Remark: i): Our result was inspired by the work of Blondel on robust stable polynomials omitting two values. In [7], a bound connecting the range of the *leading* coefficient to the absolute value of the second coefficient was established.

The diameter bound derived previously holds for all real Schur-stable interval polynomials of arbitrary degree. The uniform bound is sharp at least for the diameter of the second coefficient $|a_{n-1}^{+} - a_{n-1}^{-}|$ as the following example shows.

Example (Reproduced from [4]): Consider the family of polynomials of degree $n \geq 2$ given as the real polynomials $k(z) = (1+\epsilon) \cdot z^n + q \cdot z^{n-1} + z^{n-2}$ where q varies in the interval $[-2, 2]$, and ϵ is positive. The family is Schur-stable as the nonzero roots belong to the quadratic $(1+\epsilon) \cdot z^2 + q \cdot z + 1$ and lie inside the unit circle. Normalizing the family to be monic the second coefficient's diameter becomes $4/(1+\epsilon)$. Thus, the constant 4 in Theorem 1 is the best possible.

Remark: ii): The perturbation limits of Theorem 1 might be refined using explicit expressions for the higher coefficients g_v; see [3]. It might be instructive to compare the above diameter bound to the exact (symmetric) perturbation bound (based on the complete polynomial information) for real robust, Schur-stable polynomials given in [8].

III. CONCLUDING REMARKS

A sharp constant bound for the coefficient diameter of real Schur-stable interval polynomials has been established. The bound holds uniformly for all polynomials and all coefficients. It may be used as a check criterion for stability given only limited information, and for the explicit study of perturbation effects on the allowable coefficient range. Another application is the study of gain margins for invariant structures which will be presented in detail in a forthcoming paper.

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Global Stabilization and Convergence of Nonlinear Systems With Uncertain Exogenous Dynamics

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***Abstract*—In this note, a class of nonlinear uncertain systems are considered, and uncertainties in the systems are assumed to be generated by exogenous dynamics. Robust control is designed by employing nonlinear observers to estimate the uncertainties. It is shown that, if a partial knowledge of the exogenous system is available and its known dynamics meet certain conditions or if input channel of the plant has certain properties, global stability and global estimation convergence can be achieved. In the latter case, the results on stability and convergence hold even if exogenous dynamics are completely unknown but bounded by some known function.**

***Index Terms*—Exosystem, Lyapunov direct method, robust control, uncertainty estimation.**

I. INTRODUCTION

Uncertainties are commonly present in most control systems due to modeling errors, parameter variations, unknown dynamics, disturbances, and unmodeled dynamics. To ensure stability and performance, feedback control must be robust. Typically, a nonlinear robust control is designed by first describing size variations of the uncertainties using a norm bound (or bounding function) and then compensating for all the variations within the bound by size domination in a Lyapunov argument [5], [1]. Many existing results on robust stability and robust control of nonlinear uncertain systems can be found in [6], [7], [9], [4], [18], and [12].

While domination by a control is essential to achieve robust stability and performance, precisely knowing the bound on uncertainties is often practically impossible. Since underestimating the bound will jeopardize robustness and since overestimating the bound will make robust control too conservative, estimating uncertainties or their bounding function online while guaranteeing robustness would be the alternative. There have been several results along this line of research. If the bounding function has a known functional expression and can be parameterized linearly in terms of finite unknown constants, an adaptive version of robust control was designed in [2] to adaptively estimate the unknowns in the bounding function. Recently, extensions have been made by using robust adaptive control [3] and robust and adaptive controls [14] so that robust stability can be achieved for systems whose uncertain dynamics or their bounding functions have nonlinear parameterization in terms of unknown constants.

In many applications, source(s) of uncertainties in the plant can be identified. In such cases, plant uncertainties can be modeled as output (or the state) of an exogenous system whose dynamics may be either completely or partially unknown. By adopting this dynamic augmentation, uncertainties or their bounding function can be estimated. It was shown in [13] that, if uncertainty bounding function is parameterized linearly in terms of time varying outputs of a known or partially known exogenous system and if known dynamics of the exogenous system has certain stability properties, uncertainties can be estimated using an adaptation law and a globally stabilizing robust control can be found. If explicit and useful stability property of the exogenous system is not

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available, it was shown in [15] that robust control based on nonlinear observer can be designed to ensure semiglobal stability of uniform ultimate boundedness.

In this note, we search for observer-based robust controls that ensures not only global robust stability but also global convergence of uncertainty estimation. The proposed controls are designed by exploring the properties of both the plant and the exogenous system, and two different robust controls are obtained: one based on properties of the partially known exogenous system, and the other based solely on properties of the input channel of the plant. Compared to the existing results, the proposed controls achieve better performance (in particular, uncertainty estimation is globally convergent), and the proposed design approach is different. Specifically, global convergent observers along that in [20] are used in this note, while adaptation laws (of standard forms in [16] and [9]) are used in [13] and high-gain observers (similar to those in [8]) are used in [15].

It is worth noting that unstructured nonlinear uncertainties can directly be estimated online to guarantee semi-global robust stability [17]. By restricting uncertainties to be those generated by an exogenous system, performance is improved in this note to be global stability for the state and global convergence for estimation.

The note is organized as follows. In Section II, a class of nonlinear plants and their associated exogenous systems are described, the control problem is defined, and basic technical assumptions are introduced. In Section III, an observer-based robust control is designed for the case that the exogenous system is partially known and its nominal part has certain stability properties. In Section IV, the case of the exogenous system being completely uncertain is considered, and an observer-based robust control is proposed based on certain condition on input channel of the plant. Conclusions are drawn in Section V.

