Weiwei Jin
Corning Cable Systems LLC,
9275 Denton Highway,
Keller, Texas 76248
e-mail: weiwei.jin@corning.com

Zhihua Qu
Professor,
School of Electrical Engineering and Computer Science,
College of Engineering and Computer Science,
University of Central Florida,
Orlando, FL 32816
e-mail: qu@pegasus.cc.ucf.edu

Kuo-Chi Lin
Associate Professor,
Department of Mechanical, Materials and Aerospace Engineering,
College of Engineering and Computer Science,
University of Central Florida,
Orlando, FL 32816
e-mail: klin@ist.ucf.edu

Design and Simulation of Robust and Adaptive Controls for a Nonlinear String System[1]

In this paper, vibration control of a nonlinear string system is considered. The system consists of a nonlinear string, two boundary supporting mechanisms, and a moving transporter at the base. To suppress the vibration, boundary control designs are carried out. A new robust and adaptive boundary controller is designed using the Lyapunov direct method. The proposed control is implemented at the two ends supporting the string to compensate for vibration induced by the base motion. It is shown that the adaptive/robust boundary control can asymptotically stabilize the nonlinear string. Numerical simulation of the closed loop system demonstrates the effectiveness of the proposed control.
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1 Introduction

A string-based system can be extracted from many physical systems, such as magnetic tapes, cables, belts, wires, chains. This and other kinds of flexible systems will have induced vibrations, especially in the presence of system uncertainties and/or disturbances. Vibrations are harmful in many applications. There are extensive research results on various aspects of vibration suppression. Recent developments on control of nonlinear string systems can be found in [1–6], and [7].

In this paper, a transported nonlinear string system is introduced to represent potential applications in the area of material handling and manufacturing automation, and the corresponding boundary control problem is studied. The string system was first introduced in [3]. The main differences between this paper (as well as [1]) and other results (in [2–6], and [7]) are twofold: nonlinear uncertainties in string dynamics, and moving transport. As in [1], robust/adaptive control is designed to suppress the vibration induced by the variations in base motion. Adaptive control algorithm is used to estimate online the unknown parameters describing the base motion. At the same time, the control is also made to be robust in order to compensate for uncertain dynamics in string dynamics. Control design and stability analysis are done using the Lyapunov direct method. Compared to [1], two major progresses have been achieved in this paper. First, the boundary controller in [1] is improved to be a robust adaptive version for the overall system. That is, all system dynamics (known or unknown) are considered in the design. Second, performance of the robust controller in [1] and the newly proposed robust adaptive controller are studied using numerical simulation. In the simulation, variable separation, modal analysis and their extension to handle nonlinear functional dependence are developed. And, the simulation results demonstrate the effectiveness of the proposed control in damping out vibrations in the presence of disturbance and uncertain string dynamics.

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2 Problem Statement

The string system considered in the paper is shown in Fig. 1. It consists of three parts: a stretched nonlinear string, two supporting mechanisms that are driven independently by actuators along two parallel tracks, and a transporter at the base. The two supporting mechanisms provide the boundary control forces/translations to the string. Vibrations of the string could be induced either by its non-zero initial condition or by speed variation of the transporter. This setup has many applications in material handling and process automation.

2.1 System Dynamics. The motion of the transporter $y_b(t)$ is characterized by a constant cruising speed (c_b) plus a variation $\delta_b(t)$, that is, $dy_b(t)/dt=c_b+\delta_b(t)$. Although dynamics of $y_b(t)$ satisfy the Newton's law, their explicit expressions are uncertain (for example, friction) and are not needed for the proposed control design. It is the speed variation of the transporter and/or change of the cruising speed will cause the string to have transverse vibration.

It follows from the derivations in [1] that the equation of motion for a stretched nonlinear string is

$$m(x)\frac{\partial^2 y(x,t)}{\partial t^2}=\frac{\partial}{\partial x}\left[T(x,t)\frac{\partial y(x,t)}{\partial x}\right].$$

Since the string is supported on a moving transporter, the motion equation of the string should be that with $y(x,t)$ replaced by $y(x,t)+y_b(t)$. That is, the equation of motion becomes

$$m(x)\frac{\partial^2 Y(x,t)}{\partial t^2}=\frac{\partial}{\partial x}\left\{T(x,t)\frac{\partial Y(x,t)}{\partial x}\right\}, \tag{1}$$

where $Y(x,t)=y(x,t)+y_b(t)$, or equivalently,

$$m(x)\frac{\partial^2 y(x,t)}{\partial t^2}=\frac{\partial}{\partial x}\left[T(x,t)\frac{\partial y(x,t)}{\partial x}\right]-m(x)\frac{d\delta_b(t)}{dt}. \tag{2}$$

It is assumed that tension of string is of the following form: for all $x\in[0,l]$

$$T(x,t)=T_0(x)+w(x)\left[\frac{\partial y(x,t)}{\partial x}\right]^2 \tag{3}$$

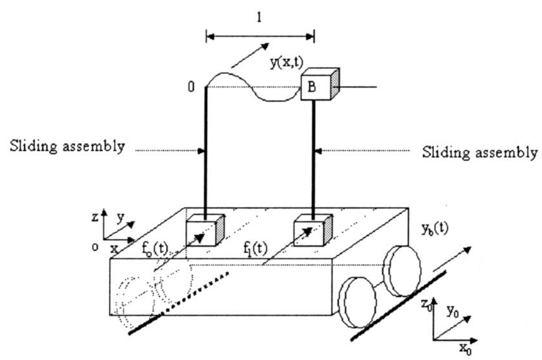

Fig. 1 A stretched string on a moving transporter

where $T_0(x)>0$ is the initial tension, and $w(x)\geqslant 0$ is the nonlinear elastic modulus.

To find a solution to the string equation, initial conditions of displacement and velocity and boundary conditions of the string are needed. It can be assumed without loss of any generality that initial conditions are given by

$$y(x,0)=c_1(x) \quad \text{and} \quad \left.\frac{\partial y(x,t)}{\partial t}\right|_{t=0}=c_2(x). \tag{4}$$

Boundary conditions in terms of displacements are provided by

$$y(0,t)=p_0(t) \quad \text{and} \quad y(l,t)=p_l(t), \tag{5}$$

where $p_0(t)$ and $p_l(t)$ are the solutions to the following dynamic equations for control mechanisms:

$$M_0\left[\frac{d^2p_0(t)}{dt^2}+\frac{d\delta_b(t)}{dt}\right]=f_0(t)-T(0,t)\left.\frac{\partial y(x,t)}{\partial x}\right|_{x=0}-b_0\frac{dp_0(t)}{dt}, \tag{6}$$

and

$$M_l\left[\frac{d^2p_l(t)}{dt^2}+\frac{d\delta_b(t)}{dt}\right]=f_l(t)+T(l,t)\left.\frac{\partial y(x,t)}{\partial x}\right|_{x=l}-b_l\frac{dp_l(t)}{dt}, \tag{7}$$

M_0, M_l are the masses of supporting mechanisms, and b_0 and b_l are dynamic friction coefficients between the control mechanism and the transporter. Variables $f_0(t)$ and $f_l(t)$ are the boundary controls to be designed in the next section so that robust stability and performance can be achieved.

2.2 Vibration Suppression. The control problem studied in the paper is characterized by the following assumptions and control objective.

