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# Cooperative control with distributed gain adaptation and connectivity estimation for directed networks

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## SUMMARY

This paper addresses the problem of how to achieve superior performance by adaptively and distributively adjusting control gains of a cooperative control system. It is shown that according to distributed observations of changing network topologies and on the basis of online estimation of network connectivity, cooperative controls with adaptive gains can be synthesized to making the time derivative of the cooperative control Lyapunov function more negative and hence to improve stability and convergence of the overall system. For undirected networks, the proposed adaptive design reduces to improving the Fiedler eigenvalue (algebraic connectivity) as well as other eigenvalues. On the other hand, connectivity of a directed network is characterized by the property of the first left eigenvector(s) associated with its dominant eigenvalue, and in this paper, a distributed high-gain observer design is proposed for each of the networked systems to utilize the same communication network among the systems. It is shown that even in the presence of transmission delays, the distributed estimators converge fast to the first left eigenvector(s) of the network. In addition, the expected consensus value(s) of the overall cooperative system under control is also estimated in a distributive manner. Rigorous analysis is carried out on estimation convergence and observer gain selection. It is shown that the proposed estimation and adaptive control designs are fully distributed, have guaranteed performance for all possible varying topologies as long as their dwelling times are bounded away from zero, and are robust with respect to excessively fast topology changes. Simulation results are included to demonstrate effectiveness of the proposed estimation and control schemes. 

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## 1. INTRODUCTION

Cooperative control is a distributed control methodology for networked systems; it utilizes advances in wireless communication and ad hoc networks, and it provides better robustness, scalability, and efficiency while requiring only local information. Cooperative control has been applied to problems in formation control [1–4], attitude synchronization [5–7], flocking [8, 9], and smart grids [10]. Common to these applications is the fact that individual physical systems share a local communication network whose topology changes intermittently and in an unpredictable manner. It is important both theoretically and practically that the design and analysis tools admit varying topologies and that the overall networked system can adaptively improve its performance in the absence of global information. Hence, the goal of this paper is to present a new method of enabling distributed adaptation of cooperative control gains using both the Lyapunov direct method and online connectivity estimation.

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In terms of methodologies, networked systems and their cooperative control can be designed and analyzed using either the graph theoretical method [1, 11, 12], or the matrix theoretical method [2, 13], or topology-based Lyapunov arguments [13–17] and generalized passivity [18]. For both linear and nonlinear networked systems, the necessary and sufficient condition on convergence to a consensus is qualitatively determined by cumulative information flow within the overall system. That is, convergence depends upon whether the composite graphs over a consecutive sequence of composite time intervals have globally reachable nodes, or equivalently whether the corresponding matrix sequence is sequentially complete. Quantitatively, the convergence rate depends upon all the details, such as the topologies and their corresponding time intervals. Hence, it is very useful to study how to distributively estimate the connectivity of the overall network and, if successful, how the result of distributed estimation can be used to improve global performance.

Connectivity of directed networks or the associated digraphs is captured by the first left eigenvector corresponding to the eigenvalue of 0. Unfortunately, there is no result on distributed estimation of network connectivity (in terms of irreducibility, etc.) or its first left eigenvector(s). Instead, as is shown in [19], standard methods of estimating the left eigenvector of digraph Laplacian require that each node knows all the information about not only in-neighbors (matrix row) but also out-neighbors (matrix column), and hence, they are inherently ill-suited for distributed estimation. As one of the main results in this paper, a distributed estimator is proposed to estimate in real time the first left eigenvector(s) so that each of the networked systems knows the instantaneous network connectivity.

In the event that network is both irreducible and symmetric (i.e., the graph is connected and undirected) or double stochastic (i.e., the digraph is balanced), the first left eigenvector becomes trivial as it is unique and the same as the first right eigenvector of all $\frac{1}{\sqrt{n}}$s, and convergence of the overall system depends upon the rest of the real-valued eigenvalues excluding 0. In particular, the dominant eigenvalue for convergence is the second smallest eigenvalue, often known as algebraic connectivity or Fiedler eigenvalue [20]. For these undirected and connected networks, several approaches and corresponding results are available on estimating their eigenvectors and eigenvalues. For instance, numerical solutions requiring global information of the network [21, 22] can be applied to directly compute the Fiedler eigenvalue; a decentralized orthogonal iteration approach is proposed in [23] to estimate the leading $k$ eigenvectors, but this proposed method is not scalable and also requires a centralized initialization. Eigenvalues of Laplacian matrix are estimated using fast Fourier transform (FFT) [24] by constructing distributed oscillators whose states oscillate at frequencies corresponding to the eigenvalues of graph Laplacian; however, the FFT technique is not appropriate for real-time implementation or for handling switching topologies. The best available result in estimating the Fiedler eigenvalue is arguably [25] in which distributed nonlinear dynamic estimators are designed using a decentralized power iteration approach to real-time estimate components of the Fiedler eigenvector and the Fiedler eigenvalue of an undirected and connected networks, although the estimators require estimation of several other consensus values and they may not be able to handle fast changing topologies.