II. Problem Formulation

In this note, an uncertain system under consideration is of form

$$\dot{x} = F(x,t) + B(x,t)[w(x,\upsilon,t) + u] \tag{1}$$

where $x \in \Re^n$ is the state, $u \in \Re^m$ is the control, $w(\cdot)$ has a known function expression and could be nonlinear in υ, and $\upsilon \in \Re^l$ is an uncertainty generated by the exogenous system

$$\dot{\upsilon} = G(\upsilon,x,t) + \Delta G(\upsilon,x,t). \tag{2}$$

The objective of this note is to develop sets of simple conditions on uncertain plant (1) and on exogenous dynamics (2) under which an observer-based robust control can be designed to ensure global stability of the state and global convergence of estimating uncertainty $\upsilon(t)$. To this end, let x_i, $f_i(\cdot)$ and $b_i(\cdot)$ are the ith row of x, $F(\cdot)$, and $B(\cdot)$, respectively. Also, let

$$x = [\phi^T, z^T]^T \quad \phi \triangleq [x_1 \quad x_2 \quad \cdots \quad x_{n-p}]^T$$
$$z \triangleq [x_{n-p+1}, \ldots x_n]^T \quad F(x,t) = \left[F_1^T(x,t) F_2^T(x,t)\right]^T$$

and

$$B(x,t) = \left[B_1^T(x,t) \quad B_2^T(x,t)\right]^T$$

where $p \geq 0$ is an integer, and $z, F_2(\cdot)$ and $B_2(x,t)$ are the bottom pth-order vector blocks (all of which are empty if $p = 0$) in x, $F(x,t)$ and $B(x,t)$, respectively.

Remark 2.1: In this note, the state x of the plant is assumed to be measured. Observers will be designed to generate $\hat{\upsilon}$, an estimate of uncertainty vector υ, and an estimation-based control $u = u(x,\hat{\upsilon},t)$ will be synthesized. To gauge the effectiveness of uncertainty estimation, one can use an auxiliary observer to generate $\hat{z}$ even though z is available. This is because, according to (1)

$$B_2(x,t)[w(x,\upsilon,t) + u(x,\hat{\upsilon},t)] = \dot{z} - F_2(x,t)$$

(as in the case of a reduced-order observer designed for linear systems) z can be viewed as "output" of the exogenous system, and hence $(z-\hat{z})$ can be used as an "output estimation error." In short, an observer of υ is a closed loop one if it utilizes "output feedback" of $(z-\hat{z})$; otherwise, it is open loop. Note that $p = m$ is a common choice under which, if matrix $B_2(x,t)$ is invertible, impact function $w(x,\upsilon,t)$ of uncertainty υ on the plant can be solved (with the aid of the auxiliary observer). More discussions on the choices of p can be found in Remarks 3.8, 4.2, and 4.4. $\diamond$

The following two technical assumptions are commonly made in robust control. Assumption 1 says that the origin $x = 0$ is globally asymptotically stable for the uncontrolled nominal system of (1), $\dot{x} = F(x,t)$, and that the fictitious system $\dot{x} = F(x,t) + u'$ is input-to-state stable [19] with respect to u'.

Assumption 1: There exists a C^1 function $V(x,t)$: $\Re^n \times \Re \to \Re^+$ such that

$$\begin{aligned}\gamma_1(\|x\|) \leq V(x,t) \leq \gamma_2(\|x\|) \quad & \frac{\partial V(x,t)}{\partial t} \\ & + \nabla_x^T V(x,t) F(x,t) \\ & \leq -\gamma_3(\|x\|) \quad \left\|\nabla_x^T V(x,t)\right\| \leq \gamma_4(\|x\|)\end{aligned} \tag{3}$$

where γ_i: $\Re^+ \to \Re^+$ are class $\mathcal{K}_\infty$ functions and, for some constants $\beta_1 > 0$ and $0 < \beta_2 < 1$

$$\gamma_4(\|x\|) \leq \beta_1 \gamma_3^{\beta_2}(\|x\|). \tag{4}$$

Furthermore, it is assumed that Lyapunov function $V(x,t)$ be found.

Assumption 2: All functions in system (1) and (2) are Caratheodory, locally Lipschitzian with respect to x and υ, uniformly bounded with respect to t, and locally uniformly bounded with respect to x or υ.

The second assumption is needed to ensure existence of a classical solution under a continuous feedback control. As a result, it can be assumed without loss of any generality that

$$\|B(x,t)\| \leq c_b(\|x\|) \quad \text{and} \quad \|B_2(x,t)\| \leq c_b'(\|x\|) \tag{5}$$

where $c_b(\cdot)$ and $c_b'(\cdot)$ are nonnegative functions.

Without knowing a bounding function on the size of uncertainty υ, robust stabilization will require more conditions, and so will convergence of estimation. In the next two sections, two sets of such conditions are found.

III. Control Design Based on Properties of the Exogenous System

In this section, we proceed with robust control design by exploiting properties of the exogenous system if such properties are known apriori. Among the following three assumptions, Assumption 4A on a certain stability property of the nominal exogenous system is the key. As will be shown, Lyapunov matrix P_g in the assumption is only used in the analysis but not in the proposed control (to be given in the subsequent theorem), thus it does not have to be solved explicitly as long as $\lambda_1(\cdot)$ and c_g are available. Since function expression of $w(\cdot)$ is known, Assumption 5A can easily be verified. If the proposed control is made adaptive, constants β_3 and β_4 in Assumption 3 need not be known either.