Assumption 1: *Functions $m(x)$, $T_0(x)$ and $w(x)$ in the string dynamics may be uncertain to the control designer, but they are bounded by known, constant lower and upper bounds as follows.*

• *Bounded in size: for all $x\in[0,l]$, $\underline{m}\leqslant m(x)\leqslant\overline{m}$, $\underline{c}_{T_0}\leqslant T_0(x)\leqslant\overline{c}_{T_0}$, and $\underline{w}\leqslant w(x)\leqslant\overline{w}$.*

• *Bounded in the rate of spatial change: for all $x\in[0,l]$, values of $\partial m(x)/\partial x$, $\partial T_0(x)/\partial x$, and $\partial w(x)/\partial x$ are known to be within a certain range.*

Assumption 2: *Speed variation in the motion of the transporter can be parameterized as*

$$\delta_b(t)=\eta_1\sin(\omega_b t+\eta_2),$$

where ω_b is a known oscillation frequency, η_1 and η_2 represent unknown magnitude and phase angle, respectively. Furthermore, friction coefficients b_0 and b_l in (6) and (7) are unknown but constant.

Design Objective. Given the string system described by (6), (7) and (1) with nonlinear tension (3), find boundary controls $f_0(t)$ and $f_l(t)$ and the corresponding adaptation laws so that, under assumptions 1 and 2, the nonlinear string will asymptotically converge to the equilibrium.

3 Robust and Adaptive Control

In this section, robust control and robust/adaptive control are designed for $f_0(t)$ and $f_l(t)$ using the Lyapunov direct method. To this end, consider the following Lyapunov function candidate for the string system:

$$\begin{aligned}V_s(t)=\int_0^l m(x)\Bigg\{&\left[\frac{\partial y(x,t)}{\partial t}+\delta_b(t)\right]^2+\frac{T_0(x)}{m(x)}\left[\frac{\partial y(x,t)}{\partial x}\right]^2\\&+\frac{w(x)}{2m(x)}\left[\frac{\partial y(x,t)}{\partial x}\right]^4+\frac{\alpha(x)x}{l}\left[\frac{\partial y(x,t)}{\partial t}\right.\\&\left.+\delta_b(t)\right]\frac{\partial y(x,t)}{\partial x}\Bigg\}dx\end{aligned} \tag{8}$$

where its initial condition can be computed using the initial conditions in (4), and $\alpha(x)$ is a positive scalar function satisfying the following inequalities: for all $x\in[0,l]$ and for some constant $\epsilon>0$,

$$x^2\alpha^2(x)\overline{m}<\underline{c}_{T_0}l^2, \tag{9}$$

$$9\alpha^2(l)\overline{m}<16\underline{c}_{T_0}, \tag{10}$$

$$11\alpha^2(l)\overline{m}\leqslant\left[4\sqrt{2\underline{c}_{T_0}}+\sqrt{32\underline{c}_{T_0}-11\alpha^2(l)\overline{m}}\right]^2, \tag{11}$$

$$\frac{\partial[\alpha(x)m(x)x]}{\partial x}>\epsilon, \tag{12}$$

$$\frac{\partial[\alpha(x)x]}{\partial x}T_0(x)>\alpha(x)x\frac{\partial T_0(x)}{\partial x}+\epsilon, \tag{13}$$

and

$$3\frac{\partial[\alpha(x)x]}{\partial x}w(x)>\alpha(x)x\frac{\partial w(x)}{\partial x}+2\epsilon. \tag{14}$$

It is straightforward to show the following result.

Lemma 1 *Under assumption 1, inequalities (9) up to (14) can all be satisfied by simply choosing $\alpha(x)=\beta_1e^{x/\beta_2}$ (with β_1, $\beta_2>0$) whose value can be made sufficiently small and whose partial derivative can be made arbitrarily large. In addition, Lyapunov functional in (8) is positive definite with respect to $[\partial y(x,t)/\partial t+\delta_b(t)]$ and $\partial y(x,t)/\partial x$.*

Under assumption 1 and under the condition that $\delta_b(t)$ is known, a successful robust control design is available in [1] and is now restated as Theorem 1.

Theorem 1 *Consider the string system described by (6), (7) and (1) with the boundary condition (5). Under assumption 1, the following boundary controls together are robust and globally exponentially stabilizing with respect to the equilibrium of the string provided that scalar function $\alpha(x)$ is chosen to satisfy inequalities (9) up to (14) and that parameters b_0, b_l, η_1* and *η_2 are known*:

$$f_0(t)=-k_0\frac{\partial Y(0,t)}{\partial t}+3c_f(0,t)+b_0\left[\frac{\partial Y(0,t)}{\partial t}-\delta_b(t)\right], \tag{15}$$

and

$$\begin{aligned}f_l(t)=-k_l\left[\frac{3}{8}\alpha(l)\left.\frac{\partial y(x,t)}{\partial x}\right|_{x=l}+\frac{\partial Y(l,t)}{\partial t}\right]-3c_f(l,t)+b_l\left[\frac{\partial Y(l,t)}{\partial t}\right.\\\left.-\delta_b(t)\right]-\frac{3}{8}M_l\alpha(l)\left.\frac{\partial^2y(x,t)}{\partial x\partial t}\right|_{x=l},\end{aligned} \tag{16}$$

where $k_0 \geqslant 0$ and $k_l \geqslant \alpha(l)\overline{m}$ are positive control gains, $c_f(0,t) = T(0,t)[\partial y(x,t)/\partial x]_{x=0}$ and $c_f(l,t) = T(l,t)[\partial y(x,t)/\partial x]_{x=l}$ are boundary contacting forces.

The control defined by (15) and (16) is called *robust* as, by the choice of gains $\alpha(l)$ and k_l, these two fixed control laws can maintain stability under admissible uncertainties in system dynamics (and as defined by assumption 1). While the string system is structurally symmetric, the choice of Lyapunov functional (8) yields different expressions for the two boundary control laws.

The above control requires the perfect knowledge of such parameters as b_0, b_l, η_1 and η_2. Since $\delta_b(t)$ is an unknown perturbation and since b_0 and b_l may be unknown, we propose the following new robust adaptive controller to compensate for these uncertainties. The proof is included as Appendix A.

Theorem 2 *Consider the string system described by (6), (7) and (1) with the boundary condition (5). Under assumption 1, the following boundary controls are robust and globally asymptotically stabilizing with respect to the equilibrium of the string provided that scalar function $\alpha(x)$ is chosen to satisfy inequalities (9) up to (14)*:

$$f_0(t) = -k_0 \frac{\partial Y(0,t)}{\partial t} + 3c_f(0,t) + \hat{b}_0 \frac{\partial Y(0,t)}{\partial t} - \hat{\xi}_1 \sin(\omega_b t) - \hat{\xi}_2 \cos(\omega_b t), \tag{17}$$

and

$$f_l(t) = -k_l \left[\frac{3}{8}\alpha(l) \left.\frac{\partial y(x,t)}{\partial x}\right|_{x=l} + \frac{\partial Y(l,t)}{\partial t} \right] - 3c_f(l,t) + \hat{b}_l \frac{\partial Y(l,t)}{\partial t} - \hat{\xi}_3 \sin(\omega_b t) - \hat{\xi}_4 \cos(\omega_b t) - \frac{3}{8} M_l \alpha(l) \left.\frac{\partial^2 y(x,t)}{\partial x \partial t}\right|_{x=l}, \tag{18}$$

