Under additional conditions, the stability result can be enhanced to be either semiglobal, or global, or asymptotic, or asymptotic and global.

II. Problem Formulation

In this note, we consider the class of uncertain systems that are of form

$$\dot{x} = F(x,t) + B(x,t)[\Delta F_m(x,\upsilon,t) + u] \tag{1}$$

where $x(t) \in \Re^n$ is the state of the system, $X(t) = \{x(\tau), 0 \le \tau \le t\}$, $\Omega(X) \subset \Re^p$ is an unknown set that may be dependent upon state trajectory and is bounded if X is uniformly bounded, $\upsilon(t) \in \Omega(X)$ denotes the vector of uncertainties (and it is also the state of the so-called exogenous subsystem to be defined shortly), $u(t) \in \Re^m$ (with $m \le n$) is the control to be designed, $F(x,t)$ and $B(x,t)$ are known parts of the system dynamics, and $\Delta F_m(x,\upsilon,t)$ denotes the matched uncertainties.

The robust control problem is to design a control $u(x,t)$ such that, under the following four assumptions, the resulting closed-loop system is stable (in the sense of either asymptotic stability or stability of uniform ultimate boundedness (which also called practical stability) [5], [21]) for all possible values of uncertain vector $\upsilon(t)$ in the *unknown* set $\Omega(X)$.

Assumption 1: All functions in (1) are Caratheodory, locally Lipschitzian with respect to x and υ, uniformly bounded with respect to t, and locally uniformly bounded with respect to x or υ. Furthermore, matrix $B(x,t)$ has the properties that $B(x,t) = [\, B_1^T(x,t) \quad B_2^T(x,t)\,]^T$ where $B_2^{-1}(x,t) \in \Re^{m\times m}$ exists, and first-order partial derivatives of $B_1(x,t)$, $B_2(x,t)$ and $B_2^{-1}(x,t)$ are well defined everywhere and locally uniformly bounded. Specifically, for all (x,υ,t) and for all i

$$\begin{aligned} \underline{c}_b \le& \|B_2(x,t)\| \quad \|B(x,t)\| \le c_b(\|x\|) \\ \left\|\frac{\partial B(x,t)}{\partial x_i}\right\| \le& c_b'(\|x\|) \quad \left\|\frac{\partial B(x,t)}{\partial t}\right\| \le c_b''(\|x\|) \end{aligned} \tag{2}$$

where $c_b(\cdot)$ and $c_b'(\cdot)$ are nonnegative and nondecreasing functions, and $\underline{c}_b > 0$ is a constant.

Assumption 2: The origin $x = 0$ is globally asymptotically stable for nominal system of (1), $\dot{x} = F(x,t)$. Therefore, by the Lyapunov converse theorem [12], there exist a C^1 function $V(x,t)\colon \Re^n \times \Re \to \Re^+$ such that

$$\begin{aligned} \gamma_1(\|x\|) \le& V(x,t) \le \gamma_2(\|x\|) \\ \frac{\partial V(x,t)}{\partial t} + \bigtriangledown_x^T V(x,t) F(x,t) \le& -\gamma_3(\|x\|) \\ \left\|\bigtriangledown_x^T V(x,t)\right\| \le& \gamma_4(\|x\|) \end{aligned} \tag{3}$$

where $\gamma_i\colon \Re^+ \to \Re^+$ are class $\mathcal{K}_\infty$ functions and, for some constants $\beta_1 > 0$ and $0 < \beta_2 < 1$,

$$\gamma_4(\|x\|) \le \beta_1 \gamma_3^{\beta_2}(\|x\|). \tag{4}$$

The first two are typical (and in line with the standard ones in [12]). Existence of $0 < \beta_2 < 1$ in (4) is equivalent to stability of boundedness under a constant-bounded disturbance for the uncontrolled nominal system $\dot{x} = F(x,t)$. Assumption 2 can be relaxed such that, if $\dot{x} = F(x,t) + B(x,t)u$ is unstable when $u = 0$, there is a known stabilizing control. The next two assumptions are regarding the uncertainties, including those on an exogenous system.

Assumption 3: Set $\Omega(X)$ is bounded if X is bounded, and the uncertainties are generated by an exogenous system as follows:

$$\begin{aligned} \Delta F_m(x,\upsilon,t) =& W(x,t)\upsilon(t) \\ \dot{\upsilon} =& G(\upsilon,x,t) + \Delta G(\upsilon,x,t) \end{aligned} \tag{5}$$

where $W(x,t)$ is a known functional matrix bounded by a nonnegative, nondecreasing function $c_w(\cdot)$ as, for all (x,t)

$$\|W(x,t)\| \le c_w(\|x\|). \tag{6}$$

$G(\upsilon,x,t) + \Delta G(\upsilon,t)$ represents dynamics of the exogenous subsystem, $G(\upsilon,x,t)$ has a known functional form, and $\Delta G(\upsilon,x,t)$ is completely unknown except that

$$\|\Delta G(x,\upsilon,t)\| \le \rho_g(x,\upsilon,t) \tag{7}$$

and $\rho_g(\cdot)$ is known, $\rho_g(\cdot)$ and $G(\upsilon,x,t)$ are Caratheodory and locally Lipschitzian functions that are uniformly bounded with respect to t and locally uniformly bounded with respect to x and υ.

Note that $\Delta G(\upsilon,x,t)$ in the exogenous system is unknown except for its bounding function and there is no restriction on the magnitude of the bounding function and that all of the exogenous dynamics could be unknown (i.e., $G(\upsilon,x,t) = 0$). In essence, it is only required that exogenous dynamics be bounded-input–bounded-state if x were viewed as the input.

The fourth assumption is introduced in order to ensure nonlinear observability and to expose the main idea of the note without undue complexity. It is easy to see that $p < m$ can be treated by simply augmenting exogenous system (5) using $\dot{\upsilon}_j = 0$ $(j = p+1,\ldots,m)$. In case that $p > m$, estimation of υ would impose certain observability conditions on matrix $W(x,t)$ and on dynamics $\Delta G(\upsilon,x,t)$. The observability condition can be readily developed by noting that, instead of imposing a bounding function on uncertainty $\Delta F_m(\cdot)$, a bounding function can be introduced on $\Delta G(\cdot)$ and that a new reduced-order vector $\upsilon'(t) = W(x,t)\upsilon(t)$ can be defined and its dynamics can be derived. Hence, the process of combining these two facts should be used to overcome the restriction on the dimension of the exogenous system in the sense that bounding function on $\Delta G(\cdot)$ is not needed if $\Delta G(\cdot)$ is parameterized by outputs of another exogenous subsystem and/or that only a reduced-order exogenous system needs to be estimated in robust control design.