Assumption 3: If x remains in a bounded subset (within an invariant set of $x(\tau), \tau \in [t_0, T]$ for an arbitrary $T \geq t_0$), state v of exogenous system (2) is uniformly bounded. Furthermore, there exist known constants $\beta_3 \geq 0$ and $\beta_4 > 0$ such that

$$\|\Delta G(v, x, t)\| \leq \beta_3 \gamma_3^{\beta_4}(\|x\|). \tag{6}$$

Assumption 4A: Function $G(\cdot)$ has the property that, for all bounded x and for all (v_1, v_2, t)

$$(v_1 - v_2)^T P_g[G(v_1, x, t) - G(v_2, x, t)] \leq -\lambda_1(\|x\|)\|v_1 - v_2\|^2 \tag{7}$$

where $\lambda_1(\|x\|) > 0$ is a positively valued or positive–definite known function, P_g is a positive–definite matrix, and $\|P_g\| \leq c_g$ for some known constant c_g.

Assumption 5A: Function $w(\cdot)$ has the property that, for all bounded x and for all (v_1, v_2, t)

$$\|w(x, v_1, t) - w(x, v_2, t)\| \leq \beta_5\|v_1 - v_2\|\gamma_3^{\beta_6}(\|x\|) \tag{8}$$

where $\beta_5 \geq 0$ and $0 \leq \beta_6 < 1$ are constants.

Under the assumptions, global stability and convergence stated in the following theorem can be achieved under a robust observer-based control. In contrast, robust adaptive control in [13] does not yield any result on convergence of uncertainty estimation, and observer-based robust control in [15] cannot ensure global stability and its closed-loop performance is characterized by either uniform ultimate boundedness or uniform boundedness.

Theorem 1: Consider system (1) satisfying Assumptions 1–3, 4A, and 5A. If there exist constants $c_1 \geq 0$ and $c_2 > 0$ such that, for all $s > 0$

$$\lambda_1(s)\gamma_3(s) \geq \frac{9}{4}\left[c_2\beta_1\beta_5 c_b(s)\gamma_3^{\beta_2+\beta_6}(s) + \frac{\beta_3 c_g}{c_2}\frac{\gamma_3^{\beta_4}(s)}{\gamma_1^{c_1}(s)}\right]^2 \tag{9}$$

then global asymptotic stability of state x, global convergence of estimating uncertainty (i.e., convergence of $\tilde{v} = v - \hat{v}$), and boundedness of all variables are ensured under the observer-based robust control

$$u = -w(x, \hat{v}, t) \tag{10}$$

where $\hat{z}$ and $\hat{v}$ are defined by

$$\dot{\hat{z}} = k(x)(z - \hat{z}) + F_2(x, t) \tag{11}$$
$$\dot{\hat{v}} = G(\hat{v}, x, t) \tag{12}$$

and

$$k(x) = \frac{9\beta_5^2\gamma_3^{2\beta_6}(\|x\|)[c_b'(\|x\|)]^2}{4\lambda_1(\|x\|)} \tag{13}$$

is a scalar gain function.

Proof: It follows from control (10) and from observer (11) and (12) that dynamics of estimation error are

$$\dot{\tilde{z}} = -k(x)\tilde{z} + B_2(x, t)[w(x, v, t) - w(x, \hat{v}, t)] \tag{14}$$
$$\dot{\tilde{v}} = G(v, x, t) - G(\hat{v}, x, t) + \Delta G(v, x, t). \tag{15}$$

Consider the Lyapunov function

$$L(x, \tilde{z}, \tilde{v}, t) = \frac{\alpha_1}{1+\alpha_2}V^{1+\alpha_2}(x, t) + \frac{1}{2}\alpha_3\|\tilde{z}\|^2 + \frac{1}{2}\alpha_3\tilde{v}^T P_g\tilde{v}$$

where $\alpha_1, \alpha_3 > 0$, and $\alpha_2 \geq 0$ are parameters used only in stability analysis. It follows from (1), (3), (5)–(8), (14), and (15) that

$$\begin{aligned}\dot{L} \leq& -\alpha_1 V^{\alpha_2}(x, t)\gamma_3(\|x\|) + \alpha_1 V^{\alpha_2}(x, t)\nabla_x^T V(x, t)B(x, t)\\ &\times [w(x, v, t) - w(x, \hat{v}, t)] - \alpha_3 k(x)\|\tilde{z}\|^2\\ &+ \alpha_3\tilde{z}^T B_2(x, t)[w(x, v, t) - w(x, \hat{v}, t)]\\ &- \alpha_3\lambda_1(\|x\|)\|\tilde{v}\|^2 + \alpha_3\tilde{v}^T P_g\Delta G(v, x, t)\\ \leq& -\alpha_1 V^{\alpha_2}(x, t)\gamma_3(\|x\|) + \alpha_1\beta_5 c_b(\|x\|)V^{\alpha_2}(x, t)\\ &\times \gamma_4(\|x\|)\gamma_3^{\beta_6}(\|x\|)\|\tilde{v}\| - \alpha_3 k(x)\|\tilde{z}\|^2\\ &+ \alpha_3\beta_5\gamma_3^{\beta_6}(\|x\|)c_b'(\|x\|)\|\tilde{z}\|\|\tilde{v}\|\\ &- \alpha_3\lambda_1(\|x\|)\|\tilde{v}\|^2 + \alpha_3\beta_3 c_g\gamma_3^{\beta_4}(\|x\|)\|\tilde{v}\|.\end{aligned} \tag{16}$$