with adaptation laws

$$\begin{aligned}
\frac{d\hat{b}_0}{dt} &= -2k_a \left[\frac{\partial Y(0,t)}{\partial t} \right]^2, \\
\frac{d\hat{b}_l}{dt} &= -2k_a \left[\frac{\partial Y(l,t)}{\partial t} + \frac{3}{8}\alpha(l) \left.\frac{\partial y(x,t)}{\partial x}\right|_{x=l} \right] \frac{\partial Y(l,t)}{\partial t}, \\
\frac{d\hat{\xi}_1}{dt} &= 2k_a \frac{\partial Y(0,t)}{\partial t} \sin(\omega_b t), \\
\frac{d\hat{\xi}_2}{dt} &= 2k_a \frac{\partial Y(0,t)}{\partial t} \cos(\omega_b t), \\
\frac{d\hat{\xi}_3}{dt} &= 2k_a \left[\frac{\partial Y(l,t)}{\partial t} + \frac{3}{8}\alpha(l) \left.\frac{\partial y(x,t)}{\partial x}\right|_{x=l} \right] \sin(\omega_b t), \\
\frac{d\hat{\xi}_4}{dt} &= 2k_a \left[\frac{\partial Y(l,t)}{\partial t} + \frac{3}{8}\alpha(l) \left.\frac{\partial y(x,t)}{\partial x}\right|_{x=l} \right] \cos(\omega_b t)
\end{aligned} \tag{19}$$

where $k_0 \geqslant 0$ and $k_l \geqslant 2\alpha(l)\overline{m}$ are positive control gains, and $k_a > 0$ is adaptation gain.

4 Performance Verification

In this section, effectiveness of robust and robust/adaptive controllers presented in the previous section will be evaluated using simulation. Mathematically, the resulting closed loop system under the proposed control is a problem consisting of homogeneous partial differential equation with non-homogeneous boundary conditions. To the best of our knowledge, there is no commercially available software package that is capable of simulating the system under the proposed control. Thus, we shall first develop necessary equations for us to simulate the proposed control using Matlab. In what follows, the method of modal analysis [8] will be applied by taking the following steps. First, transform the boundary-value problem consisting of a homogeneous differential equation with non-homogeneous boundary conditions into a problem consisting of a nonhomogeneous differential equation with homogeneous boundary conditions. Second, solve the homogeneous boundary value problem by separating the time and spatial dependence of the solution. This leads to the eigenvalue problem yielding the normal modes and the associated natural frequencies of the system. The solution to the non-homogeneous differential equation is obtained by the expansion theorem, which assumes that the solution can be expressed as a superposition of normal modes. The above process leads to a numerical simulation model.

4.1 Simulation Model.

The purpose of the simulation study is to demonstrate the robustness and the capability of on-line adaptation of the proposed control. Functions $m(x)$ and $T(x,t)$ are the potential sources of uncertainty in string dynamics. Meantime, modal analysis calls for superposition of the modes. Therefore, we choose to make the following assumption.[2]

Assumption 3: *Tension of the string is constant (i.e., $T(x,t) = T$ for all (x,t)), and mass per unit length $m(x)$ is the source of uncertainty.*

Under assumption 3, the string equation becomes

$$\frac{\partial^2 y(x,t)}{\partial x^2} - \frac{m(x)}{T} \frac{\partial^2 y(x,t)}{\partial t^2} = \frac{m(x)}{T} \frac{d\delta_b(t)}{dt} \tag{20}$$

with the boundary condition (5). Following the steps outlined above, one needs to separate the time and spatial dependence by representing the velocity at any point along the string with respect to the ground as[3]

$$\frac{\partial Y(x,t)}{\partial t} = \sum_{i=1}^{\infty} \phi_i(x) \frac{d\gamma_i(t)}{dt} + \frac{dp_0(t)}{dt} + \frac{x}{l}\left[\frac{dp_l(t)}{dt} - \frac{dp_0(t)}{dt} \right] + \delta_b(t), \tag{21}$$

where $\phi_i(x)$ are the modal functions to be determined. For the purpose of performance studies, we need to assume the types of uncertainties in $m(x)$ considered in the simulation so that the modal functions can be found.

Assumption 4: *The mass per unit length of string $m(x)$ has a maximum of 25% variation from its nominal value. Furthermore, its spatial dependence changes in such a way that, given a modal function $\phi_i(x)$ in (21), modal function with respect to the decomposition of $m(x)[\partial^2 y(x,t)/\partial t^2]/T$ will be*

$$\frac{1 + 4\delta \cos \dfrac{i\pi x}{l}}{1 + \delta \cos \dfrac{i\pi x}{l}} \phi_i(x), \tag{22}$$

where $i = 1,2,3,\ldots,$ and $0 \leqslant \delta < 0.25$.

Theorem 3 *If assumption 4 holds, modal functions in (21) for the simulated string equation (20) are given by*

$$\phi_i(x) = \sin \frac{i\pi x}{l} \left(1 + \delta \cos \frac{i\pi x}{l} \right),$$

where $i = 1,2,3,\ldots$. If the first four modes are chosen, Matlab-based simulation model will be in terms of $\gamma_i(t)$ governed by the following ordinary differential equations:

[2]This assumption is made to find a class of modal functions with or without uncertainties in $m(x)$. The same process applies if $T(x,t)$ is not a constant. In the latter case, other techniques such as finite difference method can also be applied to yield a numerical simulation model, and the results of simulation under those conditions can be found in another paper [9].

[3]Its derivation can be seen from Appendix B.