Assumption 4: Exogenous system (5) is of dimension p where $p = m$, and matrix $W(x,t)$ and its inverse are differentiable and well defined everywhere, that is, for a nondecreasing function $c'_w(\cdot)$ and a constant $\underline{c}_w > 0$

$$\underline{c}_w \leq \|W(x,t)\|, \text{ and } \left\|\frac{\partial W(x,t)}{\partial x_i}\right\| \leq c'_w(\|x\|). \tag{8}$$

III. Nonlinear Observer-Based Robust Controls

The proposed robust controls are capable of compensating for uncertainties generated by an unknown exogenous system because they are based on a robust observer estimating uncertainties. While not necessary due to the absence of backstepping or high-order observation, we choose to introduce function $\mathrm{CDS}[\cdot]$ as, for any given pair of constants μ, $\delta > 0$ and for any vector argument $y \in \Re^m$, $\mathrm{CDS}[s,\mu,1+\delta]: \Re^m \to \Re^m$ and its ith element is defined by the equation shown at the bottom of the page. CDS stands for continuously differentiable saturation, the function has a continuous derivative bounded by μ, and it makes the resulting robust controls continuously differentiable. It becomes the standard saturation function $\mathrm{SAT}[\mu y, 1]$ in the limit of $\delta \to 0$, and tends to the standard sign function in the limit of both $\delta \to 0$ and $\mu \to \infty$. For the proposed observer design, design constants are chosen such that $0 < \mu \ll 1$ and $\delta > 0$, and the guaranteed stability region will depend on μ. Note that, in both observer designs, state x is available for feedback but it will be estimated by $\hat{x}$ as an "output" because the output estimation error $(x - \hat{x})$ is required as the feedback by the observers in order to estimate uncertainty υ.

A. Reduced Order Nonlinear Observer and Robust Control

Let x_i, $f_i(\cdot)$ and $b_i(\cdot)$ denote the ith rows of x, $F(\cdot)$ and $B(\cdot)$, respectively. Also, let

$$\begin{aligned} x &= [\phi^T,\ z^T]^T \quad \phi \triangleq [x_1 x_2 \ldots x_{n-m}]^T \\ z &\triangleq [x_{n-m+1}, \ldots x_n]^T \\ F(x,t) &= \left[F_1^T(x,t) F_2^T(x,t)\right]^T \end{aligned}$$

where z and $F_2(\cdot)$ be the bottom mth order vector blocks in x and $f(x,t)$, respectively.

If $B_1(x,t) = 0$, state variables in ϕ are independent of uncertainties and need not be estimated. In this case, let $\hat{z}$ be the estimate of z and set $\hat{x} = [x_1\ x_2 \cdots x_{n-m}\ \hat{z}^T]^T$ be the estimate of x. Then, the following reduced-order observer is proposed to estimate the uncertainties generated by unknown exogenous system (5): with $\hat{z}(t_0) = z(t_0)$ and $\hat{\upsilon}(t_0) = 0$

$$\begin{aligned} \dot{\hat{z}} =& F_2(\hat{x},t) + B_2(x,t)[u + W(x,t)\hat{\upsilon}] \\ &+ \frac{l_1}{\mu^2}\mathrm{CDS}[\tilde{z},\mu,1+\delta] \end{aligned} \tag{9}$$

$$\dot{\hat{\upsilon}} = G(\hat{\upsilon},x,t) + \frac{l_2}{\mu^2}\mathrm{CDS}[y,\mu,1+\delta] \tag{10}$$

where $\hat{\upsilon}$ is the estimate of uncertainty vector υ, $\tilde{z} = z - \hat{z}$ and $\tilde{\upsilon} = \upsilon - \hat{\upsilon}$ are estimation errors, $\tilde{z} = z - \hat{z}$ is also the "output error," $l_i > 0$ are scaling factors of reduced-order observer gains; y is the output of the auxiliary system given by

$$\begin{aligned} \mu\dot{\eta} =& -\eta + \frac{l_1}{\mu^2}S^{-1}(x,t)\mathrm{CDS}[\tilde{z},\mu,1+\delta] \\ &- \frac{1}{\mu}S^{-1}(x,t)\tilde{z} - \frac{\partial S^{-1}(x,t)}{\partial t}\tilde{z} \\ &- \sum_{i=1}^{n}\frac{\partial S^{-1}(x,t)}{\partial x_i}\tilde{z}\left\{f_i(x,t) + b_i(x,t)[u + W(x,t)\hat{\upsilon}]\right\} \\ &+ S^{-1}(x,t)[F_2(\hat{x},t) - F_2(x,t)] \\ y =& \frac{1}{\mu}S^{-1}(x,t)\tilde{z} + \eta \end{aligned} \tag{11}$$

with $\eta(t_0) = 0$ [and, hence, $y(t_0) = 0$ for (15)] and $S(x,t) = B_2(x,t)W(x,t)$.

Now, consider the following observer-based robust control:

$$u = -W(x,t)\hat{\upsilon} - \frac{l_1}{\mu^2}B_2^{-1}(x,t)\mathrm{CDS}[z - \hat{z},\mu,1+\delta] \tag{12}$$

where $\hat{z}$ and $\hat{\upsilon}$ are defined by (9) and (10). Then, under robust control (12), equation (9) becomes $\dot{\hat{z}} = F_2(\hat{x},t)$, and the estimation error system becomes

$$\begin{aligned} \dot{\tilde{z}} =& [F_2(x,t) - F_2(\hat{x},t)] + B_2(x,t)W(x,t)\tilde{\upsilon} \\ &- \frac{l_1}{\mu^2}\mathrm{CDS}[\tilde{z},\mu,1+\delta] \end{aligned} \tag{13}$$

$$\begin{aligned} \dot{\tilde{\upsilon}} =& [G(\upsilon,x,t) - G(\hat{\upsilon},x,t)] + \Delta G(\upsilon,x,t) \\ &- \frac{l_2}{\mu^2}\mathrm{CDS}[y,\mu,1+\delta]. \end{aligned} \tag{14}$$

On the other hand, it follows from auxiliary system (11) that, using differential operator $s = d/dt$

$$\begin{aligned} y =& \frac{1}{\mu s + 1}S^{-1}(x,t)s\tilde{z} \\ &+ \frac{1}{\mu s + 1}\frac{l_1}{\mu^2}S^{-1}(x,t)\mathrm{CDS}[\tilde{z},\mu,1+\delta] \\ &+ \frac{1}{\mu s + 1}S^{-1}(x,t)\left[F_2(\hat{x},t) - F_2(x,t\right] \\ &+ \frac{1}{\mu s + 1}\sum_{i=1}^{n}\frac{\partial S^{-1}(x,t)}{\partial x_i}\tilde{z}b_i(x,t)W(x,t)\tilde{\upsilon}. \end{aligned} \tag{15}$$

Substituting error dynamics (14) into (15) yields

$$\mu\dot{y} = -y + \tilde{\upsilon} + \sum_{i=1}^{n}\frac{\partial S^{-1}(x,t)}{\partial x_i}\tilde{z}b_i(x,t)W(x,t)\tilde{\upsilon} \tag{16}$$

which shows that the auxiliary output y in (11) is a filtered version of $\tilde{z}$ through a combination of low-pass and high-pass filters, saturation, and nonlinear weighting. This property is instrumental in establishing the following theorem on stability of an observer-based robust control and its associated closed loop system. The proof of the theorem is included as Appendix A.

Theorem 1: Consider (1) satisfying assumptions 1, 2, 3, and 4. If $B_1(x,t) = 0$, the following stability properties are ensured by robust control (12) with any fixed value of $l_2 > 0$.