It follows from the Cauchy–Schwarz inequality that

$$-\frac{1}{3}\alpha_1 V^{\alpha_2}\gamma_3(\|x\|) + \alpha_1\beta_5 c_b(\|x\|)V^{\alpha_2}\gamma_4(\|x\|)\gamma_3^{\beta_6}(\|x\|)\|\tilde{v}\| + \alpha_3\beta_3 c_g\gamma_3^{\beta_4}(\|x\|)\|\tilde{v}\| - \frac{1}{3}\alpha_3\lambda_1(\|x\|)\|\tilde{v}\|^2 \leq 0$$

and that

$$-\frac{1}{3}\alpha_3 k(x)\|\tilde{z}\|^2 + \alpha_3\beta_5\gamma_3^{\beta_6}(\|x\|)c_b'(\|x\|)\|\tilde{z}\|\|\tilde{v}\| - \frac{1}{3}\alpha_3\lambda_1(\|x\|)\|\tilde{v}\|^2 \leq 0$$

provided that

$$\begin{aligned}&\frac{2}{3}\sqrt{\alpha_1\alpha_3\lambda_1(\|x\|)V^{\alpha_2}\gamma_3(\|x\|)}\\ &\quad\geq \alpha_1\beta_5 c_b(\|x\|)V^{\alpha_2}\gamma_4(\|x\|)\gamma_3^{\beta_6}(\|x\|)\\ &\qquad + \alpha_3\beta_3 c_g\gamma_3^{\beta_4}(\|x\|)\end{aligned} \tag{17}$$

and that

$$\frac{2}{3}\sqrt{k(x)\lambda_1(\|x\|)} \geq \beta_5\gamma_3^{\beta_6}(\|x\|)c_b'(\|x\|). \tag{18}$$

It follows from the choice of $k(x)$ and from (9) and (4) that inequalities (17) and (18) hold for some α_i. Therefore, we have

$$\dot{L} \leq -\frac{2}{3}\alpha_1\gamma_1^{\alpha_2}(\|x\|)\gamma_3(\|x\|) - \frac{2}{3}\alpha_3 k(x)\|\tilde{z}\|^2 - \frac{1}{3}\alpha_3\lambda_1(\|x\|)\|\tilde{v}\|^2. \tag{19}$$

Stability, convergence, and boundedness can be readily concluded from (19) and from Assumption 3. □

The following remarks provide further elaboration or relaxation on stability analysis and conditions.

Remark 3.1: To ensure that gain $k(x)$ in (13) is bounded at the origin, function $\lambda_1(\|x\|)$ must be infinitesimal of order equal to or lower than that of $\gamma_3^{2\beta_6}(\|x\|)[c_b'(\|x\|)]^2$. As a sufficient condition, one could assume that $\lambda_1(\|x\|) \geq \underline{\lambda}_1$ for some constant $\underline{\lambda}_1 > 0$. ◇

Remark 3.2: In the proof of Theorem 1, stability and convergence is obtained for $x, \tilde{z}$ and $\tilde{v}$. Thus, the requirement of v being uniformly bounded, as stated in Assumption 3, is a sufficient condition for internal stability and for boundedness of the control. On the other hand, it is obvious from system dynamics that v being uniformly bounded is also necessary for closed-loop stability of state x under a bounded control. $\diamondsuit$

Remark 3.3: Inequality (6) in Assumption 3 can be relaxed so that

$$\|\Delta G(v, x, t)\| \leq \lambda_{\Delta G}\|v\| + \beta_3 \gamma_3^{\beta_4}(\|x\|)$$

where $\lambda_{\Delta G} > 0$. In this case, the product

$$\lambda_{\Delta G} c_g \alpha_3 \|v\| \|\tilde{v}\|$$

will appear on the right-hand sides of (16) and (19) as an additional term. Recalling from Assumption 3 that $\|v\|$ can be bounded by a function of x, one can show (by using the same Lyapunov function and, if necessary, by increasing α_2) that the previous term can be dominated (at least outside a neighborhood around the origin) by the negative definite terms in (19). Therefore, in the presence of $\lambda_{\Delta G} > 0$, global stability of uniform and ultimate boundedness of all variables can be maintained under same control. Unless v is known to converge to zero once x is zero, an counterexample can be used to show that, in general, global convergence of uncertain estimation cannot be achieved for the case that $\lambda_{\Delta G} > 0$. $\diamondsuit$

Remark 3.4: In (9), β_3 and β_4 represent the magnitude and the type of nonlinearities in uncertainty $\Delta G(\cdot)$ as related to those of the nominal system. The uncertainty specified by inequality (6) can be compensated for without any degradation of performance because the uncertainty is equivalently matched (as defined by [10]) and its bounding function depends only on x. Along this line, Assumption 3 and condition (9) can also be relaxed further. For example, if bounding function on uncertainty $\Delta G(v, x, t)$ could be decomposed as, for some constants $\beta_3' > 0$ and $\beta_4' > 1$

$$\|\Delta G(v, x, t)\| \leq \beta_3 \gamma_3^{\beta_4}(\|x\|) + \beta_3' \gamma_1^{\frac{c_1}{2}}(\|x\|) \left\|\nabla_x^T V(x, t) B(x, t)\right\|^{\beta_4'}$$

then the value of β_4 would equivalently be reduced. Then, the second term newly introduced previously into the bounding function on $\Delta G(\cdot)$ can be compensated for by changing control (10) into

$$u = -w(x, \hat{v}, t) - \frac{4}{c_2^2} \frac{[\beta_3']^2 c_g^2 \|\nabla_x^T V(x, t) B(x, t)\|^{2(\beta_4' - 1)}}{\lambda_1(\|x\|)} \times B^T(x, t) \nabla_x V(x, t) \quad (20)$$

while its associated observer remains to be (11) and (12). In this case, $\lambda_1(\|x\|) \geq \underline{\lambda}_1 > 0$ and $\beta_4' > 1$ are usually required. $\diamondsuit$