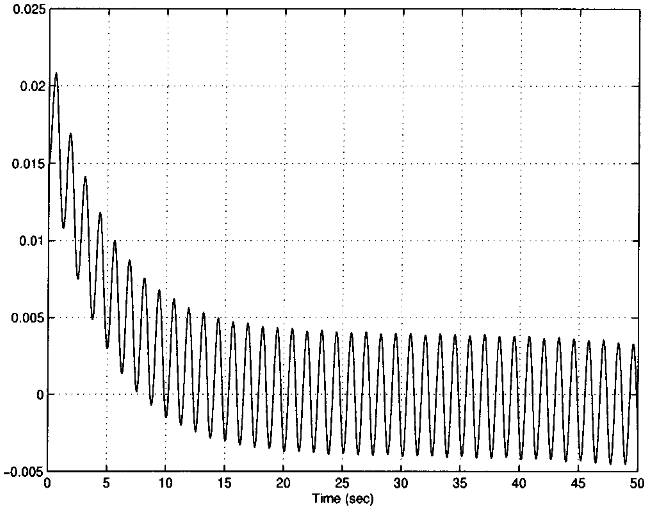

Fig. 2 The open-loop velocity at point $x=0$

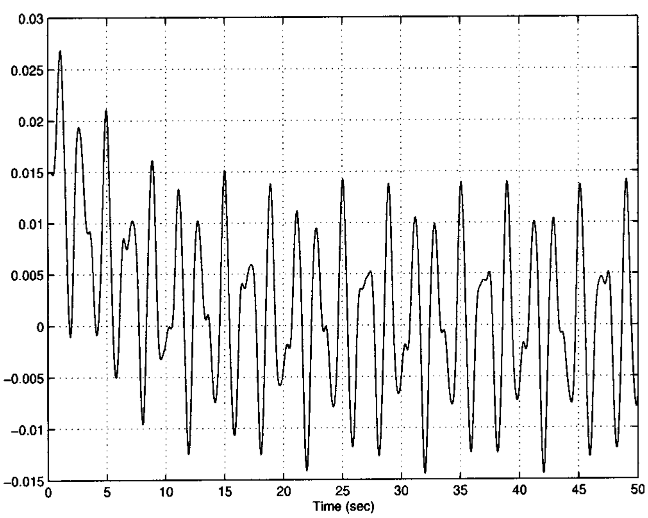

Fig. 3 The open-loop velocity point $x=0.5l$

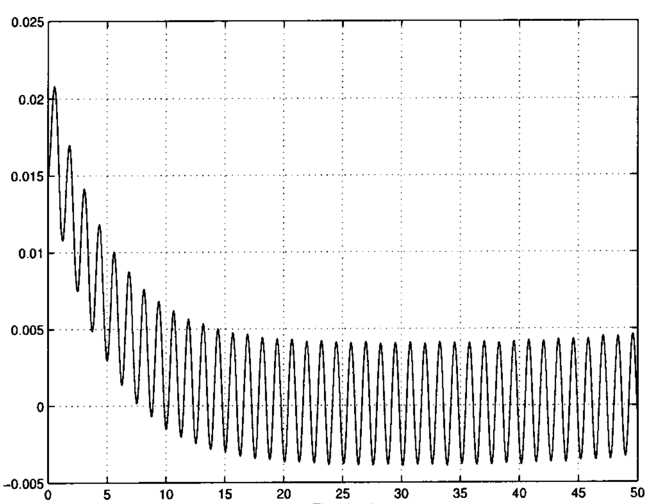

Fig. 4 The open-loop velocity at point $x=l$

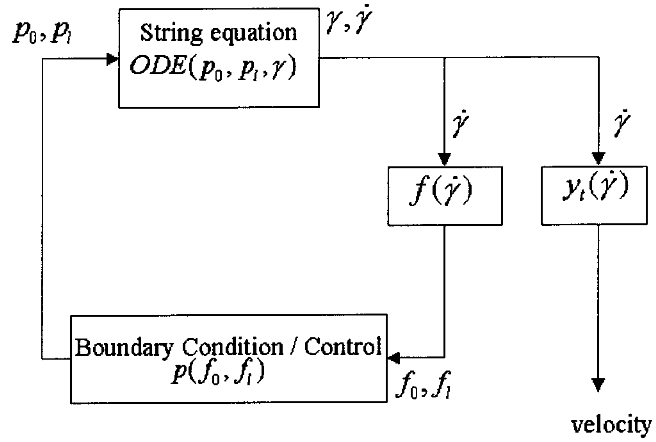

Fig. 5 The signal flow graph for robust control where $\dot{\gamma} \triangleq d\gamma(t)/dt$

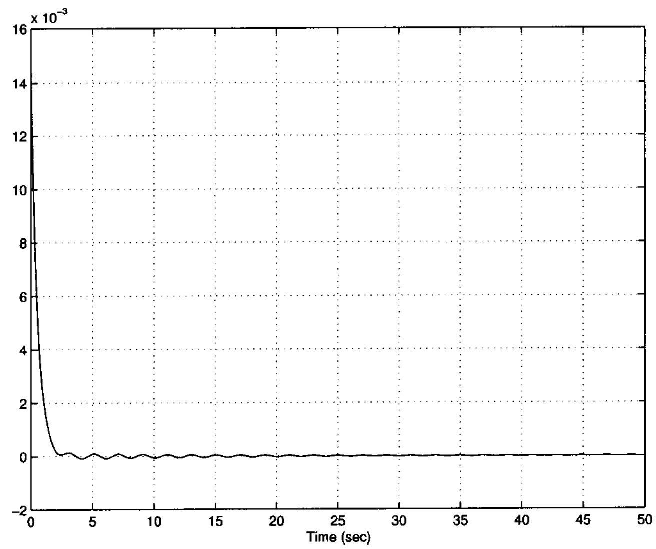

Fig. 6 The closed-loop velocity at point $x=0$ under robust control

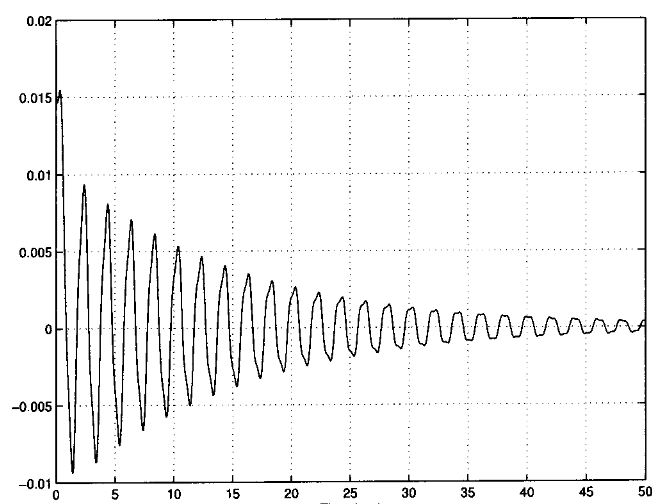

Fig. 7 The closed-loop velocity at point $x=0.5l$ under robust control

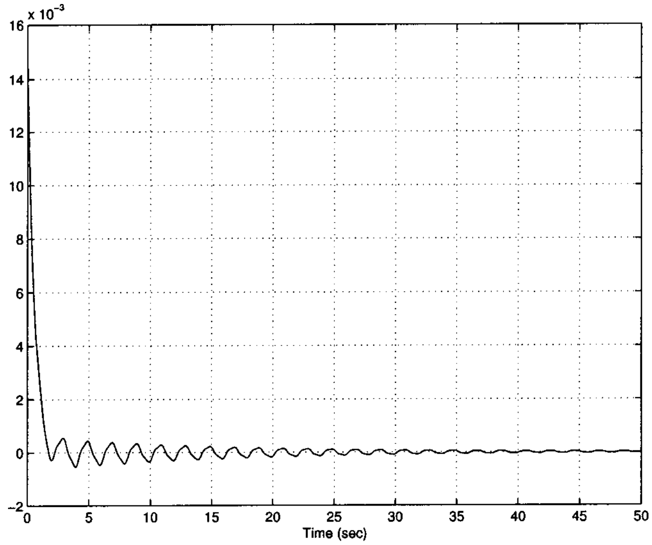

Fig. 8 The closed-loop velocity at point $x=l$ under robust control

$$\begin{bmatrix} \underline{W_1} & -\underline{W_3} \\ 0 & \underline{W_4} \end{bmatrix} \begin{bmatrix} \dfrac{d^2\gamma(t)}{dt^2} \\ \dfrac{d^2\underline{p}}{dt^2} \end{bmatrix} = \begin{bmatrix} -\underline{W_2}\cdot\gamma(t)+\underline{u_1} \\ \underline{u_2} \end{bmatrix}, \tag{23}$$

where $i=1,2,3,4$ *and* $j=1,2,3,4$,

$$\underline{W_1}(j,i)=\int_0^l \phi_i(x)\phi_j(x)dx,$$

$$\underline{W_2}(j,i)=-\int_0^l \frac{1+\delta\cos\left(\dfrac{i\pi x}{l}\right)}{1+4\delta\cos\left(\dfrac{i\pi x}{l}\right)} \frac{d^2\phi_i(x)}{dx^2}\phi_j(x)dx,$$

