Adaptive and Robust Controls of Uncertain Systems With Nonlinear Parameterization

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***Abstract*—Two classes of partially known systems are considered in this note; both of them have a fractional parameterization of the unknowns. The first class consists of nonlinear systems whose uncertainties are bounded by a function of fractional parameterization, and the second class compromises those systems whose unknown dynamics can directly, but nonlinearly, be parameterized. It is shown that adaptive robust control can be extended to accommodate nonlinearly parameterized bounding functions and that, with the aid of a robust auxiliary system, new adaptation laws and a simple adaptive control can be designed for the unknowns in the fractional parameterization. Practical stability (in terms of uniform boundedness and ultimate boundedness) is shown; global for adaptive robust control and semiglobal for the new adaptive control.**

***Index Terms*—Adaptive control, nonlinear parameterization, observer, robust control, uncertain systems.**

I. Introduction

In many applications, dynamics of the plant are partially known, and estimation and robustness are the key in designing a successful control. Adaptive control, robust control, and their combinations represent the means of achieving online estimation and robustness. Roughly speaking, a control system is adaptive if unknown parameters (of either the plant or its corresponding controller) are estimated online and the estimates are used to synthesize a stabilizing control; and a control system is robust if stability and performance under a fixed controller is guaranteed for a specific class of uncertainties (which could be unknown functionals, parameter variations, unmodeled dynamics, disturbances, etc).

Robust control of nonlinear uncertain systems has been a focus of research in the recent years. Conceptually, a control system is made to be robust if a specific class of uncertainties has been taken into consideration in control design and stability analysis. Typically, robust control design requires that the uncertainties be bounded in some norm and have a certain structural property in terms of their functional dependence and locations in system dynamics. Classes of stabilizable uncertain systems have been found, and several robust control design procedures have been proposed [4]–[6], [8], [10], [11], [13], [16], [19], [20], [21], [24], [27]. On the other hand, adaptive control is the technique of choice if the uncertainties can be expressed linearly in terms of unknown constants. Its popularity is due to the fact that standard adaptive control results [16] are concerned about how to estimate the unknowns and to use the estimates in control design.

Adaptive and robust controls are often combined as uncertainties are unknown by nature and, in a plant, several types of them may be present. It is straightforward to design a control containing both components to handle different kinds of unknowns, for example, a part of the control is adaptive to estimate unknown but constant parameters while the rest of the control is robust to compensate for bounded uncertainties. In addition, there are so-called robust adaptive control and adaptive robust control. Robust adaptive control is a modified adaptive control in order to gain certain robustness property. For example, robust adaptive controls have been proposed for slowly time varying systems [26], fast time varying systems [18], [22], and systems with internal dynamics [9], [11], [27]. Adaptive robust control is an adaptive version of robust control, and the first of such results is [6] in which uncertainties of the plant is bounded by a function linearly parameterized in terms of unknown constant parameters and robust control is made adaptive to estimate these parameters.

In this note, we continue to explore the benefits of adaptive and robust controls. The technical problem addressed in the note is how to design a stabilizing control for systems whose uncertainties or their bounding functions are parameterized not linearly but of a fractional expression. It is shown that, in case that the bounding function on uncertainties has a fractional parameterization, adaptive robust control can be designed and that, if the uncertainties can be parameterized directly, a new class of adaptive control based on robust observer can be designed to estimate all unknown parameters in the nonlinear parameterization. In both cases, the control problem in the presence of nonlinear parameterization is solved.

The proposed results are related to several topics in systems and control and are benefited from the recent developments therein. Clearly, the idea of designing adaptive robust control can be traced back to [6], and extensions have been made recently [22], [23] so that the bounding function can be parameterized in terms of time varying parameters defined by exogenous systems. On the other hand, the newly proposed adaptive control does not contain any robust control part but utilizes a robust observer to estimate the nonlinear parameterization as a whole, and it is this nonlinear observer that keeps the adaptive control be of standard form and enables the design of adaptation laws for parameter estimation. Some of existing results on nonlinear observer designs can be found in [1]–[3], [14], and [15].

II. Problem Formulation

The following class of affine uncertain systems is considered in this note:

$$\dot{x} = f(x,t) + g(x,t)\left[\Delta f_m(x,v,t) + u\right] \tag{1}$$

where $x(t) \in \Re^n$ is the state of the system, $X(t) = \{x(\tau), 0 \le \tau \le t\}$, $\Omega(X) \subset \Re^p$ is an unknown but bounded set provided that set X is uniformly bounded, $v(t) \in \Omega(X)$ is the vector of uncertainties, $u(t) \in \Re^m$ (with $m \le n$) is the control vector, $f(x,t)$ and $g(x,t)$ are dynamics of the so-called nominal system, and $\Delta f_m(x,v,t)$ is the vector of matched uncertainties.

In order to design adaptive and robust controls to stabilize system (1), the following assumptions are introduced. It is distinct that, in this note, uncertain dynamics of the system have a nonlinear parameterization (as described by either Assumption 2A or 2B).

Assumption 1: There is a known C^1 function $V(x,t)$: $\Re^n \times \Re \to \Re^+$ such that

$$\gamma_1(\|x\|) \le V(x,t) \le \gamma_2(\|x\|) \quad \text{and} \quad \frac{\partial V(x,t)}{\partial t} + \nabla_x^T V(x,t) f(x,t) \le -\gamma_3(\|x\|) \tag{2}$$

where γ_i: $\Re^+ \to \Re^+$ are class $\mathcal{K}_\infty$ functions.

Assumption 2A: Uncertain dynamics in the system are bounded in norm as

$$\|\Delta f_m(x,v,t)\| \le \rho(x,\phi,t) \tag{3}$$

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where $\rho(x,\phi,t)$ is a nonnegative bounding function and has a fractional parameterization of form

$$\rho(x,\phi,t) = \frac{W_1^T(x,t)\phi_1}{W_2^T(x,t)\phi_2} \tag{4}$$

$W_i(x,t)$ are known functional vectors, Ω_{ϕ_i} are compact sets, $\phi_i \in \Omega_{\phi_i}$, $\phi = [\phi_1^T\ \phi_2^T]^T \in \Re^q$ is a vector of unknown constants, and there exist a known function $c(x)$ and a known constant $\underline{c} > 0$ such that, for all (x,t) and for all $\phi_2 \in \Omega_{\phi_2}$

$$W_2^T(x,t)\phi_2 \geq c(x) \geq \underline{c}. \tag{5}$$

Assumption 2B: Uncertain dynamics in the system are parameterized as the matrix fractional expression

$$v = \phi = \left[\phi_1^T\ \phi_2^T\right]^T$$
$$\Delta f_m(x,v,t) = \frac{1}{W_2^T(x,t)\phi_2} W_1(x,t)\phi_1 \tag{6}$$

where $W_1(x,t)$ and $W_2(x,t)$ are known functional matrix and vector, respectively, inequality (5) holds, and vector ϕ contains unknown constants.