Remark 3.5: Assumption 4A states that the nominal system of estimation error system (15) is asymptotically stable. For the nominal exogenous system $\dot{v} = G(v, x, t)$, the assumption means that the mapping from "input" x (if arbitrarily given) to "state" v is asymptotically stable for all initial condition $v(t_0)$. Therefore, Assumption 4A can be referred to as *asymptotic stability of input-to-state mapping*.

Extension of the assumption can be made so that Assumption 4A admits a nonquadratic Lyapunov function of argument $(v_1 - v_2)$, and such a Lyapunov function may also depend on x. In addition, if $G(v, x, t)$ has a matrix representation of the lower triangular structure [11], condition (7) can be relaxed so that it needs to hold only for the diagonal entries. However, such extensions require more detailed information about the exogenous system. $\diamondsuit$

Remark 3.6: If (9) holds only for small $\|x\|$, then stability and convergence become local. If (9) holds only for $\|x\|$ over some threshold, global stability of uniform and ultimate boundedness of all variables can be ensured, but not convergence of uncertainty estimation.

It follows from (1) and (10) that

$$\lim_{s \to +\infty} \frac{c_b(s) \gamma_4(s) \gamma_3^{\beta_6}(s)}{\gamma_3(s)} = 0$$

is needed for input-to-state stability [19] of x with respect to $\tilde{v}$ (in terms of the Lyapunov function $V(x, t)$). Similarly, input-to-state stability of (15) can be concluded for all x if

$$\lim_{s \to +\infty} \frac{\gamma_3^{\beta_4}(s)}{\lambda_1(s)} = 0.$$

Thus, the previous two conditions are combined and then balanced into condition (9) for global convergence. $\diamondsuit$

Remark 3.7: If the uncertain system is of form

$$\dot{x} = F(x, t) + B(x, t)[\Delta f(x, v, t) + u]$$

and if unstructured uncertainty $\Delta f(x, v, t)$ has a bounding function of known functional expression as

$$\|\Delta f(x, v, t)\| \leq w(x, v, t)$$

then the same stability results in Theorem 1 can be concluded provided that control (10) and observer (11) and (12) are replaced by

$$\begin{aligned} u &= -w(x, \hat{v}, t) \mathrm{sign}[B^T(x, t) \nabla_x V(x, t)] \\ \dot{\hat{z}} &= k(x)(z - \hat{z}) \\ &\quad + F_2(x, t) + B_2(x, t) w(x, \hat{v}, t) \frac{B_2^T(x, t) \tilde{z}}{\|B_2^T(x, t) \tilde{z}\|} + B_2(x, t) u \\ \dot{\hat{v}} &= G(\hat{v}, x, t) \end{aligned}$$

where gain $k(x)$ remains to be the same. $\diamondsuit$

Remark 3.8: Note that control (10) and observer (12) do not depend on $\hat{z}$ from auxiliary observer (11). Therefore, observer (12) is open loop, which is feasible due to Assumption 4A. Thus, except for the case in Remark 3.7, one can remove the auxiliary observer by setting $p = 0$. Besides the need for Remark 3.7, auxiliary observer (11) is kept for the purpose of comparing the results in Theorems 1 and 2 (the latter is presented in the subsequent section). $\diamondsuit$

IV. Control Design Based on Properties of the Plant

Assumption 4A implies that, given any fixed trajectory of $x(t)$, various trajectories of $v(t)$ of the nominal exogenous system will be asymptotically convergent to the same trajectory for all initial conditions. This convergence property does not very likely hold for uncertainties based upon physical observations. More importantly, modeling of an exogenous system is typically as difficult as (if not more than) identifying a bounding function on the magnitude of uncertainties. Without any prior knowledge of the exogenous system, we have to set $G(v, x, t) = 0$ (in which case Assumption 4B is trivially satisfied nonetheless), do not use Assumption 4A and its associated control design, and find another way of estimating uncertainties and to design robust control. In what follows, a different set of conditions are obtained by exploring properties of the plant.

Assumption 4B: Function $G(\cdot)$ has the property that, for all bounded x and for all (v_1, v_2, t)

$$\|G(v_1,x,t) - G(v_2,x,t)\| \leq \beta_7 \gamma_3^{\beta_8}(\|x\|)\|v_1 - v_2\| \tag{21}$$

where $\beta_7 \geq 0$ and $\beta_8 \geq 0$ are constants.

Assumption 5B: There exist a known constant rank-l matrix $C \in \Re^{l\times p}$ and a known positive–definite matrix $P_w \in \Re^{l\times l}$ such that, for all bounded x and for all (v_1, v_2, t)

$$(v_1 - v_2)^T P_w C B_2(x,t)[w(x,v_1,t) - w(x,v_2,t)] \leq -\lambda_2(\|x\|)\|v_1 - v_2\|^2 \tag{22}$$

where $\lambda_2(\|x\|) > 0$ is a known positively valued or positive definite function (or a positive constant), $c_c = \|C\|$, and $c_w = \|P_w\|$.