$$\underline{W_3}(j,1)=-\int_0^l \phi_j(x)dx+\frac{1}{l}\int_0^l x\phi_j(x)dx,$$

$$\underline{W_3}(j,2)=-\frac{1}{l}\int_0^l x\phi_j(x)dx,$$

$$\underline{u_1}(j,1)=-\frac{d\delta_b(t)}{dt}\int_0^l \phi_j(x)dx$$

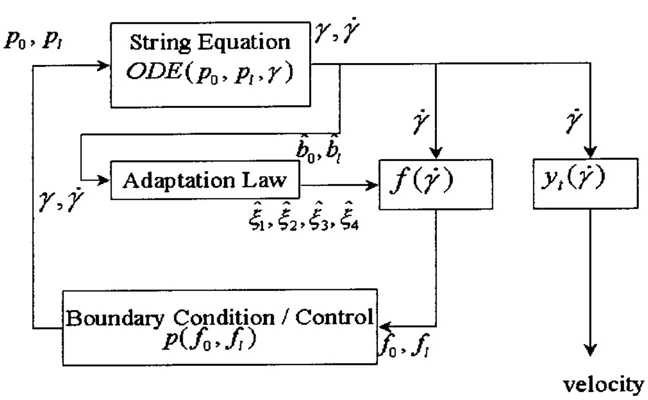

Fig. 9 The signal flow graph for robust adaptive control

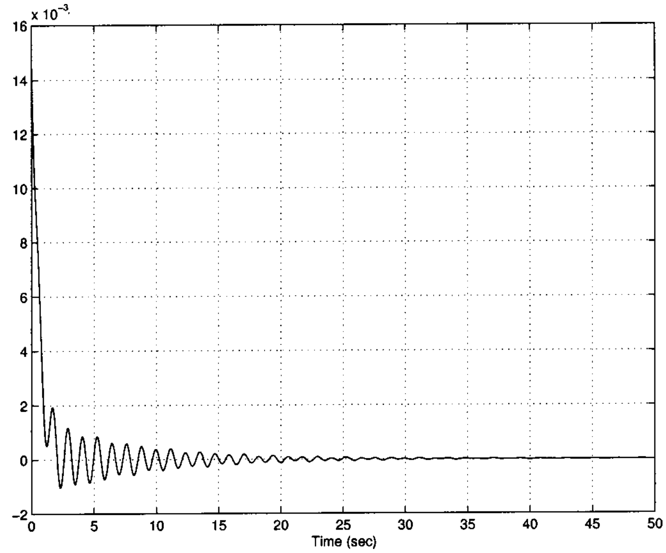

Fig. 10 The closed-loop velocity at point $x=0$ under robust adaptive control

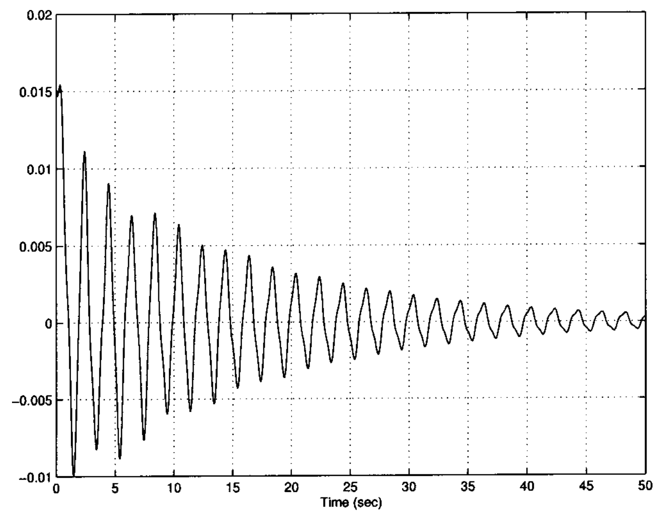

Fig. 11 The closed-loop velocity at point $x=0.5l$ under robust adaptive control

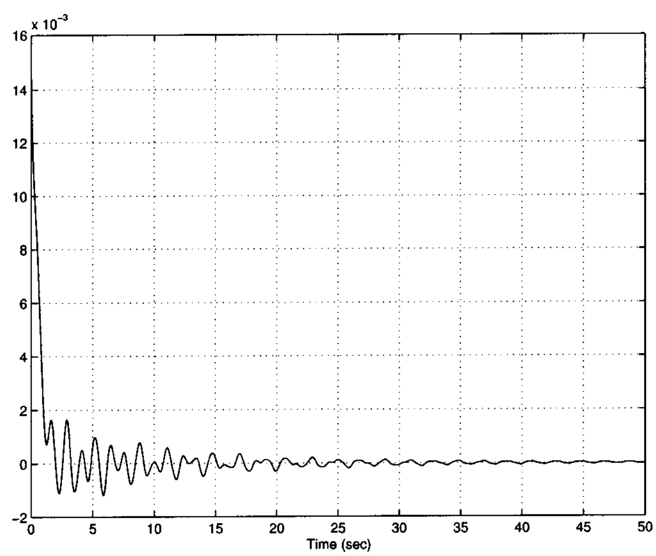

Fig. 12 The closed-loop velocity at point $x=l$ under robust adaptive control

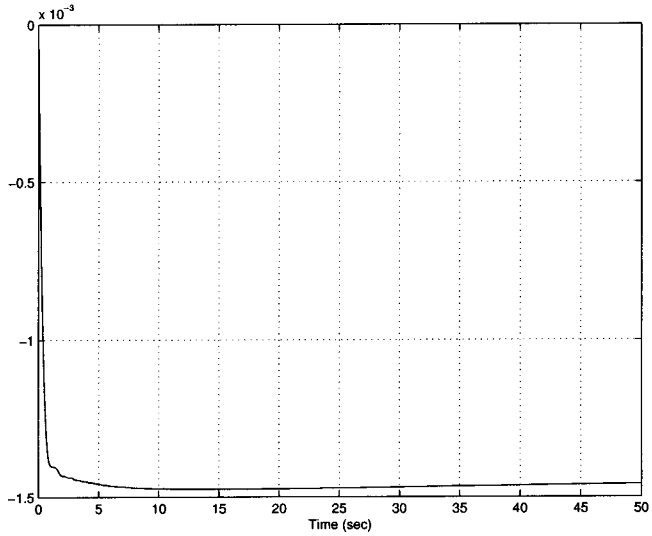

Fig. 13 The estimate of b_0

$$\gamma(t)=[\gamma_1(t) \quad \gamma_2(t) \quad \gamma_3(t) \quad \gamma_4(t)]^T,$$

$$\underline{p}(t)=\begin{bmatrix} p_0(t) \\ p_l(t) \end{bmatrix}, \quad \underline{W_4}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and

$$u_2=\begin{bmatrix} \dfrac{1}{M_0}\left[-M_0\dfrac{d^2\delta_b(t)}{dt^2}+f_0(t)-T\left.\dfrac{\partial y(x,t)}{\partial x}\right|_{x=0}-b_0\dfrac{d\underline{p}(t)}{dt}\right] \\ \dfrac{1}{M_l}\left[-M_l\dfrac{d^2\delta_b(t)}{dt^2}+f_l(t)-T\left.\dfrac{\partial y(x,t)}{\partial x}\right|_{x=l}-b_l\dfrac{d\underline{p}(t)}{dt}\right] \end{bmatrix}.$$

4.2 Simulation Parameters. In the simulation, the following setup is used:

- The initial conditions of ordinary differential equations in (23) are set to be zero.
- The following parameter values are used: $L=1$ m, $M_0=5$ kg, $M_l=5$ kg, $\omega_b=5$ rad/sec, and $T=0.1$ N.
- Values assumed by "uncertainties": $\eta_1=0.1$, $\eta_2=0.15$, $b_0=1$, $b_l=1$, and $\delta\in[0,0.25]$.