Assumption 3: All functions in system (1) and in the previous assumptions are Caratheodory, uniformly bounded with respect to t, and locally uniformly bounded with respect to x and v.

Assumption 4: Matrix $g(x,t)$ has the properties that $g(x,t) = [g_1^T(x,t)\ g_2^T(x,t)]^T$ where $g_2^{-1}(x,t) \in \Re^{m\times m}$ is well defined everywhere and locally uniformly bounded. Furthermore, partial derivatives of $g(x,t)$, $W_1(x,t)$ and $W_2(x,t)$ with respect to x and t are well defined and locally uniformly bounded.

Remark 2.1: In Assumption 2A, inequality (5) implies that uncertainties to be compensated for are locally uniformly bounded. Knowledge of $\underline{c}$ can be assumed without loss of any generality as parameterization (4) can be kept unchanged through scaling both its denominator and numerator. Assumption 2B can be viewed to be a special case of Assumption 2A. Both parameterizations have their roots in matrix-fraction description [12]. ◇

Remark 2.2: Existence of a matrix inverse and partial derivatives is required in Assumption 4 in order to estimate $\Delta f_m(x,t)$ directly using a robust observer (that is nonlinear in x). Differentiability ensures that $\Delta f_m(x,t)$ has a locally bounded rate of change in order to be estimated, and invertibility of $g_2(x,t)$ implies that $\Delta f_m(x,t)$ could be solved from dynamic equation of the system if all other terms were known. Thus, both of them can be interpreted as observability conditions. ◇

It follows from (4) and (5) that $W_1^T(x,t)\phi_1 \geq 0$, which together with (5) requires projection in the design of adaptation law(s). A standard assumption of existing projection algorithms [25] is that unknown parameters stay and all the constraints are satisfied in a known convex bounded set. To better handle constraints of form (5), we propose the following method: Through studying properties of the known vectors $W_i(x,t)$, find ϕ_i^* such that

$$W_1^T(x,t)\phi_1^* \geq 0, \text{ and } \hat{\phi}_{1j} \geq \phi_{1j}^* \text{ for all } j \text{ implies } W_1^T(x,t)\left(\hat{\phi}_1 - \phi_1^*\right) \geq 0 \tag{7}$$

and

$$W_2^T(x,t)\phi_2^* \geq c(x), \text{ and } \hat{\phi}_{2j} \geq \phi_{2j}^* \text{ for all } j \text{ implies } W_2^T(x,t)\left(\hat{\phi}_2 - \phi_2^*\right) \geq 0 \tag{8}$$

where $\hat{\phi}_{ij}$ is the jth element of $\hat{\phi}_i$. In essence, ϕ_i^* consists of lower bounds on the elements of ϕ_i. Using this approach, the set of admissible ϕ_i is still known and convex but not compact and, by having lower bounds ϕ_{ij}^* in adaptation laws, explicit projection of parameter estimates is no longer needed.

The control problem studied in the note is twofold: 1) design a robust control $u(x,t)$ such that the resulting closed-loop system is stable (in the sense of either asymptotic stability or practical stability, that is, stability of uniform ultimate boundedness [5], [21]) for all possible values of uncertain vector $v(t)$ in the *unknown* set $\Omega(X)$ and for all uncertainties whose bounding function is described by fractional parameterization (4), and 2) if the uncertainty itself has a fractional parameterization, design an adaptive control to ensure closed-loop stability (as previously described) and to estimate all the unknown parameters in the nonlinear parameterization.

III. Adaptive Robust Control Design

It is known that, if uncertainties in the system are matched and bounded by a known bounding function, robust control can be designed through size domination in a Lyapunov argument. It is also shown in [6] that, in case that the bounding function is unknown but parameterizable linearly in terms of constants, an adaptive robust control can be designed using the certainty-equivalence principle as did in a standard adaptive control design. The following lemma extends the existing results to the case that the bounding function on uncertainties has a nonlinear parameterization.

Theorem 1: Suppose that system (1) satisfies Assumptions 1, 2A, and 3 as well as condition (7). Then, the closed-loop system is uniformly ultimate bounded under the adaptive robust control

$$u(x,t) = -\hat{\rho}_m(x,t)\frac{\hat{\mu}(x,t)}{\|\hat{\mu}(x,t)\| + \epsilon} \tag{9}$$

where $\hat{\rho}_m(x,t) = W_1^T(x,t)\hat{\phi}_1/[W_2^T(x,t)\hat{\phi}_2]$, $\hat{\mu}(x,t) = g^T(x,t)\ \nabla_x\ V(x,t)W_1^T(x,t)\hat{\phi}_1$, ϵ is a design parameter given by

$$\dot{\epsilon} = -k_\epsilon \epsilon \tag{10}$$

with $\epsilon(t_0) > 0$, $\hat{\phi}_2$ is the estimate of ϕ_2 and its value can be arbitrarily chosen as long as inequality $0 < W_2(x,t)\hat{\phi}_2 \leq c(x)$ is satisfied, $\hat{\phi}_1$ is the estimate of ϕ_1 and is generated by adaptation law

$$\dot{\hat{\phi}}_1 = \frac{1}{c(x)}W_1(x,t)\left\|g^T(x,t)\nabla_x V(x,t)\right\| - k_a\left(\hat{\phi}_1 - \phi_1^*\right),\ \hat{\phi}_{1j}(t_0) \geq \phi_{1j}^* \tag{11}$$

and $0 \leq k_a \ll 1$ and $k_\epsilon \geq 0$ are gains. Furthermore, if $k_\epsilon > 0$ and $k_a = 0$, the original state $x(t)$ converges to zero.