- For any initial conditions of $x(t_0)$, $\hat{z}(t_0)$, $y(t_0)$, $\upsilon(t_0)$, and $\hat{\upsilon}(t_0)$, the corresponding state variables will be uniformly

$$\mathrm{CDS}_i[y,\mu,1+\delta] = \begin{cases} \mu y_i, & \text{if } |\mu y_i| \leq 1 \\ 1 + \delta\left[1 - e^{-(1/\delta)(\mu y_i - 1)}\right], & \text{if } \mu y_i > 1 \\ -1 - \delta\left[1 - e^{-(1/\delta)(-\mu y_i - 1)}\right], & \text{if } \mu y_i < -1. \end{cases}$$

bounded in a hyper-ball (whose radius is a class-$\mathcal{K}$ function of their initial conditions) for all sufficiently large values of l_1 and $1/\mu$.
- Given any positive constant ϵ^* as the ultimate bound, state variables x, $\tilde{z}$, y, and $\tilde{v}$ will be uniformly ultimately bounded with respect to ϵ^* for all sufficiently large values of l_1 and $1/\mu$.
- If $\beta_2 \geq 0.5$ and if $\Delta G(x, v, t) = 0$, state variables $x, \tilde{z}, y, \tilde{v}$, and $\hat{v}$ will be asymptotically stable for all sufficiently large values of l_1 and $1/\mu$.
- Stability of uniform boundedness and asymptotic stability will be global if $F(x,t) = A(x,t)x$ and $G(v,x,t) = H(x,t)v$ for some uniformly bounded matrices $A(x,t)$ and $H(x,t)$, if $B_2(x,t)W(x,t) = D(t)$ for a matrix $D(t)$, if $0 < \beta_2 \leq 0.5$, and if $\rho_g(x,v,t)$ is also uniformly bounded by a constant.

B. Full Order Nonlinear Observer and Robust Control

For the general case that $B_1(x,t) \neq 0$, it is necessary to account for the impact of uncertainties on ϕ (the top partition of x) in stability analysis and control design. To this end, a full-order observer is to be designed to generate the estimate of x, i.e., $\hat{x} = [\hat{\phi}^T \, \hat{z}^T]^T$. The proposed full-order observer is described by (9)–(11), and

$$\begin{aligned}\dot{\hat{\phi}} =& F_1(x,t) + B_1(x,t)\left[u + W(x,t)\hat{v}\right] \\ &+ \frac{l_1}{\mu^2} B_1(x,t) B_2^{-1}(x,t) \mathrm{CDS}[\tilde{z}, \mu, 1+\delta] \\ &+ \frac{l_0}{\mu^2} \mathrm{CDS}[\tilde{\phi}, \mu, 1+\delta] \end{aligned} \tag{17}$$

where $\tilde{\phi} = \phi - \hat{\phi}$ is the estimation error, and $l_0 > 0$ is an observer gain. Then, the corresponding observer-based robust control and its stability properties are provided by the following theorem. The proof of the theorem is included in the Appendix.

Theorem 2: Consider (1) satisfying assumptions 1, 2, 3, and 4. Then, if $V(x,t)$ is a C^2 function and if $\gamma_4^2(\|x\|)/\gamma_1(\|x\|)$ is locally Lipschitzian in any closed and bounded set, the stability properties in theorem 1 can be restated for the observer-based robust control (12) together with the full order observer (17), (9) and (10) under the following additional conditions.

- Local stability of both uniform and ultimate boundedness or local asymptotic stability can be ensured by letting $l_1 \gg 1$ and $l_0 \gg l_1$ and by choosing μ to be sufficiently small.
- Local stability results can be made semiglobal or global stability can be achieved if $\gamma_4^2(\|x\|)/\gamma_1(\|x\|)$ and $B_1(x,t)B_2^{-1}(x,t)$ are uniformly bounded by some constants and by setting l_0/l_1 be a fixed large number and by choosing l_1 and $1/\mu$ sufficiently large.

IV. Simulation Example

The proposed robust control is applied to control a simple pendulum. As shown in [5], pendulum dynamics are described by differential equations

$$\dot{x}_1 = x_2 \quad \dot{x}_2 = -\frac{g}{l}\sin x_1 - \frac{v}{l}\cos x_1 + U$$

where l is the length, $g = 9.8$, v is the uncertainty, and U is the control. It is different from [22] that uncertainty v is generated here by an unknown exogenous system and it must be estimated by a nonlinear observer.

Consider the control $U = -k_1 x_1 - k_2 x_2 + u$, where u is robust control (12) in Theorem 1. Assumption 2 holds, where the Lyapunov function is

$$\begin{aligned} V(x) =& x^T P x + \frac{2g}{l}(1 - \cos x_1) \text{ with} \\ P =& \begin{bmatrix} k_1 + 0.5k_2^2 & 0.5k_2 \\ 0.5k_2 & 1 \end{bmatrix}. \end{aligned}$$

The corresponding class-$\mathcal{K}_\infty$ functions are

$$\begin{aligned} \gamma_1(\|x\|) =& \lambda_{\min}(P)\|x\|^2 \\ \gamma_2(\|x\|) =& \lambda_{\max}(P)\|x\|^2 \\ &+ \begin{cases} \frac{2g(1-\cos\|x\|)}{l} & \|x\| \leq \pi \\ \frac{4g}{l} & \|x\| > \pi \end{cases} \\ k_1' =& k_1 - \frac{g}{l} \sup_{x_1 \in \Re} \frac{\sin x_1}{x_1} \\ \gamma_3(\|x\|) =& \min\{k_2, k_1' k_2\}\|x\|^2 \\ \gamma_4(\|x\|) =& 2\left[\lambda_{\max}(P) + \frac{g}{l}\right]\|x\| \\ \gamma_5(\|\Psi\|) =& \min\left\{\lambda_{\min}(P), \frac{1}{2}, \frac{1}{l_2}, \frac{l_2}{2}\right\}\|\Psi\|^2 \\ \gamma_6(\|\Psi\|) =& \max\left\{\lambda_{\max}(P), \frac{1}{2}, \frac{l_2+1}{l_2}, \frac{l_2+2}{2}\right\}\|\Psi\|^2 \end{aligned}$$

where $\lambda_{\min}(P)$, $\lambda_{\max}(P) = 0.5\left[1 + k_1 + 0.5k_2^2 \pm \sqrt{(1 + k_1 + 0.5k_2^2)^2 - 4(k_1 + 0.25k_2^2)}\right]$, and $k_1' > 0$.

Estimation of the uncertainty is done for the worst case that, in (5), $G(v,x,t) = 0$. It follows from assumption 3 that $\Delta F_m(x,v,t) = W(x,t)v(t)$ and $W(x,t) = -\cos x_1/l$. Once the regions of attraction and ultimate boundedness are given, parameters ξ_i can be computed according to (20)–(22), gain l_1 and design constant μ can be determined using (37) and (38). In the simulation, initial conditions are set to be $(x_1, x_2, v) = (0.3, 0.5, 0.12)$ and $(\hat{x}_1, \hat{x}_2, \hat{v}, y) = (0.3, 0.5, 0, 0)$, and the following choices are made/calculated:

$$\begin{aligned} k_1 =& 5 \quad k_2 = 1 \quad l = \sqrt{g} \quad \Delta G(v,x,t) \\ =& -\|x\|v + 0.2\|x_1\|^2 + 0.3\cos(2t) \\ \underline{c}_b =& 1.0 \quad c_b = 1.02 \quad c_b' = c_b'' = 0.02 \\ \beta_1 \geq& 2\frac{\left(\lambda_{\max}(P) + \frac{g}{l}\right)}{\sqrt{\min\{k_2, k_1' k_2\}}} \quad \beta_2 = 0.5 \\ c_\phi =& c_{\hat{\phi}} = 1.5, c_{\tilde{\phi}} = 2c_\phi \quad c_w = c_w' = \frac{1}{l} \\ \underline{c}_\omega =& \cos\frac{(1.5|x_1(t_0)|)}{l} \quad \rho_g(x,v,t) \\ =& \|x\| \cdot \|v\| + 0.5\|x_1\|^2 + 0.5 \\ c_z =& c_{\hat{z}} = 2.0 \quad c_{\tilde{z}} = 2c_z \\ c_x =& c_{\hat{x}} = \left(c_\phi^2 + c_z^2\right)^{1/2} \\ c_{\tilde{x}} =& 2c_x \quad c_v = c_{\hat{v}} = 1.0 \quad c_{\tilde{v}} = 2c_v \quad c_y = 1.0 \\ \xi_0 =& \sqrt{2}\max\{1, k_1 + k_2\}c_x + \frac{g}{l} \quad \xi_1 = 1 \\ \xi_2 =& \sqrt{2}\max\left\{k_1 + \frac{g}{l}, k_2\right\} \quad \xi_3 = 0.01 \quad \xi_4 = 1.13 \\ \delta =& 0.3 \quad \epsilon^* = 0.6 \quad \lambda_1 = 1 \quad \lambda_2 = 2 \\ \lambda_3 =& 10 \quad l_2 = 1 \quad \mu = 0.002\,14, \text{ and } l_1 = 78.6. \end{aligned}$$

Simulation results are shown in Figs. 1 and 2, and they demonstrate the effectiveness of the proposed control.

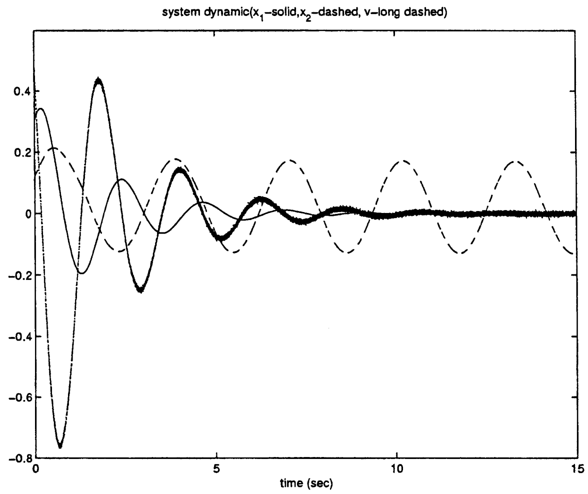

Fig. 1. System variables: x_1 (solid), x_2 (dotted), and v (dashed).

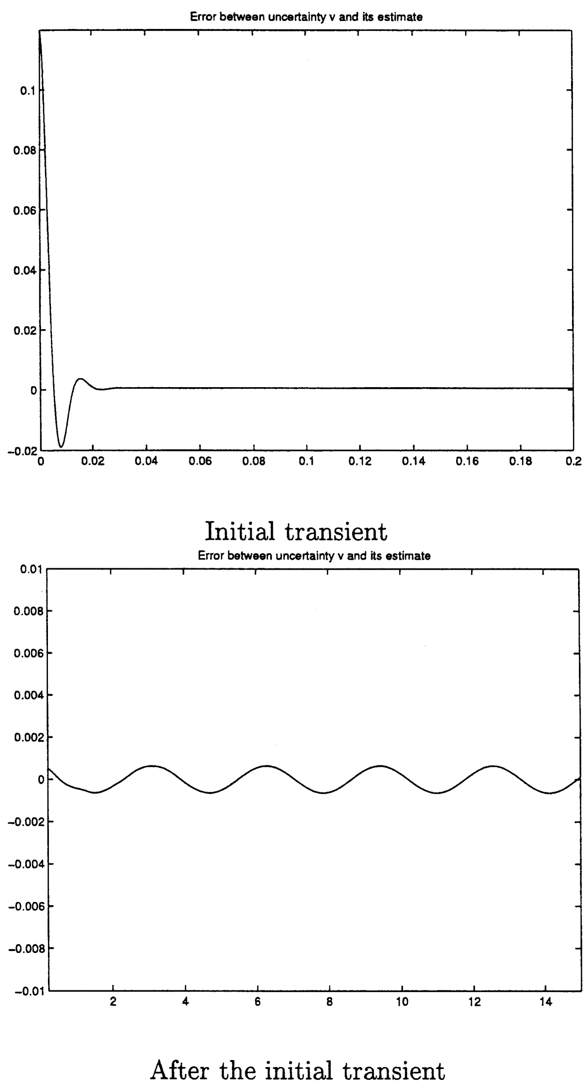

Fig. 2. Estimation error of uncertainty v.

V. Conclusion

In this note, a robust control is designed for a class of uncertain systems. In the systems, uncertainties are *not* assumed to be bounded by a known function of the state. Instead, they are generated by an exogenous system and remain bounded if the state of the system under control is bounded. The exogenous system itself could be completely uncertain as long as its unknown dynamics are bounded by a known function. It is shown that, without any additional information about the exogenous system, a stabilizing robust control can be designed by incorporating either a full- or reduced-order nonlinear observer. Conditions are found to guarantee either local practical stability, or local asymptotic stability, or their counterparts of being global/semiglobal.

Appendix A

A. Proof of Theorem 1

To establish local asymptotic stability or local uniform ultimate boundedness, let us consider closed and bounded sets defined by

$$\|x\| \le c_x \quad \|y\| \le c_y \quad \|\hat{z}\| \le c_{\hat{z}} \quad \|\hat{v}\| \le c_{\hat{v}} \tag{18}$$

where c_x, c_y, $c_{\hat{z}}$, and $c_{\hat{v}}$ are arbitrary but positive constants. It follows from Assumption 4 and from the definitions of $\hat{x}$ and z that inequalities

$$\|v\| \le c_v \quad \|z\| \le c_z, \text{ and } \|\hat{x}\| \le c_{\hat{x}} \tag{19}$$

hold for some constants $c_v, c_z, c_{\hat{x}} > 0$. Stability analysis will be done in three steps, and it is to show that, if initial conditions are within the closed and bounded sets in (18) and (19), the state variables will remain in these sets. If so, semiglobal stability is shown since the sets in (18) are arbitrary.