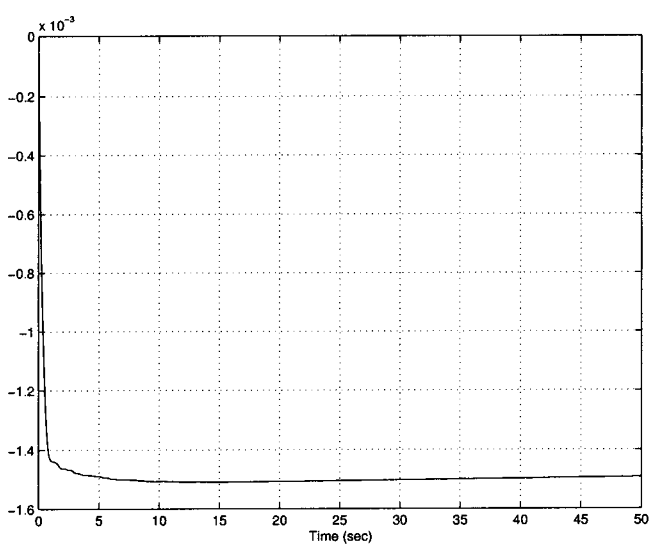

Fig. 14 The estimate of b_l

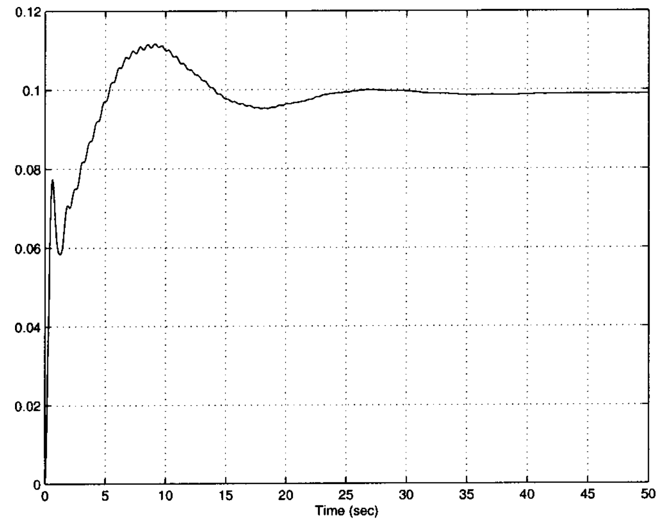

Fig. 15 The estimate of ξ_1

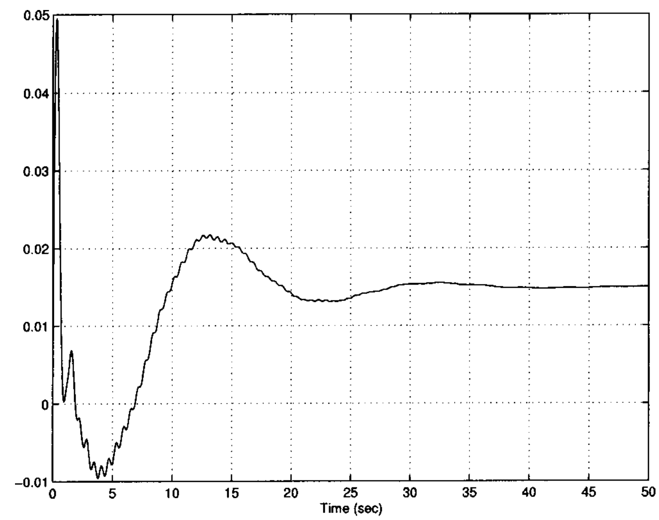

Fig. 16 The estimate of ξ_2

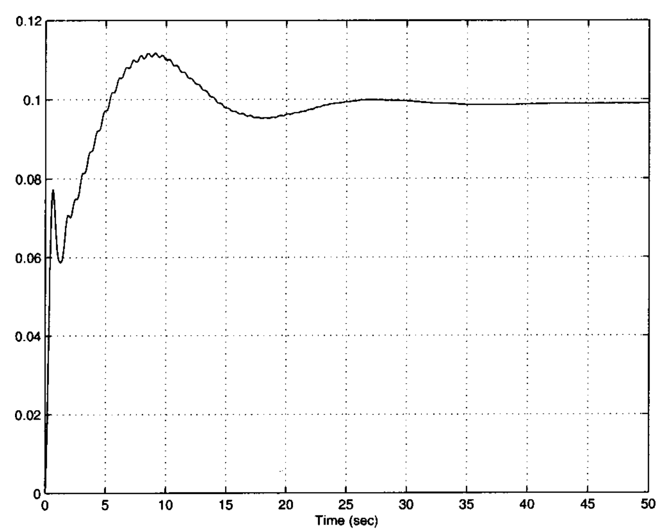

Fig. 17 The estimate of ξ_3

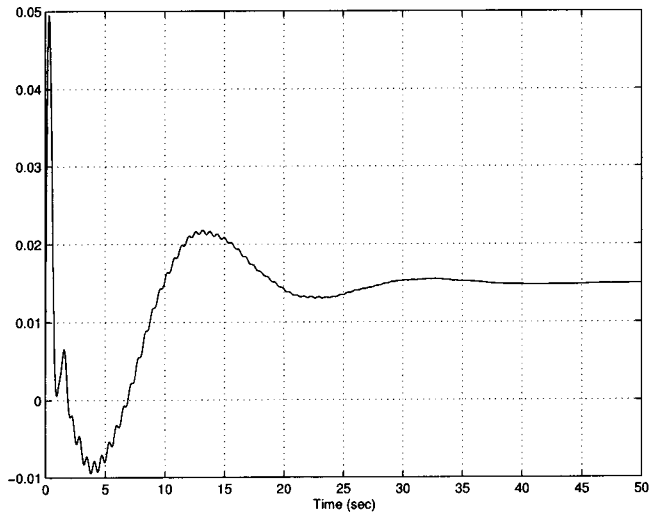

Fig. 18 The estimate of ξ_4

- Bounds: $\underline{c}_{T_0} = \overline{c}_{T_0} = 0.1$ N, $\underline{m} = 0.0545$ kg/m, and $\overline{m} = 0.1556$ kg/m.
- Design parameter: $\alpha(l) = 0.16$ with $\beta_1 = 0.059$ and $\beta_2 = 1$ (to satisfy inequalities (9) up to (14)).
- Control gains: $k_0 = 10$, $k_l = 10$, and $k_a = 10$.

Due to limited space, simulation results for the value of $\delta = 0.1$ will be presented although several runs were done to verify that the same behavior is observed for $\delta \in [0, 0.25]$.

4.3 Open-Loop Responses. Without applying any control, velocities at points $x = 0$, $0.5l$, l with respect to the reference frame are shown in Figs. 2, 3 and 4 respectively. As expected, the maximum oscillation occurred at the middle of the string.

The frequency of "input," 5 rad/sec or 0.8 Hz, is clearly seen in the open loop responses. Applying pointwise linear analysis, one can conclude that the first mode frequency is between 0.2 and 0.4, which is the "low frequency" element seen in Fig. 3.

While only 4 modes are simulated in this and the next two subsections, convergence is ensured by studying the magnitudes of time responses corresponding to these modes. Specifically, magnitudes for the first, second, third and fourth modes are of order 10^{-3}, 10^{-4}, 10^{-4}, and 10^{-6}, respectively. This verifies both our choice of the four modes and convergence of high modes. The same phenomenon is also observed in the closed-loop responses, and hence there is no spill-over problem.