Proof: Consider the Lyapunov function $L(x,t,\phi_1,\hat{\phi}_1) = V(x,t) + 0.5\|\tilde{\phi}_1\|^2 + k_{l\epsilon}$, where $\tilde{\phi}_1 = \phi_1 - \hat{\phi}_1$ is the parameter estimation error, and $k_l = 0$ if $k_\epsilon = 0$ and $k_l = 1/(k_\epsilon\underline{c})$ if otherwise. It follows from (2)–(5) and (9)–(11) that, letting $\mu(x,t) = \nabla_x^T V(x,t)g(x,t)W_1^T(x,t)\phi_1$

$$\begin{aligned}\dot{L} \leq& \frac{\partial V(x,t)}{\partial t} + \nabla_x^T V(x,t)f(x,t) \\ &+ \nabla_x^T V(x,t)g(x,t)\Delta f_m(x,t) \\ &+ \nabla_x^T(x,t)g(x,t)u(x,t) + \tilde{\phi}_1^T\dot{\tilde{\phi}} + k_{l\dot{\epsilon}} \\ \leq& -\gamma_3(\|x\|) + \frac{1}{c(x)}\|\mu(x,t)\| + \frac{\epsilon}{\underline{c}} \\ &- \frac{1}{c(x)}\|\hat{\mu}(x,t)\| + \tilde{\phi}_1^T\dot{\tilde{\phi}} + k_{l\dot{\epsilon}} \\ \leq& -\gamma_3(\|x\|) - \frac{k_a}{2}\|\tilde{\phi}_1\|^2 + \frac{k_a}{2}\|\phi_1 - \phi_1^*\|^2 \\ &+ \left(\frac{1}{\underline{c}} - k_l k_\epsilon\right)\epsilon\end{aligned} \tag{12}$$

from which the stability claims stated can be concluded using stability theorems in [5] and [21]. Also, note from (11) that

$$\frac{d}{dt}\left\|\hat{\phi}_1-\phi_1^*\right\|^2=\frac{2}{c(x)}\left\|g^T(x,t)\nabla_x V(x,t)\right\| \times W_1^T(x,t)\left(\hat{\phi}_1-\phi_1^*\right)-2k_a\left\|\hat{\phi}-\phi_1^*\right\|^2$$

which, as $k_a \to 0^+$, approaches $2\|g^T(x,t)\nabla_x V(x,t)\|W_1^T(x,t)(\hat{\phi}_1-\phi_1^*)/c(x)$. Thus, we know from (7) that $\|\hat{\phi}_1-\phi_1^*\|$ does not become zero and, hence, $W_1^T(x,t)\hat{\phi}_1 \geq 0$. □

Remark 3.1: This theorem shows that nonlinear parameterization can be easily handled in the context of robust control. This is due to the fact that, while an adaptive robust control has an online estimation algorithm, robust control compensates for the unknowns primarily through size domination. As a result, unknown parameters in ϕ_2 are not explicitly estimated (via an adaptation law). In fact, nonlinear parameterization in the more general form of (3) (rather than just fractional one) can be directly handled by an adaptive robust control of form (9) and with $\hat{\rho}(x,\hat{\phi},t)$ as long as $\|\rho(x,\phi,t)-\hat{\rho}(x,\hat{\phi},t)\|$ is bounded by a function of x and compensated for by another robust control part. While such a design is simple, the resulting control (i.e., the adaptive robust control) may be and usually is conservative. For systems whose dynamics have a nonlinear fractional parameterization, control design can be proceeded by estimating unknown parameters in both the numerator and denominator, and its development is the subject of the next section and the main objective of the note. ◇

IV. New Adaptive Control Design

In the event that unknown dynamics have a nonlinear parameterization as defined in Assumption 2B, adaptive robust control can also be applied. As shown in the previous section, the adaptive robust control does not explicitly produce any estimate of ϕ_2. To estimate all parameters (in particular, ϕ_2) on line, one should proceed with an adaptive control design. In the presence of nonlinear parameterization, it remains to be preferred that an adaptive control be designed to achieve both objectives of ensuring stability and estimating ϕ_i while *not employing any size-dominating robust part explicitly in the control expression*. The new adaptive control proposed in this note is given by

$$u=-\frac{1}{W_2^T(x,t)\hat{\phi}_2}W_1^T(x,t)\hat{\phi}_1 \tag{13}$$

where $\hat{\phi}_i$ are estimates generated by adaptation laws

$$\dot{\hat{\phi}}_1=\frac{1}{W_2^T(x,t)\hat{\phi}_2}\nabla_x^T V(x,t)g(x,t)W_1(x,t)-k_a\hat{\phi}_1 \tag{14}$$

$$\dot{\hat{\phi}}_2=-\frac{1}{W_2^T(x,t)\hat{\phi}_2}W_2(x,t)\nabla_x^T V(x,t)g(x,t)g_2^{-1}(x,t)y - k_a\left(\hat{\phi}_2-\phi_2^*\right) \quad \hat{\phi}_{2j}(t_0)\geq\phi_{2j}^* \tag{15}$$

$0 \leq k_a \ll 1$ is an adaptation gain, $y \in \Re^m$ is the output of the following auxiliary dynamics:

$$\dot{\eta}=-\frac{1}{\mu}(x_2+\eta)-f_2(x,t)-g_2(x,t)u$$
$$y=\frac{1}{\mu}(x_2+\eta) \quad \eta(t_0)=-x_2(t_0). \tag{16}$$

$\mu > 0$ is a design parameter, and x_2, $f_2(x,t)$ and $g_2(x,t)$ are bottom mth order blocks of x, $f(x,t)$ and $g(x,t)$, respectively. That is, $x=[x_1^T,x_2^T]^T$, $x_2\in\Re^m$, $f(x,t)=[f_1^T(x,t)\,f_2^T(x,t)]^T$, $f_2(x,t)\in\Re^m$, $g(x,t)=[g_1^T(x,t)\,g_2^T(x,t)]^T$, and $g_2(x,t)\in\Re^{m\times m}$.

Remark 4.1: Obviously, control (13) is an adaptive control of the simplest form, which makes it possible for the unknown parameter vector/matrix ϕ_i in both denominator and numerator to be explicitly and properly estimated. What is new is that adaptation laws (14) and (15) are in terms of not only the system state but also the output of a robust auxiliary system whose dynamics are nonlinear in x. As will be shown in the proof of Theorem 2, the fractional parameterization as a whole is also "estimated" (through size domination in a Lyapunov argument) by the output of the auxiliary system (16). Thus, the proposed new design is in essence to combine robust observer into adaptation laws. ◇

Remark 4.2: Nonlinear parameterization can be handled by introducing a robust part directly into adaptive control (13). Such a control, often called robust adaptive control, has the same shortcomings of adaptive robust controls (explained in Remark 3.1) as the robust control part tends to be dominating and, thus, makes parameter estimation less likely and the overall control conservative. A recent result proposed in [7] is along this line of robust adaptive control. ◇

Remark 4.3: Auxiliary system (16) is a relative-degree-one observer, and the choice of its initial condition [i.e., $y(t_0)=0$] makes the peaking phenomenon absent. In an extension to high-relative-degree observers (if required by system dynamics), introduction of saturation becomes necessary as in the previous work on high-gain observers. ◇

Properties of robust-observer-based adaptive control (13) and stability of the closed loop system are summarized in the following theorem. Its proof will be facilitated by the following lemma. In the lemma, $\|\cdot\|_\tau$ is the truncated functional norm defined by $\|w(t)\|_\tau \triangleq \sup_{0\leq t\leq\tau}\|w(t)\|$.