Assumption 5B says that, if used to construct a fictitious dynamic system, a part of the input-channel dynamics of the plant has the same property explained aforementioned (as that of function $G(v,\cdot,t)$). Matrix C can be freely chosen to find positive–definite solutions P_w and $\lambda_2(\cdot)$. If the solutions exist, robust control and uncertainty estimator are given by the following theorem.

Theorem 2: Consider system (1) satisfying Assumptions 1–3, 4B, 5A, and 5B and under the observer-based robust control

$$u = -w(x, \hat{v} - C\tilde{z}, t) \tag{23}$$

where $\tilde{z} = z - \hat{z}$, $\hat{z}$, and $\hat{v}$ are defined by

$$\dot{\hat{z}} = k(x)\tilde{z} + F_2(x,t) \tag{24}$$
$$\dot{\hat{v}} = G(\hat{v} - C\tilde{z}, x, t) - k(x)C\tilde{z} \tag{25}$$

and

$$k(x) = \frac{9c_c^2\beta_5^2\gamma_3^{2\beta_6}(\|x\|)[c_b'(\|x\|)]^2}{4\lambda_2(\|x\|)}. \tag{26}$$

Then, global asymptotic stability of the state and global convergence of estimating uncertainty can be ensured as long as the following inequalities hold: For all $s > 0$

$$\lambda_2(s)\gamma_3(s) \geq \frac{9}{4}\left[c_2\beta_1\beta_5 c_b(s)\gamma_3^{\beta_2+\beta_6}(s) + \frac{\beta_3 c_w}{c_2}\frac{\gamma_3^{\beta_4}(s)}{\gamma_1^{c_1}(s)}\right]^2 \tag{27}$$

and

$$\lambda_2(s) > 3\beta_7 c_w \gamma_3^{\beta_8}(s) \tag{28}$$

where $c_1 \geq 0$ and $c_2 > 0$ are constants.

Proof: It follows from observer (24) and (25) that dynamics of estimation error are

$$\dot{\tilde{z}} = -k(x)\tilde{z} + B_2(x,t)[w(x,v,t) - w(x,\hat{v} - C\tilde{z},t)] \tag{29}$$
$$\dot{\tilde{v}} = G(v,x,t) - G(\hat{v} - C\tilde{z},x,t) + \Delta G(v,x,t) + k(x)C\tilde{z}. \tag{30}$$

Now, choose the Lyapunov function to be

$$\begin{aligned} L'(x,\tilde{z},\tilde{v},t) &= \frac{\alpha_1}{1+\alpha_2}V^{1+\alpha_2}(x,t) + \frac{1}{2}\alpha_3(C\tilde{z})^T P_z C\tilde{z} \\ &\quad + \frac{1}{2}\alpha_3(C\tilde{z}+\tilde{v})^T P_w (C\tilde{z}+\tilde{v}) \end{aligned} \tag{31}$$

where $\alpha_1, \alpha_3 > 0$ and $\alpha_2 \geq 0$ are constants to be determined, and so is positive–definite matrix P_z. It follows from (1), (3), (29), and (30) that

$$\begin{aligned} \dot{L}' \leq & -\alpha_1 V^{\alpha_2}(x,t)\gamma_3(\|x\|) + \alpha_1 V^{\alpha_2}(x,t)\nabla_x^T V(x,t) \\ & \times B(x,t)[w(x,v,t) - w(x,\hat{v} - C\tilde{z},t)] \\ & - \alpha_3 k(x)(C\tilde{z})^T P_z C\tilde{z} + \alpha_3 (C\tilde{z})^T P_z C B_2(x,t) \\ & \times [w(x,v,t) - w(x,\hat{v} - C\tilde{z},t)] + \alpha_3(C\tilde{z}+\tilde{v})^T P_w \\ & \times \{CB_2(x,t)[w(x,v,t) - w(x,\hat{v} - C\tilde{z},t)] + G(v,x,t) \\ & - G(\hat{v} - C\tilde{z},x,t) + \Delta G(v,x,t)\}. \end{aligned} \tag{32}$$

Setting $P_z = I$, using inequalities (3)–(6), (8), (21), and (22), and then applying the Cauchy–Schwarz inequality and (27) yields

$$\begin{aligned} \dot{L}' \leq & -\alpha_1 V^{\alpha_2}(x,t)\gamma_3(\|x\|) + \alpha_1\beta_5 c_b(\|x\|)V^{\alpha_2}(x,t) \\ & \times \gamma_4(\|x\|)\gamma_3^{\beta_6}(\|x\|)\|C\tilde{z}+\tilde{v}\| - \alpha_3 k(x)\|C\tilde{z}\|^2 \\ & + \alpha_3 c_c\beta_5\gamma_3^{\beta_6}(\|x\|)c_b'(\|x\|)\|C\tilde{z}\|\|C\tilde{z}+\tilde{v}\| \\ & - \alpha_3\lambda_2(\|x\|)\|C\tilde{z}+\tilde{v}\|^2 + \alpha_3 c_w\Big[\beta_7\gamma_3^{\beta_8}(\|x\|)\|C\tilde{z}+\tilde{v}\| \\ & + \beta_3\gamma_3^{\beta_4}(\|x\|)\Big]\|C\tilde{z}+\tilde{v}\| \\ \leq & -\frac{2}{3}\alpha_1\gamma_1^{\alpha_2}(\|x\|)\gamma_3(\|x\|) - \frac{2}{3}\alpha_3 k(x)\|C\tilde{z}\|^2 \\ & - \frac{1}{3}\alpha_3\left[\lambda_2(\|x\|) - 3\beta_7 c_w\gamma_3^{\beta_8}(s)\right]\|C\tilde{z}+\tilde{v}\|^2 \end{aligned}$$

from which stability, convergence, and boundedness can be concluded. □

The following remarks explain several details of design choices and stability conditions, and also expose possible extensions.