The first step of stability analysis is to use the local lipschitzian property stated in Assumptions 1 and 3. It follows that, in closed and bounded subsets (18) and (19), the following inequalities hold for some nonnegative constants $\xi_i(\cdot)$: for $i = 1, 2$:

$$\begin{aligned} \|F(x,t)\| &\le \xi_0(c_x) \\ \|F(x,t)\| &\le \xi_0'(c_x)\gamma_3^{1-\beta_2}(\|x\|) \\ \|F_i(x,t) - F_i(\hat{x},t)\| &\le \xi_i(c_x, c_{\hat{x}})\|x - \hat{x}\| \qquad (20) \\ \|G(v,x,t) - G(\hat{v},x,t)\| &\le \xi_3(c_x, c_v, c_{\hat{v}})\|\tilde{v}\| \qquad (21) \end{aligned}$$

and

$$\rho_g(x, v, t) \le \xi_4(c_x, c_v). \tag{22}$$

As the second step, we adopt the following Lyapunov function to study stability analysis of the closed-loop system consisting of (9), (13), (14) and (16)[1]: $L(x, \hat{z}, v, \hat{v}, y) = L_1(\hat{x}, t) + L_2(\tilde{z}) + L_3(\tilde{v}, y)$, where $L_1(\hat{x}, t) = V(\hat{x}, t)$ is given by (3), $L_2 = 1/2\|\tilde{z}\|^2$, and $L_3 = 1/2\|\tilde{v} - y\|^2 + 1/l_2\|\tilde{v}\|^2 + l_2/2\|y\|^2$. It is apparent that the Lyapunov function is globally positive definite and radially unbounded with respect to its arguments as $\gamma_5(\|\Psi\|) \le L(x, \hat{z}, v, \hat{v}, y) \le \gamma_6(\|\Psi\|)$, where $\Psi = [\hat{x}^T\, \tilde{z}^T\, \tilde{v}^T\, y^T]^T$, and $\gamma_5, \gamma_6\colon \Re^+ \to \Re^+$ are class $\mathcal{K}_\infty$ functions (that can be defined in terms of $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$).

The time derivative of Lyapunov function can easily be evaluated using (9), (13), (14) and (16). For example, it follows from robust control (12), from reduced-order observer (9), from $B_1(x, t) = 0$, and from inequalities (3) and (4) that, under robust control (12)

$$\begin{aligned} \dot{L}_1 &= \frac{\partial V(\hat{x},t)}{\partial t} + \nabla_{\hat{x}}^T V(\hat{x},t)\dot{\hat{x}} \\ &= \frac{\partial V(\hat{x},t)}{\partial t} + \nabla_{\hat{x}}^T V(\hat{x},t)\left[F_1^T(x,t)\, F_2^T(\hat{x},t)\right]^T \\ &\le -\gamma_3(\|\hat{x}\|) + \beta_1\gamma_3^{\beta_2}(\|\hat{x}\|) \\ &\quad \times \|F_1(x,t) - F_1(\hat{x},t)\|. \end{aligned} \tag{23}$$

[1]Note that stability of (16) and (11) are equivalent, that stability of both (9) and (13) ensures stability of (1), and that, under Assumption 3, stability of (14) implies boundedness of (10).

As the third step, the positive but otherwise arbitrary constants c_x, c_y, $c_{\hat{z}}$, and $c_{\hat{v}}$ in (18) and (19) can always be increased such that, at the initial time instant t_0, the following inequalities hold:

$$\begin{aligned} \|x(t_0)\| &\le \gamma_6^{-1}\circ\gamma_5(c_x) \quad \|y(t_0)\| \le \gamma_6^{-1}\circ\gamma_5(c_y) \\ \|\hat{z}(t_0)\| &\le \gamma_6^{-1}\circ\gamma_5(c_{\hat{z}}) \quad \|\hat{v}(t_0)\| \le \gamma_6^{-1}\circ\gamma_5(c_{\hat{v}}) \end{aligned} \tag{24}$$

and

$$\begin{aligned} \|v(t_0)\| &\le \gamma_6^{-1}\circ\gamma_5(c_v) \quad \|z(t_0)\| \le \gamma_6^{-1}\circ\gamma_5(c_z) \\ \text{and } \|\hat{x}(t_0)\| &\le \gamma_6^{-1}\circ\gamma_5(c_{\hat{x}}) \end{aligned} \tag{25}$$

where ${}^{-1}$ and $\circ$ denote the inverse function and the composition of functions, respectively. It follows from (20) and (23) and from Holder's inequality that, for initial conditions satisfying (24) and (25) and as long as the state stays in the region defined by (18) and (19)

$$\begin{aligned} \dot{L}_1 \le & -\gamma_3(\|\hat{x}\|) + \beta_1\gamma_3^{\beta_2}(\|\hat{x}\|)\xi_1(c_x, c_{\hat{x}})\|\tilde{z}\| \\ \le & -\frac{1}{2}\gamma_3(\|\hat{x}\|) + 2^{\beta_2/(1-\beta_2)}(1-\beta_2) \\ & \times \beta_1^{1/(1-\beta_2)}\beta_2^{\beta_2/(1-\beta_2)}\xi_1^{1/(1-\beta_2)} \\ & \times (c_x, c_{\hat{x}})\|\tilde{z}\|^{1/(1-\beta_2)}. \end{aligned} \tag{26}$$

Similarly, it follows from (13), (2), and (6) that, for initial conditions satisfying (24) and (25) and as long as the state remains in the region defined by (18) and (19)

$$\begin{aligned} \dot{L}_2 \le & -\frac{l_1}{\mu^2}\tilde{z}^T\mathrm{CDS}[\tilde{z},\mu,1+\delta] + \xi_2\|\tilde{z}\|^2 \\ & + c_b(c_x)c_w(c_x)\|\tilde{z}\|\|\tilde{v}\| \\ \le & -\frac{l_1-\lambda_2}{\mu}\|\tilde{z}\|^2 - \frac{1}{\mu}\|\tilde{z}\|^2\left(\lambda_2 - \mu\xi_2 - \mu\lambda_1 c_b^2 c_w^2\right) \\ & + \frac{1}{4\lambda_1}\|\tilde{v}\|^2 \end{aligned} \tag{27}$$

$$\le -\frac{l_1-\lambda_2}{\mu}\|\tilde{z}\|^2 + \frac{1}{4\lambda_1}\|\tilde{v}\|^2 \tag{28}$$

provided that μ is chosen such that

$$0 < \mu < \min\left\{\frac{1}{c_z + c_{\hat{z}}}, \frac{\lambda_2}{\xi_2 + \lambda_1 c_b^2 c_w^2}\right\} \triangleq \underline{\mu}_1. \tag{29}$$

In (28), $\lambda_1, \lambda_2 > 0$ are design parameters that could be chosen arbitrarily, and $\dot{L}_2$ has a negative definite term with respect to $\tilde{z}$ (i.e., the first term) as long as $l_1 > \lambda_2$. Combining (26) and (28) and making $l_1 > \lambda_2 + \lambda_3$ (for an arbitrarily chosen design parameter $\lambda_3 > 0$) yield

$$\begin{aligned} \dot{L}_1 + \dot{L}_2 \le & -\frac{1}{2}\gamma_3(\|\hat{x}\|) - \frac{l_1-\lambda_2-\lambda_3}{\mu}\|\tilde{z}\|^2 \\ & + \frac{1}{4\lambda_1}\|\tilde{v}\|^2 - \frac{1}{\mu}\|\tilde{z}\| \\ & \times\Big[\lambda_3\|\tilde{z}\| - \mu 2^{\beta_2/(1-\beta_2)}(1-\beta_2)\beta_1^{1/(1-\beta_2)} \\ & \quad \times \beta_2^{\beta_2/(1-\beta_2)}\xi_1^{1/(1-\beta_2)}\|\tilde{z}\|^{\beta_2/(1-\beta_2)}\Big] \end{aligned} \tag{30}$$

in which the first two terms are negative definite with respect to both $\hat{x}$ and $\tilde{z}$. The last term in the aforementioned inequality also assumes a nonpositive value in the region

$$0 \le \gamma_6^{-1}\circ\gamma_5(\epsilon_{\tilde{z}}) \le \|\tilde{z}\| \le c_x + c_{\hat{x}} \tag{31}$$

for the equation shown at the bottom of the page, if μ is chosen such that (32), shown at the bottom of the next page, holds.