Lemma: Consider the first-order, linear, vector differential equation of form $c_1\dot{w}+c_2w=v(t)$, where c_1, $c_2>0$ are constants. Then, the input–output relationship from $v(t)$ to $w(t)$ satisfies the following inequality: for all $\tau\geq t_0$, $\|w(t)\|_\tau\leq\|w(t_0)\|+(1/c_2)\|v(t)\|_\tau$.

Proof: The solution is $w(t) = e^{-(c_2/c_1)t}w(t_0) + \int_0^t e^{-(c_2/c_1)(t-s)}(1/c_1)v(s)ds$. Taking the norm on both sides yields $\|w(t)\|\leq\|w(t_0)\|+\|v(s)\|_t\int_0^t e^{(c_2/c_1)(t-s)}(1/c_1)ds$, from which the statement can be concluded. □

Theorem 2: Consider system (1) satisfying Assumptions 1, 2B, 3, and 4 as well as (8). Under robust-observer-based adaptive control (13), the following stability properties can be ensured. i) For any initial conditions of $x(t_0)$, $\hat{\phi}(t_0)$, and $\eta(t_0)$ (or $y(t_0)$), the corresponding state variables will be uniformly bounded in a hyperball (whose radius is a class-$\mathcal{K}$ function of their initial conditions) for all sufficient large values of $1/\mu$. ii) The state variables will also be ultimately bounded with respect to a bound whose value is a class-$\mathcal{K}$ function of k_a and μ. iii) Stability of uniform boundedness and ultimate boundedness is semiglobal.

Proof: It follows from (1) and (16) that

$$\mu\dot{y}=\psi(x,t)-y \tag{17}$$

where $\psi(x,t)=g_2(x,t)\Delta f_m(x,t)$. Let $\tilde{\psi}(x,y,t)=\psi(x,t)-y$. Equation (17) shows that, by properly choosing μ (to be small), output y of auxiliary system (16) can be made to approach $\psi(x,t)$, that is, $\tilde{\psi}(x,y,t)\to 0$. Since matrix $g_2(x,t)$ is invertible, output y can be viewed as a robust estimate of $\Delta f_m(x,t)$ despite of its nonlinear parameterization. It is this online estimate that makes adaptive estimation of both ϕ_1 and ϕ_2 possible. In the sequel, stability will be analyzed for the closed loop system consisting of (1), (17), (14), and (15).

Asymptotic stability or uniform ultimate boundedness will be established by induction and in four steps. The main idea of the proof is as follows. Given any positive constants c_x, $c_{\hat{\phi}}$, c_y, $c_{\tilde{\phi}}$ and $c_{\tilde{\psi}}$ such that, for some $\tau\geq t_0$

$$\|x\|_\tau<c_x,\ \|\hat{\phi}\|_\tau<c_{\hat{\phi}},\ \|y\|_\tau<c_y,\ \|\tilde{\phi}\|_\tau<c_{\tilde{\phi}},\ \|\tilde{\psi}\|_\tau<c_{\tilde{\psi}} \tag{18}$$

and that their initial conditions are within subsets of the above compact sets [as described by (26) and (27)], the time derivative of a proper Lyapunov function [given by (22)] is made (by choice of μ) negative definite in a hyper annulus of a sufficient width [as described by (31)]. By doing so and by noting that τ is arbitrary, it can be shown that the state variables in the Lyapunov function (i.e., x, $\tilde{\phi}$ and $\tilde{\psi}$) will remain in the region defined by (18) and they will ultimately converge to a smaller subset [described by an ultimate bound which is a class-$\mathcal{K}$ function of the lower bounds in (30) and (31)], i.e., local stability can be claimed. Since c_x, $c_{\tilde{\phi}}$, and $c_{\tilde{\psi}}$ are also arbitrary, semi-global stability can then be concluded.

As the first step of stability analysis, properties stated in Assumptions 1, 2B, 3, and 4 are used to develop bounds or bounding functions on the dynamics of the closed-loop system. Specifically, values of non-negative bounds $\xi_i (i = 1, \cdots, 8)$ and a positive bound $0 < \beta < 1$ are to be determined. It follows from assumptions 1 and 3 that, given $\|x\| < c_x$, inequalities

$$\|f(x,t)\| \leq \xi_0 \gamma_3^{(1-\beta)}(\|x\|), \text{ and } \left\|\nabla_x^T V(x,t)\right\| \leq \xi_1 \gamma_3^{\beta}(\|x\|) \tag{19}$$

hold for some constants ξ_0, $\xi_1 > 0$ and $0 < \beta < 1$ (whose values may depend upon c_x). Given $\|x\| < c_x$, bounds ξ_i $(i = 2, \cdots, 8)$ can also be found according to Assumptions 4 and 2B such that $\|g_2(x,t)\| \leq \|g(x,t)\| \leq \xi_2$, $\|g_2^{-1}(x,t)\| \leq \xi_3$, $\|W_1(x,t)\| \leq \xi_4$, $\|W_2(x,t)\| \leq \xi_5$, $\|\partial[g_2(x,t)]/\partial t\| \leq \xi_6$, $\|\partial[\Delta f_m(x,t)]/\partial t\| \leq \xi_7$, and $\|\partial[g_2(x,t)\Delta f_m(x,t)]/\partial x\| \leq \xi_8$.

In the second step, relationships among the compact sets in (18) and with respect to time τ are explored. To this end, consider the following sets of initial conditions: letting $c_\phi \triangleq \|\phi\|$

$$\|x(t_0)\| < c_{x0} \quad \left\|\hat{\phi}(t_0)\right\| < c_{\hat{\phi}0} \quad \|y(t_0)\| < c_{y0}$$
$$\left\|\tilde{\phi}(t_0)\right\| < c_{\tilde{\phi}0} + c_\phi \quad \text{and} \quad \left\|\tilde{\psi}(t_0)\right\| < c_{y0} + \frac{1}{\underline{c}}\xi_2\xi_4 c_\phi. \tag{20}$$