Remark 4.1: Counterparts of Remarks 3.1–3.5 hold here as well.◇

Remark 4.2: Assumption 5B is mathematically parallel to Assumption 4A except for the presence of matrices C and $B_2(x,t)$. If $w(x,v,t) = v$, choices of P_w and C should be made so that $P_w C B_2(x,t)$ be negative semi-definite or definite (with respect to x). In general, the negative definite property in (22) implies that $l \leq p$. Recalling $p \leq n$ and $m \leq n$, we know that, while $p = m$ is a typical choice, setting $p = n$ maximizes the chance of meeting Assumption 5B.

As a simple example, consider a mechanical system of form $\dot{x}_1 = x_2$ and $\dot{x}_2 = v + U$, where $x_1, x_2 \in \Re^k$ are the position and velocity vectors, v is the vector of lumped uncertainties that are smooth and uniformly bounded, and U is the total control (i.e., nominal plus robust). It is obvious that Assumption 4B is satisfied with $G = 0$ and that Assumption 5B is met with $P_w = I$ and $C = -I$, where $I \in \Re^{k\times k}$ is the identity matrix. Hence, Theorem 2 can readily be applied.

An extension is possible to establish global stability when $l > p$ (but not global convergence of uncertainty estimation). For instance, consider the case where

$$w(x,v,t) = \sum_{j=1}^{q}\eta(x,v_j,t)$$

where $v_j \in \Re^m$, and $q > 1$ is an integer. Let $\xi(\cdot)$ be a function satisfying the property that $a_j\xi(a_j) < 0$, for $j = 1,\ldots,q$, imply

$$\left(\sum_{j=1}^{q}a_j\right)\left[\sum_{j=1}^{q}\xi(a_j)\right] < 0$$

which is negative definite (with respect to $\sum_{j=1}^{q} a_j$). Examples of such functions include $\xi(s) = -s$ and $\xi(s) = -s^3$. If function $\eta(\cdot)$ has the property that

$$\left[\sum_{i=1}^{q}(a_i - b_i)\right]\left\{\sum_{i=1}^{q}[\eta(x, a_i, t) - \eta(x, b_i, t)]\right\} \leq \left[\sum_{i=1}^{q}(a_i - b_i)\right]\left[\sum_{i=1}^{q}\xi(a_i - b_i)\right]$$

then global stability conditions [similar to (27) and (28)] can be established by following the proof of Theorem 2 except for modifying robust control and observer to be

$$u = -\sum_{j=1}^{q}\eta\left(x, \hat{v}_j - \frac{1}{q}C\tilde{z}, t\right)$$
$$\dot{\hat{z}} = k(x)\tilde{z} + F_2(x, t)$$

while (25) remains to be the same.

It is also worth noting that an extension of Assumption 5B can be made to admit matrix $C(x)$ [and consequently a stability condition more complicated than (9) must be obtained] and that, if $n > l$, choices of p and corresponding matrix $B_2(x, t)$ may not be unique (and, thus, could be optimized). $\diamondsuit$

Remark 4.3: It follows from (32) that Assumptions 4B and 5B can be combined into the condition that, for a known constant matrix C and a known positive–definite matrix P_w, inequality

$$\begin{aligned}(v_1 - v_2)^T P_w\{CB_2(x, t)[w(x, v_1, t) - w(x, v_2, t)] \\ + [G(v_1, x, t) - G(v_2, x, t)]\} \\ \leq -\lambda_2(\|x\|)\|v_1 - v_2\|^2\end{aligned} \tag{33}$$

holds for all bounded x and for all v_1, v_2, and t. In this case, Assumption 4B is no longer required, and one can set $\beta_7 = 0$ in (28).

Assumption 4B is no longer needed and $\beta_7 = 0$ can again be set in condition (28) if Assumption 5B holds and if

$$(v_1 - v_2)^T P_w[G(v_1, x, t) - G(v_2, x, t)] \leq 0. \tag{34}$$

Note that (34) holds automatically if all dynamics of the exogenous system are uncertain. In this case, one can choose the same control but modify the observer to be

$$\dot{\hat{z}} = k(x)\tilde{z} + F_2(x, t) + B_2(x, t)[w(x, \hat{v} - 2C\tilde{z}, t) + u]$$

while (25) remains to be the same. Then, setting $P_z = P_w$ in (31) yields

$$\begin{aligned}\dot{L}' \leq & -\alpha_1 V^{\alpha_2}(x, t)\gamma_3(\|x\|) + \alpha_1\beta_5 c_b(\|x\|)V^{\alpha_2}(x, t) \\ & \times \gamma_4(\|x\|)\gamma_3^{\beta_6}(\|x\|)\|C\tilde{z} + \tilde{v}\| - \alpha_3 k(x)(C\tilde{z})^T P_w C\tilde{z} \\ & - \alpha_3\lambda_2(\|x\|)\|2C\tilde{z} + \tilde{v}\|^2 + \alpha_3 c_w\beta_3\gamma_3^{\beta_4}(\|x\|)\|C\tilde{z} + \tilde{v}\| \\ = & -\alpha_1 V^{\alpha_2}(x, t)\gamma_3(\|x\|) + \alpha_1\beta_5 c_b(\|x\|)V^{\alpha_2}(x, t) \\ & \times \gamma_4(\|x\|)\gamma_3^{\beta_6}(\|x\|)\|C\tilde{z} + \tilde{v}\| - \alpha_3 k(x)(C\tilde{z})^T P_w C\tilde{z} \\ & - \alpha_3\lambda_2(\|x\|)\|C\tilde{z} + \tilde{v}\|^2 - \alpha_3\lambda_2(\|x\|)\|C\tilde{z}\|^2 \\ & + 2\alpha_3\lambda_2(\|x\|)\|C\tilde{z}\|\|C\tilde{z} + \tilde{v}\| \\ & + \alpha_3 c_w\beta_3\gamma_3^{\beta_4}(\|x\|)\|C\tilde{z} + \tilde{v}\|\end{aligned}$$