4.4 Simulation of Robust Controller. By theorem 3, simulation of robust controls (15) and (16) should be done according to the signal flow graph shown in Fig. 5. Under the robust control, velocities measured at points $x = 0$, $0.5l$, l are shown in Figs. 6, 7 and 8, respectively. It is obvious that robust control is very effective in damping out the oscillations.

4.5 Simulation of Robust and Adaptive Controller. The proposed robust and adaptive controller can be simulated by adding an adaptation module, and the resulting signal flow graph is shown in Fig. 9. Specifically, differential equations in (23) will be integrated simultaneously with the adaptation laws in (19). Under the robust adaptive control, velocities measured at points $x = 0$, $0.5l$, l are shown in Figs. 10, 11 and 12, respectively. It is obvious that the proposed control is asymptotically stabilizing. And, the result is verified for all admissible values of the control gains. Estimates of b_0, b_l, ξ_1, ξ_2, ξ_3, and ξ_4 are shown in Figs. 13, 14, 15, 16, 17, and 18, respectively. As expected, estimation errors of system parameters (such as friction coefficients, magnitude and phase shift of the oscillation) do not converge to zero without a certain persistent excitation condition, which is of less importance than control in the setting of this paper. In case that the feedback is contaminated by noise, adaptation laws will have to be modified to be those of leakage-type so that they are robust [10].

5 Conclusion

In this paper, a new robust adaptive controller has been proposed for suppressing oscillations in a nonlinear string system. Global asymptotic stability is shown in control design. The proposed control is capable of stabilizing the nonlinear string without the perfect knowledge of string dynamics or the motion of its transporter. Compared to the robust controller in [1], all system dynamics are considered. The effectiveness of the proposed controller and the existing robust controller has been demonstrated via simulation.

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Appendix A: Proof of Theorem 2

To compensate for unknown parameters b_0, b_l, η_1 and η_2 while taking into account the effect of control mechanisms, the following Lyapunov function candidate is chosen:

$$V(t) = V_s + V_0 + V_l + L(t),$$

where V_s is given by (8), positive definite functions V_0, V_l and $L(t)$ (with respect to their arguments) are defined by

$$V_0(t) = \frac{1}{2} M_0 \left[\frac{\partial Y(0,t)}{\partial t} \right]^2,$$

$$V_l(t) = \frac{1}{2} M_l \left[\frac{\partial Y(l,t)}{\partial t} + \frac{3}{8} \alpha(l) \left. \frac{\partial y(x,t)}{\partial x} \right|_{x=l} \right]^2,$$

$$L(t) = \frac{1}{2k_a} [b_0 - \hat{b}_0(t)]^2 + \frac{1}{2k_l} [b_l - \hat{b}_l(t)]^2 + \frac{1}{2k_a} \sum_{i=1}^{4} [\xi_i - \hat{\xi}_i(t)]^2,$$

$\xi_1 = b_0 \eta_1 \cos \eta_2$, $\xi_2 = b_0 \eta_1 \sin \eta_2$, $\xi_3 = b_l \eta_1 \cos \eta_2$, $\xi_4 = b_l \eta_1 \sin \eta_2$, and $\hat{\xi}$ is the estimate of ξ.

It follows from conditions (12) to (14) that

$$\begin{aligned} \frac{dV_s(t)}{dt} \leqslant & -2T_0(0) \left. \frac{\partial y(x,t)}{\partial x} \right|_{x=0} \frac{\partial Y(0,t)}{\partial t} \\ & + 2T(l,t) \left. \frac{\partial y(x,t)}{\partial x} \right|_{x=l} \left[\frac{\partial Y(l,t)}{\partial t} + \frac{3}{8} \alpha(l) \left. \frac{\partial y(x,t)}{\partial x} \right|_{x=l} \right] \\ & - \frac{1}{4} \alpha(l) T_0(l) \left[\left. \frac{\partial y(x,t)}{\partial x} \right|_{x=l} \right]^2 + \frac{1}{2} \alpha(l) m(l) \left[\frac{\partial Y(l,t)}{\partial t} \right]^2 \\ & - \frac{\epsilon}{2l} \int_0^l \left\{ \left[\frac{\partial Y(x,t)}{\partial t} \right]^2 + \left[\frac{\partial y(x,t)}{\partial x} \right]^2 + \left[\frac{\partial y(x,t)}{\partial x} \right]^4 \right\} dx. \end{aligned}$$

Applying boundary conditions (6) and (7) and substituting controls (17) and (18), we have

$$\begin{aligned} \frac{dV_0(t)}{dt} = \frac{\partial Y(0,t)}{\partial t} \Bigg[& -k_0 \frac{\partial Y(0,t)}{\partial t} + 2T(0,t) \left. \frac{\partial y(x,t)}{\partial x} \right|_{x=0} \\ & + (\hat{b}_0 - b_0) \frac{\partial Y(0,t)}{\partial t} - (\hat{\xi}_1 - \xi_1) \sin(\omega_b t) \\ & - (\hat{\xi}_2 - \xi_2) \cos(\omega_b t) \Bigg], \end{aligned}$$

$$\frac{dV_l(t)}{dt}=\left[\frac{\partial Y(l,t)}{\partial t}+\frac{3}{8}\alpha(l)\frac{\partial y(x,t)}{\partial x}\bigg|_{x=l}\right]\left\{-2T(l,t)\frac{\partial y(x,t)}{\partial x}\bigg|_{x=l}-k_l\left[\frac{\partial Y(l,t)}{\partial t}+\frac{3}{8}\alpha(l)\frac{\partial y(x,t)}{\partial x}\bigg|_{x=l}\right](\hat{b}_l-b_l)\frac{\partial Y(l,t)}{\partial t}-(\hat{\xi}_3-\xi_3)\sin(\omega_b t)-(\hat{\xi}_4-\xi_4)\cos(\omega_b t)\right\},$$

and

$$\frac{dL(t)}{dt}=-\frac{1}{k_a}[b_0-\hat{b}_0(t)]\frac{d\hat{b}_0(t)}{dt}-\frac{1}{k_a}[b_l-\hat{b}_l(t)]\frac{d\hat{b}_l(t)}{dt}-\frac{1}{k_a}\sum_{i=1}^{4}[\xi_i-\hat{\xi}_i(t)]\frac{d\hat{\xi}_i(t)}{dt}.$$