Applying the lemma to differential equations (14), (15), and (17) yields that: since $\|x\|_\tau < c_x$, $\|y(\tau)\| \leq c_{y_0} + (1/\underline{c})\xi_2\xi_4 c_\phi \triangleq c_y$, $\|\hat{\phi}_1(\tau)\| \leq c_{\hat{\phi}_0} + (1/\underline{c}k_a)\xi_1\xi_2\xi_4\gamma_3^\beta(c_x) \triangleq c_{\hat{\phi}_1}$, $\|\hat{\phi}_2(\tau)\| \leq c_{\hat{\phi}_0} + (1/\underline{c}k_a)\xi_1\xi_2\xi_3\xi_4\xi_5\gamma_3^\beta(c_x)c_y \triangleq c_{\hat{\phi}_2}$, $\|\tilde{\phi}(\tau)\| \leq c_\phi + c_{\hat{\phi}} \triangleq c_{\tilde{\phi}}$, $\|\tilde{\psi}(\tau)\| \leq (1/\underline{c})\xi_2\xi_4 c_\phi + c_y \triangleq c_{\tilde{\psi}}$ and, consequently

$$\|u\| \leq \frac{1}{\underline{c}}\xi_4 c_{\hat{\phi}1} \triangleq c_u. \tag{21}$$

It is obvious that, once c_x is chosen, bounds c_y, $c_{\hat{\phi}}$, $c_{\tilde{\phi}}$, $c_{\tilde{\psi}}$, and c_u can be selected (as shown before) and that, while some of them are proportional to $1/k_a$, they are all independent of design parameter μ.

In the third step of analysis, the following Lyapunov function is adopted:

$$L(x, \tilde{\phi}, \tilde{\psi}, t) = V(x,t) + \frac{1}{2}\text{Trace}\left(\tilde{\phi}_1\tilde{\phi}_1^T\right) + \frac{1}{2}\|\tilde{\phi}_2\|^2 + \frac{1}{2}\|\tilde{\psi}\|^2 \tag{22}$$

which is globally positive definite and radially unbounded with respect to its arguments as $\gamma_4(\|\Psi\|) \leq L(x, \tilde{\phi}, \tilde{\psi}, t) \leq \gamma_5(\|\Psi\|)$, where $\Psi = [x^T\ \tilde{\phi}^T\ \tilde{\psi}^T]^T$, and $\gamma_4, \gamma_5 \colon \Re^+ \to \Re^+$ are class $\mathcal{K}_\infty$ functions defined by: for all $c_3, c_4, c_5 \in \Re^+$

$$\gamma_4\left(\sqrt{c_3^2 + c_4^2 + c_5^2}\right) \leq \gamma_1(c_3) + \frac{1}{2}c_4^2 + \frac{1}{2}c_5^2$$
$$\text{and } \gamma_5\left(\sqrt{c_3^2 + c_4^2 + c_5^2}\right) \geq \gamma_2(c_3) + \frac{1}{2}c_4^2 + \frac{1}{2}c_5^2.$$

It follows from (1), (2), (13)–(15), and (17) that

$$\begin{aligned}\dot{L} = &\frac{\partial V(x,t)}{\partial t} + \nabla_x^T V(x,t)\dot{x} \\ &+ \text{Trace}\left(\tilde{\phi}_1\dot{\tilde{\phi}}_1^T\right) + \tilde{\phi}_2^T\dot{\tilde{\phi}}_2 + \tilde{\psi}^T\dot{\tilde{\psi}} \\ \leq & -\gamma_3(\|x\|) + \nabla_x^T V(x,t)g(x,t) \\ &\times \left[\frac{W_1(x,t)\phi_1}{W_2^T(x,t)\phi_2} - \frac{W_1(x,t)\hat{\phi}_1}{W_2^T(x,t)\hat{\phi}_2}\right] \\ &+ \text{Trace}\left(\tilde{\phi}_1^T\dot{\tilde{\phi}}_1\right) + \tilde{\phi}_2^T\dot{\tilde{\phi}}_2 + \tilde{\psi}^T\dot{\tilde{\psi}} \\ = & -\gamma_3(\|x\|) - \frac{1}{W_2^T(x,t)\hat{\phi}_2}\nabla_x^T V(x,t)g(x,t)g_2^{-1}(x,t) \\ &\times \tilde{\psi}(x,y,t)W_2^T(x,t)\tilde{\phi}_2 - \frac{k_a}{2}\|\tilde{\phi}_1\|^2 + \frac{k_a}{2}\|\phi_1\|^2 \\ &- \frac{k_a}{2}\|\tilde{\phi}_2\|^2 + \frac{k_a}{2}\|\phi_2 - \phi_2^*\|^2 \\ &- \frac{1}{\mu}\|\tilde{\psi}\|^2 + \tilde{\psi}^T\dot{\psi}(x,t).\end{aligned} \tag{23}$$

It follows from inequalities (19)–(21) that, within the compact sets in (18)

$$\left\|\frac{1}{W_2^T(x,t)\hat{\phi}_2}\nabla_x^T V(x,t)g(x,t)g_2^{-1}(x,t)\tilde{\psi}(x,y,t)W_2^T(x,t)\tilde{\phi}_2\right\|$$
$$\leq \frac{1}{\underline{c}}\xi_1\xi_2\xi_3\xi_5 c_{\tilde{\phi}}\gamma_3^\beta(\|x\|)\|\tilde{\psi}\| \triangleq \lambda_1\gamma_3^\beta(\|x\|)\|\tilde{\psi}\|$$

and that, within the compact sets in (18)

$$\begin{aligned}\left\|\dot{\psi}(x,t)\right\| \leq &\left[\frac{1}{\underline{c}}\xi_4\xi_6 c_\phi + \xi_2\xi_7 + \frac{1}{\underline{c}}\xi_2\xi_4\xi_8 c_\phi + c_u\right] \\ &+ \xi_0\xi_8\gamma_3^{(1-\beta)}(\|x\|) \triangleq \lambda_2 + \lambda_3\gamma_3^{(1-\beta)}(\|x\|).\end{aligned} \tag{24}$$