from which $\dot{L}'$ being negative definite can be concluded under (27), and the choice of control gain in (26) can be relaxed (if smaller) to be

$$k(x) = c'_w\lambda_2(\|x\|)$$

where $c'_w = \|P_w^{-1}\|$.

Relaxation of (27) can also be made along this line. For instance, if Assumption 5B and (34) hold, if $B_1(x, t) = 0$ and $\Delta G(v, x, t) = 0$, and if C is invertible, then (27) and Assumption 5A can be completely removed by changing the control and observer to be

$$\begin{aligned}u &= -w\left(x, \hat{v} - C\tilde{z} - P_w^{-1}C^{-T}\nabla_z V(x, t), t\right) \\ \dot{\hat{z}} &= k(x)\tilde{z} + F_2(x, t)\end{aligned} \tag{35}$$

and (25), where $k(x) > 0$ can be chosen arbitrarily. $\diamondsuit$

Remark 4.4: It follows from (29) and (30) that

$$\begin{aligned}\frac{d(C\tilde{z} + \tilde{v})}{dt} &= CB_2(x, t)[w(x, v, t) - w(x, \hat{v} - C\tilde{z}, t)] + G(v, x, t) \\ &\quad - G(\hat{v} - C\tilde{z}, x, t) + \Delta G(v, x, t).\end{aligned}$$

Therefore, (33) implies that the nominal dynamics of the aforementioned system has an asymptotically stable input-to-state mapping. Matrix C provide the means of possibly achieving this stability property. For instance, consider a scalar system with $B(x, t) = 1 + x, w(x, v, t) = v$, and $G(v, x, t) = (x - x^2)v$. It is easy to see that Assumptions 4A and 5B do not hold but condition (33) holds for $C = -1$.

For multivariable systems, matrix C is an $l \times p$ matrix. The statement in Assumption 5B that matrix C is of rank l implies $p \geq l$, this assumption is needed to conclude global convergence of estimating uncertainties from the exogenous system, but can be dropped for a partial convergence. $\diamondsuit$

Remark 4.5: While control (10) remains to be one designed based on the certainty-equivalence principle, all other proposed controls such as (20), (23), and (35) are not of the same type, and they are synthesized using a Lyapunov argument to achieve stability under Assumption 5B or (34). $\diamondsuit$

V. CONCLUSION

In this note, the problem of estimating uncertainties and designing observer-based robust control is studied for a class of nonlinear uncertain systems whose uncertainties are generated by some exogenous system. It is shown that, if the exogenous system is partially known and if the input-to-state mapping of its nominal system is asymptotically stable, the uncertainties can be observed asymptotically and compensated for. Otherwise, observer and robust control designs can still be proceeded with by exploring properties of the physical plant, and a set of conditions are found to yield a pair of globally stabilizing robust control and globally convergent observer.

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Achieving Proportional Fairness Using Local Information in Aloha Networks

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***Abstract*—We address the problem of attaining proportionally fair rates using Aloha protocols at the medium access layer. We consider a wireless network where all nodes need not be in transmission ranges of each other. We show how the attempt probabilities in Aloha protocols should be set so that the achieved rates are globally proportionally fair. For both slotted and unslotted Aloha, we argue that each node can compute its optimal attempt probability just by knowing some minimal information about the network topology in its two-hop radius.**

***Index Terms*—Aloha networks, fairness, local information.**

I. Introduction

Medium access control (MAC) algorithms are used in wireless networks to control access to a shared wireless medium, and thereby reduce collisions, ensure high system throughput, and distribute the available bandwidth fairly among the competing streams of traffic. We address the issue of designing medium access protocols for attaining proportionally fair rates [2] in wireless networks. The problem of designing distributed access control for attaining fair rates in wireless networks has not been adequately addressed. Tassiulas *et al.* [7] have proposed a centralized algorithm for attaining max–min fairness in certain classes of networks. However, centralized strategies can not be used in large, dynamic *ad-hoc* networks. In another line of work, Nandagopal *et al.* [5] and Ozugur *et al.* [6] have proposed decentralized heuristic medium access strategies that try to achieve some fairness objectives, but the authors did not prove the fairness properties of these approaches.

The problem of fair rate control at the transport layer of wired networks has however been extensively researched, e.g., [3] and [4]. In this context, researchers have shown that globally fair rates can be attained via distributed approaches based on convex programming. However, these techniques can not be directly applied to wireless networks. This is because the rates attained by most wireless MAC protocols can only be indirectly controlled by regulating the transmission probabilities or back-off window sizes. It is difficult to attain the globally fair rates in wireless networks through a distributed approach as the feasible rate region is a complex, nonconvex, and nonseparable function of the attempt probabilities or back-off window sizes. In contrast, distributed rate control algorithms have been developed for wired networks, using the feature that the feasible rate region can be represented by a set of simple, separable, convex constraints.

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