Analogously, if $\mu < 1/c_y$, it follows from (14), (16), (2), (8), (7), (22), and (21) and from Cauchy–Schwarz inequality that, given any initial conditions satisfying (24) and (25) and for all trajectories of the state in the hyper-ball defined by (18) and (19),

$$\begin{aligned} \dot{L}_3 \le & -\frac{l_2}{\mu^2}\left(\tilde{v} - y + \frac{2}{l_2}\tilde{v}\right)^T \mathrm{CDS}[y,\mu,1+\delta] \\ & + \frac{1}{\mu}(y - \tilde{v} + l_2 y)^T(-y + \tilde{v}) \\ & + \left\|\tilde{v} - y + \frac{2}{l_2}\tilde{v}\right\|[\rho_g(x,v,t) + \xi_3\|\tilde{v}\|] \\ & + \frac{1}{\mu}\|y - \tilde{v} + l_2 y\|\sqrt{n}\,\underline{c}_b^{-1}\underline{c}_w^{-1} \\ & \times\left[\underline{c}_w^{-1}c_w'(\|x\|) + \underline{c}_b^{-1}c_b'(\|x\|)\right] \\ & \times\|\tilde{z}\|c_b(\|x\|)c_w(\|x\|)\|\tilde{v}\| \\ \le & -\frac{1}{\mu}(\|\tilde{v}\|^2 + \|y\|^2) + \left[\left(1 + \frac{2}{l_2}\right)\|\tilde{v}\| + \|y\|\right] \\ & \times(\xi_3\|\tilde{v}\| + \xi_4) + \frac{1}{\mu}[\|\tilde{v}\| + (1+l_2)\|y\|]\sqrt{n}\,\underline{c}_b^{-1}\underline{c}_w^{-1} \\ & \times\left[\underline{c}_w^{-1}c_w'(c_x) + \underline{c}_b^{-1}c_b'(c_x)\right] \\ & \times(c_v + c_{\hat{v}})c_b(c_x)c_w(c_x)\|\tilde{z}\| \\ \le & -\frac{1}{4\mu}(\|\tilde{v}\|^2 + \|y\|^2) - \frac{1}{4\lambda_1}\|\tilde{v}\|^2 \\ & + \frac{(2 + 2l_2 + l_2^2)n[\underline{c}_b c_w'(c_x) + \underline{c}_w c_b'(c_x)]^2}{\mu\underline{c}_b^4\underline{c}_w^4} \\ & \times(c_v + c_{\hat{v}})^2 c_b^2(c_x)c_w^2(c_x)\|\tilde{z}\|^2 \\ & - \frac{1}{\mu}\|\tilde{v}\|\Bigg[\frac{\lambda_1 l_2 - \mu l_2 - 4\mu\lambda_1(l_2+2)\xi_3 - 2\mu\lambda_1 l_2\xi_3}{4\lambda_1 l_2} \\ & \qquad \times\|\tilde{v}\| - \frac{\mu(l_2+2)}{l_2}\xi_4\Bigg] \\ & - \frac{1}{\mu}\|y\|\left[\frac{1-\mu\xi_3}{2}\|y\| - \mu\xi_4\right]. \end{aligned} \tag{33}$$

If μ is chosen to satisfy (34), as shown at the bottom of the next page, the last two terms in (33) are negative semidefinite in the hyper-annulus

$$\begin{aligned} \gamma_6^{-1}\circ\gamma_5(\epsilon_{\tilde{v}}) &\le \|\tilde{v}\| \le c_v + c_{\hat{v}}, \text{ and} \\ \gamma_6^{-1}\circ\gamma_5(\epsilon_y) &\le \|y\| \le c_y \end{aligned} \tag{35}$$

$$\epsilon_{\tilde{z}} = \begin{cases} 0, & \text{if } \le \beta_2 < 1 \\ \text{any positive number smaller than } \min\left\{\gamma_6^{-1}\circ\gamma_5(\epsilon^*)/\sqrt{3}, \gamma_6^{-1}\circ\gamma_5(c_x + c_{\hat{x}})\right\}, & \text{if } 0 \le \beta_2 < 0.5 \end{cases}$$

$$\begin{aligned} 0 < \mu &< \underline{\mu}_2 \\ &\triangleq \begin{cases} \frac{\lambda_3}{1-\beta_2}2^{-\beta_2/(1-\beta_2)}\beta_1^{-1/(1-\beta_2)}\beta_2^{-\beta_2/(1-\beta_2)}\xi_1^{-1/(1-\beta_2)}(c_x + c_{\hat{x}})^{-(2\beta_2-1)/(1-\beta_2)} & \text{if } 0.5 \le \beta_2 < 1 \\ \frac{\lambda_3}{1-\beta_2}2^{-\beta_2/(1-\beta_2)}\beta_1^{-1/(1-\beta_2)}\beta_2^{-\beta_2/(1-\beta_2)}\xi_1^{-1/(1-\beta_2)}\left[\gamma_6^{-1}\circ\gamma_5(\epsilon_{\tilde{z}})\right]^{(1-2\beta_2)/(1-\beta_2)} & \text{if } 0 \le \beta_2 < 0.5. \end{cases} \end{aligned} \tag{32}$$

where $\epsilon_{\tilde{v}}$ and ϵ_y are positive constants satisfying $\epsilon_{\tilde{v}} < \min\{\gamma_6^{-1} \circ \gamma_5(\epsilon^*)/\sqrt{3}, \gamma_6^{-1} \circ \gamma_5(c_v + c_{\hat{v}})\}$ and $\epsilon_y < \min\{\gamma_6^{-1} \circ \gamma_5(\epsilon^*)/\sqrt{3}, \gamma_6^{-1} \circ \gamma_5(c_y)\}$.

Note that the upper bounds for the regions in (31) and (35) are consistent with those of the hyper-balls in (18) and (19). Consequently, it follows from (30) and (33) that, given any initial conditions satisfying (24) and (25) and for all trajectories of the state in the hyper-ball/annulus defined by (18), (19), (31), and (35)

$$\begin{aligned}\dot{L}_1 + \dot{L}_2 + \dot{L}_3 \le &-\frac{1}{2}\gamma_3(\|\hat{x}\|) - \frac{l_1 - \lambda_2 - \lambda_3}{\mu}\|\tilde{z}\|^2 \\ &- \frac{1}{4\mu}(\|\tilde{v}\|^2 + \|y\|^2) \\ &+ \frac{(2 + 2l_2 + l_2^2)n[\underline{c}_b c_w'(c_x) + \underline{c}_w c_b'(c_x)]^2}{\mu \underline{c}_b^4 \underline{c}_w^4} \\ &\times (c_v + c_{\hat{v}})^2 c_b^2(c_x) c_w^2(c_x)\|\tilde{z}\|^2 \end{aligned} \tag{36}$$

which is negative definite provided that l_1 is chosen according to the inequality

$$l_1 \ge \frac{1}{4} + \lambda_2 + \lambda_3 + \frac{(2 + 2l_2 + l_2^2)n[\underline{c}_b c_w'(c_x) + \underline{c}_w c_b'(c_x)]^2}{\underline{c}_b^4 \underline{c}_w^4} \times (c_v + c_{\hat{v}})^2 c_b^2(c_x) c_w^2(c_x) \tag{37}$$

and that μ is chosen according to

$$\mu < \min\{\underline{\mu}_1, \underline{\mu}_2, \underline{\mu}_3\} \tag{38}$$

where $\underline{\mu}_i$ are defined by (29), (32), and (34). It follows from [21, Th. 2.15, p. 65] that, given any initial conditions satisfying (24) and (25), all state variables (including x, y, $\tilde{z}$, v, and $\tilde{v}$) will stay in the hyper-balls defined by (18), (19), (31), and (35), and that state variables $\hat{x}$, $\tilde{z}$, and y will eventually converge into a hyper-ball of radius less or equal to ϵ^*. That is, the first two statements are shown.