Combining the three inequalities from (23) up to (24) and applying the Holder's inequality yield that, given $\|x\|_\tau < c_x$

$$\begin{aligned}\dot{L} \leq & -\gamma_3(\|x\|) + \lambda_1\gamma_3^\beta(\|x\|)\|\tilde{\psi}\| + \lambda_2\|\tilde{\psi}\| + \lambda_3\gamma_3^{(1-\beta)}(\|x\|) \\ &\times \|\tilde{\psi}\| - \frac{1}{\mu}\|\tilde{\psi}\|^2 + \frac{k_a}{2}\sum_{i=1}^{2}\left[-\|\tilde{\phi}_i\|^2 + \|\phi_i\|^2\right] \\ \leq & -\frac{1}{3}\gamma_3(\|x\|) - \frac{1}{4\mu}\|\tilde{\psi}\|^2 - \frac{k_a}{2}\|\tilde{\phi}_1\|^2 - \frac{k_a}{2}\|\tilde{\phi}_2\|^2 \\ &+ \frac{k_a}{2}\|\phi_1\|^2 + \frac{k_a}{2}\|\phi_2 - \phi_2^*\|^2 + \mu\lambda_2^2 \\ &- \frac{1}{\mu}\|\tilde{\psi}\|\left[\frac{1}{4}\|\tilde{\psi}\| - \mu 3^{\frac{\beta}{1-\beta}}(1-\beta)\lambda_1^{\frac{1}{1-\beta}}\beta^{\frac{\beta}{1-\beta}}\|\tilde{\psi}\|^{\frac{\beta}{1-\beta}}\right] \\ &- \frac{1}{\mu}\|\tilde{\psi}\|\left[\frac{1}{4}\|\tilde{\psi}\| - \mu 3^{\frac{1-\beta}{\beta}}\beta\lambda_3^{\frac{1}{\beta}}(1-\beta)^{\frac{1-\beta}{\beta}}\|\tilde{\psi}\|^{\frac{1-\beta}{\beta}}\right]\end{aligned} \tag{25}$$

in which the first three terms are negative definite with respect to the variables.

As the fourth (and last) step, consider the initial conditions that are in the following sets consistent with (20)[1]:

$$\begin{aligned}\|x(t_0)\| &\leq \gamma_5^{-1}\circ\gamma_4(c_x) \\ \left\|\hat{\phi}(t_0)\right\| &\leq \gamma_5^{-1}\circ\gamma_4(c_{\hat{\phi}}) \\ \|y(t_0)\| &\leq \gamma_5^{-1}\circ\gamma_4(c_y) \\ \left\|\tilde{\phi}(t_0)\right\| &\leq \gamma_5^{-1}\circ\gamma_4(c_{\tilde{\phi}})\end{aligned} \tag{26}$$

$$\left\|\tilde{\psi}(t_0)\right\| \leq \gamma_5^{-1}\circ\gamma_4(c_{\tilde{\psi}}) \tag{27}$$

[1]Inequalities in (26) can always be satisfied for any initial conditions described by (20) as c_x, $c_{\hat{\phi}}$, c_y, $c_{\tilde{\phi}}$, and $c_{\tilde{\psi}}$ in (18) are positive but otherwise arbitrary constants and therefore can be always be increased.

where ${}^{-1}$ and $\circ$ denote the inverse function and the composition of functions, respectively. Now, given an arbitrary constant ϵ^*, choose design parameter

$$0 < \mu < \min\{\underline{\mu}_1, \underline{\mu}_2\} \tag{28}$$

where

$$\underline{\mu}_1 \triangleq \begin{cases} \frac{1}{4(1-\beta)} 3^{-\frac{\beta}{1-\beta}} \lambda_1^{-\frac{1}{1-\beta}} \beta^{-\frac{\beta}{1-\beta}} \\ \quad \times [c_{\tilde{\psi}}]^{-\frac{2\beta-1}{1-\beta}}, & \text{if } 0.5 \le \beta < 1 \\ \frac{1}{4(1-\beta)} 3^{-\frac{\beta}{1-\beta}} \lambda_1^{-\frac{1}{1-\beta}} \beta^{-\frac{\beta}{1-\beta}} \\ \quad \times \left[\gamma_5^{-1} \circ \gamma_4(\epsilon_1)\right]^{\frac{1-2\beta}{1-\beta}}, & \text{if } 0 \le \beta < 0.5 \end{cases}$$

$$\underline{\mu}_2 \triangleq \begin{cases} \frac{1}{4\beta} 3^{-\frac{1-\beta}{\beta}} \lambda_3^{-\frac{1}{\beta}} (1-\beta)^{-\frac{1-\beta}{\beta}} \\ \quad \times \left[\gamma_5^{-1} \circ \gamma_4(\epsilon_1)\right]^{-\frac{2\beta-1}{\beta}}, & \text{if } 0.5 \le \beta < 1 \\ \frac{1}{4\beta} 3^{-\frac{1-\beta}{\beta}} \lambda_3^{-\frac{1}{\beta}} (1-\beta)^{-\frac{1-\beta}{\beta}} \\ \quad \times [c_{\tilde{\psi}}]^{\frac{2\beta-1}{\beta}}, & \text{if } 0 \le \beta < 0.5 \end{cases}$$

and constants ϵ_1 and ϵ_2 are defined by $0 < \epsilon_1 < \min\{\gamma_5^{-1} \circ \gamma_4(\epsilon^*), \gamma_5^{-1} \circ \gamma_4(\|\tilde{\psi}(t_0)\|)\}$, and $\epsilon_2 = (k_a/2)\|\phi_1\|^2 + (k_a/2)\|\phi_2 - \phi_2^*\|^2 + \mu\lambda_2^2$. Note that ϵ_2 is proportional to k_a and μ.

It follows from (25) that

$$\dot{L} \le -\frac{1}{3}\gamma_3(\|x\|) - \frac{1}{4\mu}\|\tilde{\psi}\|^2 - \frac{k_a}{2}\|\tilde{\phi}_1\|^2 - \frac{k_a}{2}\|\tilde{\phi}_2\|^2 + \epsilon_2 \tag{29}$$

provided that design parameter μ satisfies inequality (28), that the intended stability region [as defined by the sets in (18)] admits the initial conditions according to (26) and (27), and that

$$\|x\|_\tau < c_x \quad \text{and} \quad \epsilon^* \le \|\tilde{\psi}\| < c_{\tilde{\psi}}. \tag{30}$$

Thus, we have $\dot{L} < 0$ if

$$\gamma_3^{-1}(3\epsilon_2) < \|x\| < c_x \quad \text{or} \quad \sqrt{4\mu\epsilon_2} < \|\tilde{\psi}\| < c_{\tilde{\psi}} \quad \text{or} \quad \sqrt{\frac{2\epsilon_2}{k_a}} < \|\tilde{\phi}\| < c_{\tilde{\phi}}. \tag{31}$$

It is important to note that the regions defined in (26) and (27) and (30), and (31) are all consistent with the hyperballs in (18). In fact, these regions can be made subsets of those sets in (18) by increasing such bounds as c_x, and the spans of hyperannulus such as those in (31) are also be increased. Therefore, it follows from [21, Th. 2.15, p. 65] that, given any initial conditions satisfying (26) and (27), all state variables (including x, y, $\tilde{\phi}$, and $\tilde{\psi}$) will be uniformly bounded [with respect to the hyperballs defined by (18)] and uniformly ultimately bounded (with respect to a hyperball whose radius is a class-$\mathcal{K}$ function of ϵ^* and ϵ_2). Recall that bounds such as c_x can be increased arbitrarily and are independent of μ, therefore, the closed-loop stability is semiglobal.