Summing up the above four inequality/equalities, we can conclude that

$$\begin{aligned}\frac{dV(t)}{dt}\leqslant&-\frac{\epsilon}{2l}\int_0^l\left\{\frac{\partial Y(x,t)}{\partial t}+\left[\frac{\partial y(x,t)}{\partial x}\right]^2+\left[\frac{\partial y(x,t)}{\partial x}\right]^4\right\}dx-\frac{1}{4}\alpha(l)T_0(l)\left[\frac{\partial y(x,t)}{\partial x}\bigg|_{x=l}\right]^2+\frac{1}{2}\alpha(l)m(l)\left[\frac{\partial Y(l,t)}{\partial t}\right]^2-k_0\left[\frac{\partial Y(0,t)}{\partial t}\right]^2\\&+(\hat{b}_0-b_0)\left[\frac{\partial Y(0,t)}{\partial t}\right]^2-(\hat{\xi}_1-\xi_1)\sin(\omega_b t)\frac{\partial Y(0,t)}{\partial t}-(\hat{\xi}_2-\xi_2)\cos(\omega_b t)\frac{\partial Y(0,t)}{\partial t}\\&+(\hat{b}_l-b_l)\frac{\partial Y(l,t)}{\partial t}\left[\frac{\partial Y(l,t)}{\partial t}+\frac{3}{8}\alpha(l)\frac{\partial y(x,t)}{\partial x}\bigg|_{x=l}\right]-(\hat{\xi}_3-\xi_3)\sin(\omega_b t)\left[\frac{\partial Y(l,t)}{\partial t}+\frac{3}{8}\alpha(l)\frac{\partial y(x,t)}{\partial x}\bigg|_{x=l}\right]\\&-(\hat{\xi}_4-\xi_4)\cos(\omega_b t)\left[\frac{\partial Y(l,t)}{\partial t}+\frac{3}{8}\alpha(l)\frac{\partial y(x,t)}{\partial x}\bigg|_{x=l}\right]-k_l\left[\frac{\partial Y(l,t)}{\partial t}+\frac{3}{8}\alpha(l)\frac{\partial y(x,t)}{\partial x}\bigg|_{x=l}\right]^2-\frac{1}{k_a}[b_0-\hat{b}_0(t)]\frac{d\hat{b}_0(t)}{dt}\\&-\frac{1}{k_a}[b_l-\hat{b}_l(t)]\frac{d\hat{b}_l(t)}{dt}-\frac{1}{k_a}\sum_{i=1}^{4}[\xi_i-\hat{\xi}_i(t)]\frac{d\hat{\xi}_i(t)}{dt}.\end{aligned}$$

Now, substituting the adaptation laws into the right hand side of the above inequality yields

$$\frac{dV(t)}{dt}\leqslant-\frac{2k_0}{M_0}V_0(t)-\frac{2k_l}{M_l}V_l(t)-\frac{1}{4}\alpha(l)T_0(l)\left[\frac{\partial y(x,t)}{\partial x}\bigg|_{x=l}\right]^2+\frac{1}{2}\alpha(l)m(l)\left[\frac{\partial Y(l,t)}{\partial t}\right]^2-\frac{\epsilon}{2l}\int_0^l\left\{\frac{\partial Y(x,t)}{\partial t}+\left[\frac{\partial y(x,t)}{\partial x}\right]^2+\left[\frac{\partial y(x,t)}{\partial x}\right]^4\right\}dx.$$

It is obvious that, under inequality (10), gain k_l satisfying $k_l \geqslant 2\alpha(l)\overline{m}$ can be chosen to establish the following inequality:

$$\frac{1}{2}k_l\left[\frac{\partial Y(l,t)}{\partial t}+\frac{3}{8}\alpha(l)\frac{\partial y(x,t)}{\partial x}\bigg|_{x=l}\right]^2+\frac{1}{4}\alpha(l)T_0(l)\times\left[\frac{\partial y(x,t)}{\partial x}\bigg|_{x=l}\right]^2\geqslant\frac{1}{2}\alpha(l)m(l)\left[\frac{\partial Y(l,t)}{\partial t}\right]^2.$$

Therefore, we have

$$\frac{dV(t)}{dt}\leqslant-\epsilon_v(V_s+V_0+V_l),$$

for some constant $\epsilon_v>0$, from which asymptotic stability of the state in $V_s(t)$ can be claimed [10]. Q.E.D.

Appendix B: Proof of Theorem 3

To yield homogeneous boundary conditions, assume that displacement $y(x,t)$ can be expressed as

$$y(x,t)=v(x,t)+H(x,t),$$

where $v(x,t)$ is subject to the boundary conditions

$$v(0,t)=0,$$

and

$$v(l,t)=0.$$

Furthermore, it is assumed that $H(x,t)$ be a linear-like function of form

$$H(x,t)=A(t)x+B(t) \tag{B1}$$

where $H(x,t)$ satisfies

$$H(0,t)=p_0(t)$$

and

$$H(l,t)=p_l(t).$$

Solving first functions $A(t)$ and $B(t)$ and then function $H(x,t)$, we have

$$y(x,t)=v(x,t)+p_0(t)+\frac{x}{l}[p_l(t)-p_0(t)], \tag{B2}$$

from which Eq. (21) can be concluded.

Substituting ($B2$) into Eq. (20), we have

$$\frac{\partial^2 v(x,t)}{\partial t^2}-\frac{T}{m(x)}\frac{\partial^2 v(x,t)}{\partial x^2}=-\frac{d^2p_0(t)}{dt^2}-\frac{x}{l}\left[\frac{d^2p_l(t)}{dt^2}-\frac{d^2p_0(t)}{dt^2}\right]-\frac{d\delta_b(t)}{dt}, \tag{B3}$$

and the corresponding eigenvalue problem is

$$\frac{\partial^2 v(x,t)}{\partial t^2}-\frac{T}{m(x)}\frac{\partial^2 v(x,t)}{\partial x^2}=0. \tag{B4}$$

The method of variable separation can now be applied by setting

$$v(x,t)=\phi(x)\gamma(t). \tag{B5}$$

Under assumption 4, the eigenvalue problem after substituting $(B5)$ into Eq. $(B4)$ becomes the following differential equation: for some constant C_i,

$$\frac{d^2\phi_i}{dx^2}+C_i\frac{1+\delta\cos\frac{i\pi x}{l}}{1+4\delta\cos\frac{i\pi x}{l}}\phi_i=0. \tag{B6}$$

Using the boundary conditions $\phi(0)=0$ and $\phi(l)=0$, we find $C_i=(i\pi/l)^2$ and

$$\phi_i(x)=\sin\frac{i\pi x}{l}\left(1+\delta\cos\frac{i\pi x}{l}\right)$$

where $i=1,2,3,\ldots$.

Letting

$$v(x,t)=\sum_{i=1}^{\infty}\phi_i(x)\gamma_i(t),$$

we can rewrite Eq. $(B3)$ as

$$\sum_{i=1}^{\infty}\phi_i(x)\frac{d^2\gamma_i(t)}{dt^2}-\sum_{i=1}^{\infty}\frac{1+\delta\cos\frac{i\pi x}{l}}{1+4\delta\cos\frac{i\pi x}{l}}\frac{d^2\phi_i(x)}{dx^2}\gamma_i(t)$$

$$=-\frac{d^2p_0(t)}{dt^2}-\frac{x}{l}\left[\frac{d^2p_l(t)}{dt^2}-\frac{d^2p_0(t)}{dt^2}\right]-\frac{d\delta_b(t)}{dt}.$$

Multiplying both sides of the above equation by $\phi_j(x)$ $(j=1,2,3,\ldots)$ and then integrating with respect to x from 0 to l yield

$$\sum_{i=1}^{\infty}\frac{d^2\gamma_i(t)}{dt^2}\int_0^l\phi_i(x)\phi_j(x)dx=\sum_{i=1}^{\infty}\gamma_i(t)\int_0^l\frac{1+\delta\cos\frac{i\pi x}{l}}{1+4\delta\cos\frac{i\pi x}{l}}\frac{d^2\phi_i(x)}{dx^2}\phi_j(x)dx-\left[\frac{d^2p_0(t)}{dt^2}+\frac{d\delta_b(t)}{dt}\right]\int_0^l\phi_j(x)dx$$

$$-\left[\frac{d^2p_l(t)}{dt^2}-\frac{d^2p_0(t)}{dt^2}\right]\int_0^l\frac{x}{l}\phi_j(x)dx.$$

By choosing the first four modes and by applying boundary conditions (6) and (7), one can conclude Eq. (23) from the above equation. Q.E.D.

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