If $\beta_2 \ge 0.5$, $\epsilon_{\tilde{z}} = 0$ can be selected. On the other hand, $\epsilon_{\tilde{v}} = \epsilon_y = 0$ can be set only when $\xi_4 = 0$ as shown in the last equation at the bottom of the page. It is easy to see that asymptotic stability can be concluded provided that $\epsilon_{\tilde{z}} = \epsilon_{\tilde{v}} = \epsilon_y = 0$.

If $F(x,t) = A(x,t)x$ and $G(v,x,t) = H(x,t)v$ for uniformly bounded matrices $A(x,t)$ and $H(x,t)$ and and if $\rho_g(x,v,t)$ is also uniformly bounded, ξ_i in (20) up to (22) are constants independent of the sets defined in (18) and (19). If $B_2(x,t)W(x,t) = D(t)$ for a matrix $D(t)$, the last term in (16) and (36) disappears. These two results together with $0 < \beta_2 \le 0.5$ make it possible to choose l_1 and μ globally in the state–space of x, y, $\tilde{z}$, v, and $\tilde{v}$. Thus, global stability can be claimed. □

Remark: Given any conservative estimate of initial conditions, the set of semi global stability can be calculated using (24) and (25). Hence, (38) and (37) together with (20) up to (22) provide the criteria for selecting control gains and design parameters. ⋄

APPENDIX B

A. Outline of the Proof of Theorem 2

Note that two parts of the full-order observer are given by (9) and (10) as before and that the same robust control (12) is applied. Consequently, the proof here can be done by mimicking that of Theorem 1, and only the differences are provided here.

The first difference is that, under control (12), the error dynamics for estimating ϕ are

$$\begin{aligned}\dot{\tilde{\phi}} = &B_1(x,t)W(x,t)\tilde{v} - \frac{l_1}{\mu^2}B_1(x,t)B_2^{-1}(x,t) \\ &\times \mathrm{CDS}[\tilde{z}, \mu, 1+\delta] - \frac{l_0}{\mu^2}\mathrm{CDS}[\tilde{\phi}, \mu, 1+\delta].\end{aligned} \tag{39}$$

Second, choose Lyapunov function to be $L'(x, \hat{\phi}, \hat{z}, v, \hat{v}, y, t) = L_0(\tilde{\phi}, \hat{x}, t) + L_2(\tilde{z}) + L_3(\tilde{v}, y)$, where $V(\hat{x}, t)$, L_2 and L_3 are the same as before, $k_0 > 0$ is a design parameter, $h(\hat{x}, t) = \left[\nabla_{\hat{x}_1} V(\hat{x}, t) \cdots \nabla_{\hat{x}_{n-m}} V(\hat{x}, t)\right]^T$, and $L_0(\tilde{\phi}, \hat{x}, t) = 1/k_0[V(\hat{x}, t) + \tilde{\phi}^T h(\hat{x}, t) - h^T(\hat{x}, t)B_1(x,t)B_2^{-1}(x,t)\tilde{z}] + 1/2\|\tilde{\phi}\|^2$.

Third, together with sets (40), (18), and (19), also consider the set

$$\|\phi\| \le c_\phi, \text{ and } \|\hat{\phi}\| \le c_{\hat{\phi}} \tag{40}$$

where c_ϕ and $c_{\hat{\phi}}$ are arbitrary but positive constants. Within these sets, inequalities (20) up to (22) can be re-established and, since $V(\cdot)$ is a C^2 function, there are functions c_{h1} and c_{h2} such that $\|\partial h/\partial \hat{x}\| \le c_{h1}(c_{\hat{x}})$ and $\|\partial h/\partial t\| \le c_{h2}(c_{\hat{x}})\gamma_4(\|\hat{x}\|)$. It is apparent from Assumption 3 that, by choosing

$$k_0 > \max_{0 \le a \le c_{\hat{x}}} \frac{\gamma_4^2(a)}{\gamma_1(a)} \max\left\{1, \frac{c_b^2(c_x)}{\underline{c}_b^2}\right\}.$$

Lyapunov function $L(\cdot)$ is semiglobally positive definite and radially unbounded with respect to its arguments as $\gamma_7(\|\Phi\|) \le L'(x, \hat{\phi}, \hat{z}, v, \hat{v}, y) \le \gamma_8(\|\Phi\|)$, where $\Phi = [\hat{x}^T \tilde{\phi}^T \tilde{z}^T \tilde{v}^T y^T]^T$, and $\gamma_7, \gamma_8 : \Re^+ \to \Re^+$ are class $\mathcal{K}_\infty$ functions.

Fourth, stability analysis is done for initial conditions satisfying

$$\|\Phi(t_0)\| < \gamma_8^{-1} \circ \gamma_7(\min\{c_x, c_\phi + c_{\hat{\phi}}, c_z + c_{\hat{z}}, c_v + c_{\hat{v}}, c_y\}). \tag{41}$$

It is obvious that, if $\gamma_4^2(\|x\|)/\gamma_1(\|x\|)$ and $B_1(x,t)B_2^{-1}(x,t)$ are uniformly bounded by some constants, Lyapunov function L' becomes globally positive definite. Furthermore, by choosing a fixed k_0 (which is possible if $\gamma_4^2(\|x\|)/\gamma_1(\|x\|)$ and $B_1(x,t)B_2^{-1}(x,t)$ are uniformly bounded by some constants and if l_0/l_1 in (43) is bounded), inequality (41) can be satisfied for any given initial condition by increasing the size of closed and bounded sets under consideration. Otherwise, condition (41) may not be satisfied semiglobally.