To have a smaller ultimate bound, both ϵ^* and ϵ_2 need to be reduced. That is, one has to reduce gain $k_a > 0$ and design parameter $\mu > 0$. Note that, as k_a decreases, bounds such as $c_{\hat{\phi}}$ become larger and, consequently, μ becomes much smaller. In the limit that both μ and k_a approach zero, state x becomes asymptotically convergent.

Finally, it follows from (15) and (17) that, as $k_a \to 0$ and $\mu \to 0$

$$\frac{d}{dt}\left\|\hat{\phi}_2 - \phi_2^*\right\|^2 \to \frac{2W_2(x,t)\left(\hat{\phi}_2 - \phi_2^*\right)}{W_2^T(x,t)\hat{\phi}_2} \times \left[-\nabla_x^T V(x,t) g(x,t) \Delta f_m(x,t)\right]. \tag{32}$$

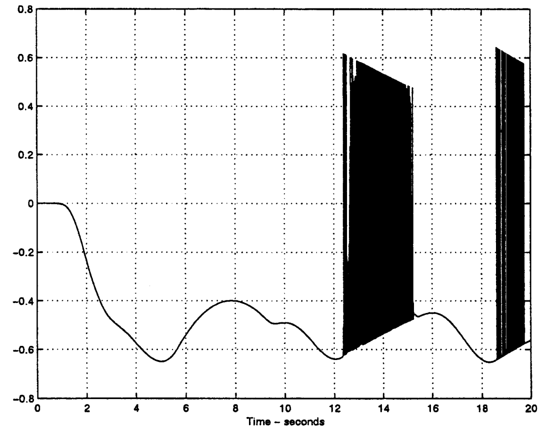

Fig. 1. Adaptive robust control (9).

It follows from the aforementioned Lyapunov analysis that, if $\nabla_x^T V(x,t) g(x,t) \Delta f_m(x,t) \le 0$, the convergence of both x and $\nabla_x^T V(x,t) g(x,t) \Delta f_m(x,t)$ to zero is facilitated; hence, it follows from (32) that $\|\hat{\phi}_2 - \phi_2^*\|$ is both increasing and convergent. On the other hand, if $\nabla_x^T V(x,t) g(x,t) \Delta f_m(x,t) \ge 0$, $\|\hat{\phi}_2 - \phi_2^*\|$ is decreasing but, according to (32), elements of $\hat{\phi}_2$ stays above those of ϕ_2^*. In summary, control and adaptation laws (13) up to (15) are made nonsingular. □

Remark 4.4: If $\beta = 0.5$, choice of μ in (28) becomes independent of ϵ^* and ϵ_1. In this case, one can set ϵ^* and ϵ_1 to be zero, and the ultimate bound depends upon ϵ_2 only. ◇

Remark 4.5: Given any conservative estimate of initial conditions, the set of semiglobal stability can be calculated using (26) and (27). Hence, (28) and (31) together with (19)–(21) provide the criteria for selecting control gain k_a and design parameter μ. ◇

Remark 4.6: It is clear from the proof of Theorem 2 that y is a robust estimate of nonlinearly parameterized function $\Delta f_m(x, \phi, t)$. As such, the observer-based robust control $u = -g_2^{-1}(x,t)y$ guarantees the same stability properties, and this control is simpler than adaptive control (13) only because it does not estimate system parameters. ◇

V. Simulation Example

To illustrate the proposed adaptive and robust controls, consider the second-order system: $\dot{x}_1 = x_2$ and $\dot{x}_2 = -x_1 - 2x_2 + \Delta f_m(x, v, t) + u$, where uncertainty $\Delta f_m(x, v, t)$ is assumed to be of form

$$\Delta f_m(x, v, t) = \frac{b_1 + b_2 \cos(2t) x_1^2 x_2^2}{a_1 + a_2 \sin(t) + a_3 x_1^2 + a_4 x_2^2}$$

and a_i and b_i are unknown constants satisfying the inequalities: $a_1 \ge a_2 + 1 > 0$, $a_3 \ge 1$, and $a_4 \ge 1$. In the simulation, the following values are used: $a_1 = 2$, $a_2 = 0.5$, $a_3 = 1.5$, $a_4 = 1.5$, $b_1 = 1$, and $b_2 = -2$. Obviously, the bounding function on the uncertainty can be chosen to be

$$\rho(x, v, t) = \frac{d_1 + d_2 x_1^2 x_2^2}{a_1 + a_2 \sin(t) + a_3 x_1^2 + a_4 x_2^2}$$

where $d_i = |b_i|$. It is straightforward to show that (5) is valid with $c(x) = 1 + x_1^2 + x_2^2$ and $\underline{c} = 1$ and that assumption 1 is met with $V(x) = 3x_1^2 + 2x_1x_2 + x_2^2$, $\gamma_1(\|x\|) = (2 - \sqrt{2})\|x\|^2$, $\gamma_2(\|x\|) = (2 + \sqrt{2})\|x\|^2$, and $\gamma_3(\|x\|) = 2\|x\|^2$.

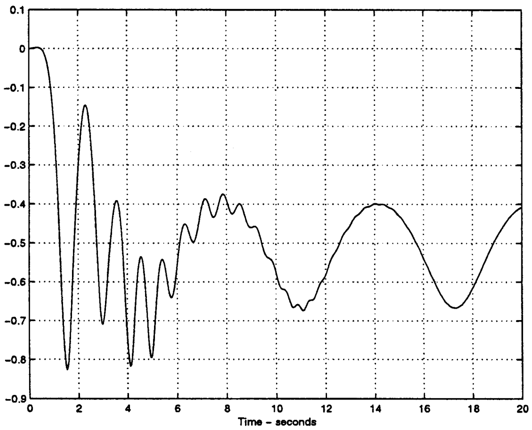

Fig. 2. Adaptive and robust-observer-based control (13).

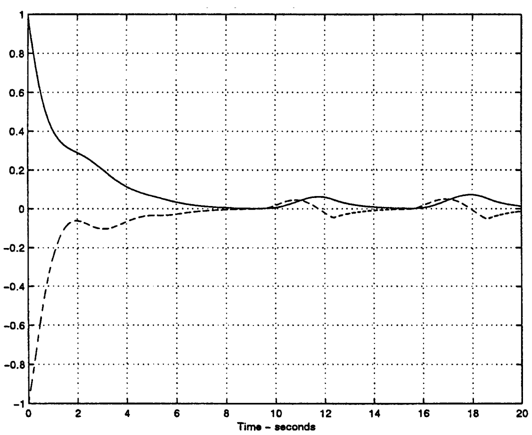

Fig. 3. State trajectory under control (9): x_1 (solid) and x_2 (dash).

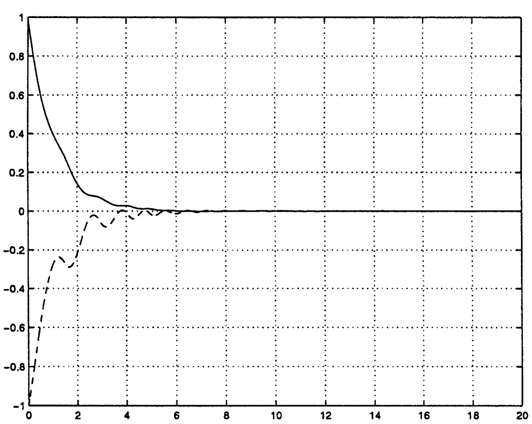

Fig. 4. State trajectory under control (13): x_1 (solid) and x_2 (dash).

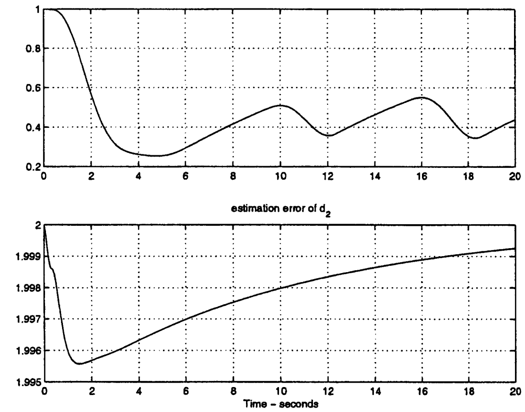

Fig. 5. Estimation errors of $\tilde{d}_i$.

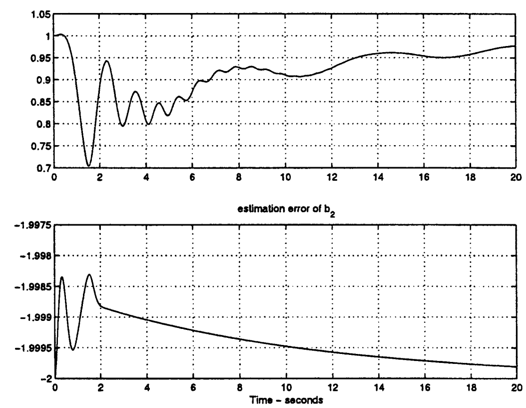

Fig. 6. Estimation errors of $\tilde{b}_i$.

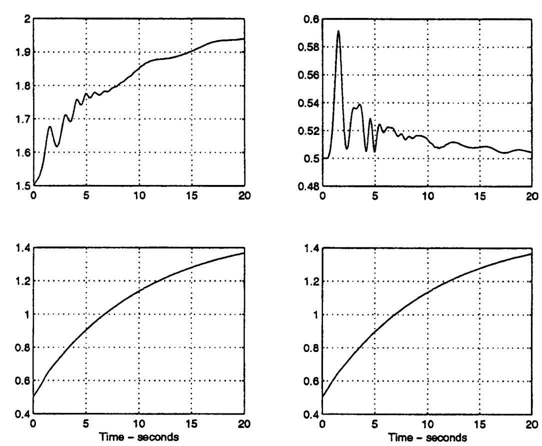

Fig. 7. Estimation errors of $\tilde{a}_j$.

Adaptive robust control (9) together with (10) and (11) is simulated by setting $e(t_0) = 1$, $\hat{\phi}_1(t_0) = \phi_1^* = 0$, $W_2^T(x,t)\hat{\phi}_2 = c(x)$, $k_\epsilon = 1$, and $k_a = 0.1$. Simulation results are shown in Figs. 1, 3, and 5. On the other hand, robust-observer-based adaptive control (13) together with adaptation laws (14) and nonlinear observer (15) is simulated using Simulink (in which variable-step ode45 solver and maximum step size of 0.0005 are set) and with the following parameters: $\eta(t_0) = 0$, $\hat{\phi}_1 = [0\ 0]^T$, $\hat{\phi}_2 = \phi_2^* = [0.5\ 0\ 1\ 1]^T$, $\mu = 0.001$, and $k_a = 0.1$. Simulation results are shown in Figs. 2, 4, and 6–8.

As demonstrated by Figs. 1 and 3, robust stability is achieved under the adaptive robust control. The fact that control (9) is conservatively magnitude dominant and sign-change sensitive makes the controller output prone to chatter extensively and/or intensively and hence induces performance degradation in both the control and the system trajectory.

It is clear from comparing Figs. 2 and 1 as well as 4 and 3 that, while a slightly larger control effort is observed in Fig. 2 during the initial six-second transient [due to the fact that control (13) has more dynamics than those of control (9)], the new adaptive control is much

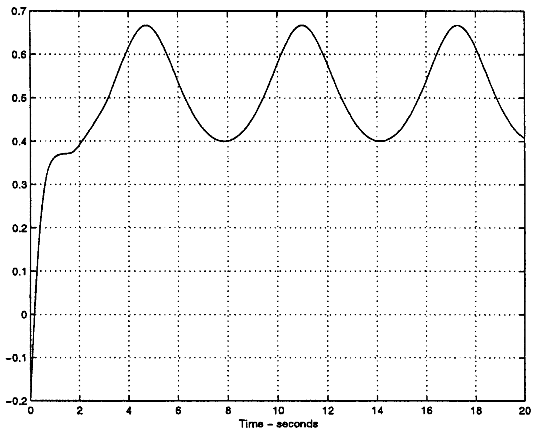

Fig. 8. Auxiliary output $y(t)$.

smoother and far less conservative overall, and it guarantees better performance (faster convergence and much smaller ultimate boundedness that is very close to asymptotic stability).

As expected, parameter estimates in either case do not converge to their true values. For control (9), parameter estimates are not convergent as the parameters to be estimated are conservative bounds. For control (13), estimation convergence would require persistent excitation, which is the subject of future research.

VI. CONCLUSION

In this note, two different control schemes are proposed for systems with nonlinear parameterizations. Nonlinear parameterization used in the note is consistent with matrix fractional description, either in terms of system dynamics or their bounding functions. It is shown that an adaptive robust control can achieve global stability but does not estimate all unknown parameters and in turn may be conservative. To estimate all unknown parameters, system dynamics must be parameterizable directly, and a new and simple adaptive control can be designed based on robust estimation of nonlinear parameterization as a whole. As a tradeoff, the newly proposed adaptive control renders semiglobal stability.

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