$$0 < \mu < \min\left\{\frac{1}{c_y}, \frac{\gamma_6^{-1} \circ \gamma_5(\epsilon_y)}{\xi_3\gamma_6^{-1} \circ \gamma_5(\epsilon_y) + 2\xi_4}, \frac{l_2\lambda_1\gamma_6^{-1} \circ \gamma_5(\epsilon_{\tilde{v}})}{[l_2 + 4(2+l_2)\lambda_1\xi_3 + 2\lambda_1 l_2 \xi_2]\gamma_6^{-1} \circ \gamma_5(\epsilon_{\tilde{v}}) + 4\lambda_1(2+l_2)\xi_4}\right\} \triangleq \underline{\mu}_3 \tag{34}$$

$$\lim_{\epsilon_{\tilde{v}}, \epsilon_y \to 0} \underline{\mu}_3 = \begin{cases} 0, & \text{if } \xi_4 > 0 \\ \min\left\{\frac{1}{c_y}, \frac{1}{2\xi_3}, \frac{l_2\lambda_1}{l_2 + 4(2+l_2)\lambda_1\xi_3 + 2\lambda_1 l_2\xi_2}\right\} > 0, & \text{if } \xi_4 = 0 \end{cases}.$$

$$\underline{\mu}_1' \triangleq \frac{\frac{1}{4}l_0-(\alpha_3')^2 l_1}{\frac{1}{k_0}(1-\beta_2)\alpha_1^{1/(1-\beta_2)}(8\beta_2)^{\beta_2/(1-\beta_2)}\alpha_6^{(2\beta_2-1)/(1-\beta_2)}+\frac{\alpha_3}{2k_0}+\frac{c_b c_w}{2}+\frac{c_{h1}\xi_1}{k_0}+\frac{1}{k_0}\beta_2(\alpha_1')^{(1/\beta_2)}[8(1-\beta_2)]^{(1-\beta_2)/(\beta_2)}(\alpha_6')^{(1-2\beta_2)/\beta_2}}$$

$$\underline{\mu}_2' \triangleq \frac{\frac{1}{4}l_1-\alpha_5}{\frac{c_b c_w}{2}+\frac{\alpha_3}{2k_0}+\frac{1}{k_0}(1-\beta_2)\alpha_2^{1/(1-\beta_2)}(8\beta_2)^{\beta_2)/(1-\beta_2)}\alpha_7^{(2\beta_2-1)/(1-\beta_2)}+\frac{1}{k_0}\beta_2(\alpha_2')^{(1/\beta_2)}[8(1-\beta_2)]^{(1-\beta_2)/\beta_2}(\alpha_7')^{(1-2\beta_2)/\beta_2}+\frac{\alpha_4}{k_0}+\xi_2+\lambda_1 c_b^2 c_w^2}$$

$$\underline{\mu}_3' \triangleq \min\left\{\frac{\gamma_8^{-1}\circ\gamma_7(\epsilon^*)}{2\xi_3\gamma_8^{-1}\circ\gamma_7(\epsilon^*)+8\xi_4}, \frac{l_2\lambda_1\gamma_8^{-1}\circ\gamma_7(\epsilon^*)}{[l_2+4(2+l_2)\lambda_1\xi_3+2\lambda_1 l_2\xi_2]\gamma_8^{-1}\circ\gamma_7(\epsilon^*)+16\lambda_1(2+l_2)\xi_4}\right\}.$$

$$\begin{cases} \alpha_6 = c_\phi + c_{\tilde{\phi}},\ \alpha_6' = \alpha_7' = \frac{1}{4}\gamma_8^{-1}\circ\gamma_7(\epsilon^*), \alpha_7 = c_z + c_{\tilde{z}}, & \text{if } 0.5 \le \beta_2 < 1 \\ \alpha_6 = \alpha_7 = \frac{1}{4}\gamma_8^{-1}\circ\gamma_7(\epsilon^*),\ \alpha_6' = c_\phi + c_{\tilde{\phi}}, \alpha_7' = c_z + c_{\tilde{z}}, & \text{if } 0 \le \beta_2 < 0.5. \end{cases}$$

The fifth and final difference is the time derivative of Lyapunov function. It follows that, choosing

$$\mu < \frac{1}{\max\{c_\phi + c_{\hat{\phi}}, c_z + c_{\hat{z}}, c_y\}} \triangleq \underline{\mu}_0 \tag{42}$$

expressions of $\dot{L}_2$ and $\dot{L}_3$ are the same as those given by (27) and (33), respectively. It is straightforward to obtain the expression of $\dot{L}'$ by deriving the expression of $\dot{L}_0$ under robust control (12) [together with (9) and (17)] and under (42) and then combining the outcome with those of $\dot{L}_2$ and $\dot{L}_3$. Then, given any $\epsilon^* < \gamma_8^{-1}\circ\gamma_7(\min\{c_{\hat{x}}, c_\phi + c_{\tilde{\phi}}, c_z + c_{\tilde{z}}, c_v + c_{\tilde{v}}, c_y\})$, the time derivative of Lyapunov function $L'(\cdot)$ can be evaluated in the region defined by $\|\hat{x}\| \le c_{\hat{x}}$, $1/2\gamma_8^{-1}\circ\gamma_7(\epsilon^*) \le \|\tilde{\phi}\| \le c_\phi + c_{\hat{\phi}}$, $1/2\gamma_8^{-1}\circ\gamma_7(\epsilon^*) \le \|\tilde{z}\| \le c_z + c_{\hat{z}}$, $1/2\gamma_8^{-1}\circ\gamma_7(\epsilon^*) \le \|\tilde{v}\| \le c_v + c_{\hat{v}}$, and $1/2\gamma_8^{-1}\circ\gamma_7(\epsilon^*) \le \|y\| \le c_y$. This region is consistent with those in (40), (18) and (19). One can show that, in the region, inequality

$$\dot{L}' \le -\frac{1}{2k_0}\gamma_3(\|\hat{x}\|) - \frac{1}{4\mu}(\|\tilde{v}\|^2 + \|y\|^2)$$

holds provided that k_0, l_0, l_1 and μ are chosen according to the following conditions:

$$\begin{aligned} & l_1 > 4\alpha_5 \qquad l_0 > 4(\alpha_3')^2 l_1 \\ & k_0 \ge \max\left\{2c_{h1}, 4\alpha_4', 2\alpha_3''\frac{\sqrt{l_0}}{\sqrt{l_1}}\right\}, \text{ and} \\ & \mu < \min\left\{\underline{\mu}_0, \underline{\mu}_1', \underline{\mu}_2', \underline{\mu}_3'\right\} \end{aligned} \tag{43}$$

where $\underline{\mu}_0$ is given in (42); see the first equation shown at the top of the page. α_i and α_i' are constants that are independent of k_0, l_1, l_0 and μ and are given by $\alpha_1 = \beta_1(\xi_1 + c_{h2} + \xi_2 c_b/\underline{c}_b)$, $\alpha_1' = c_{h1}\xi_0'$, $\alpha_2 = \beta_1\left\{\xi_1\underline{c}_b^2 + c_{h1}c_b\underline{c}_b + \xi_2 c_b\underline{c}_b + c_b''(\underline{c}_b + c_b) + c_b'(\underline{c}_b + c_b)[\xi_0 + \sqrt{n}c_b c_w(c_v + c_{\hat{v}})]\right\}/\underline{c}_b^2$, $\alpha_2' = c_{h1}\xi_0' c_b/\underline{c}_b$, $\alpha_3 = c_{h1}\xi_1(1 + c_b/\underline{c}_b)$, $\alpha_3' = c_b/\underline{c}_b$, $\alpha_3'' = c_{h1}c_b/\underline{c}_b$, $\alpha_4' = \sqrt{n}c_b c_b'(\underline{c}_b + c_b)\beta_1\gamma_3^{\beta_2}(c_{\hat{x}})/\underline{c}_b^3$, $\alpha_4 = c_{h1}c_b\xi_1/\underline{c}_b$, $\alpha_5 = (2 + 2l_2 + l_2^2)n[\underline{c}_b c_w' + \underline{c}_w c_b']^2(c_v + c_{\hat{v}})^2 c_b^2 c_w^2/(\underline{c}_b^4\underline{c}_w^4)$; and the second set of equations at the top of the page. □

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