As the dominant eigenvalue, the Fiedler eigenvalue provides a conservative estimate of the convergence rate [26]. For undirected and connected network, several results are available toward improving the Fiedler eigenvalue directly or introducing additional communication channels or nodes or having relay of information. For example, a centralized semi-definite programming solver is proposed in [27] to maximize Fiedler eigenvalue directly; a similar approach is applied to select additive relay locations [28]; as an extension, a decentralized supergradient algorithm is proposed in [29] and using the fact that the supergradient direction of algebraic connectivity is a function of the Fiedler eigenvector, but this requires the a priori knowledge of Fiedler eigenvector and significant communication overhead during each iteration.

All of the aforementioned results on distributed estimation of connectivity or on improving network performance are restricted to undirected networks. To the best of our knowledge, little is available on distributively estimating connectivity of directed networks or on improving their performance. In this paper, we focus upon the problems of distributed connectivity estimation and decentralized gain adaptation for improving convergence of directed networks. Specifically,



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connectivity of a general directed network is determined by distributively estimating the first left eigenvector(s), and the expected consensus value(s) is(are) also estimated in a distributed manner. Among the new results is a lower bound on all the nonzero eigenvalues (including Fiedler eigenvalue) of networked systems with digraphs in general. With these estimation outcomes, the gains in standard linear cooperative controls are distributively adjusted so that the time derivative of cooperative control Lyapunov function becomes more negative and hence performance of the overall system is enhanced. The Fiedler eigenvalue does not need to be explicitly estimated, and for undirected networks, it is shown that making the time derivative of cooperative control Lyapunov function more negative improves Fiedler eigenvalue as well as other eigenvalues (excluding the first one). For directed networks, the proposed technique improves the appropriate eigenvalue corresponding to the current location of the state, which is less conservative (than improving the Fiedler eigenvalue). In fact, the combination of estimation and adaptation provides a systematic way of synthesizing better cooperative control laws for varying and directed networks.

The objective of this paper is threefold: (i) develop a distributed algorithm to estimate the current first left eigenvector; (ii) design a distributed algorithm to estimate the expected consensus vector; and (iii) synthesize an online adaptive scheme to adjust the communication gains to improve convergence to the expected consensus. This paper is organized as follows. In Section 2, the estimation and control design problems are formulated for generally varying and directed networks. Section 3 summarizes the relevant mathematical results on the first left eigenvector, the general solution, the steady state solution, and the cooperative control Lyapunov function of linear, piecewise constant, time-delayed networked systems. Section 4 focuses upon distributed estimation of directed network topology. In Section 4.1, a distributed, hybrid, and high-gain estimator is presented to estimate the first left eigenvector in the presence of transmission delays, its properties and convergence are rigorously analyzed, and a conservative choice of observer gain is explicitly found. Then, a distributed estimator is developed in Section 4.2 to estimate the expected consensus value(s), and discrete-time estimation and control is outlined in Section 4.3. Section 5 addresses the estimation-based design of cooperative control. Gain adaptation is designed and analyzed first in Section 5.1 using the knowledge of the first left eigenvector and expected consensus value and then in Section 5.2 using the distributed estimators. Implications and robustness of the proposed estimation and adaptive cooperative control schemes are demonstrated by using examples along with technical developments and theoretical proofs.

## 2. PROBLEM FORMULATION

Consider a group of $n$ networked linear systems whose dynamics are described by

$$\dot{y}_i = u_i, \tag{1}$$

where $y_i \in \Re^m$ is the output of the $i$th system and $u_i \in \Re^m$ is the control to be designed. Information sharing among the group of the systems is through a local communication network whose status is described by a piecewise-constant binary communication matrix $S(t)$. Specifically, there is a time sequence $\{t_k : k \in \aleph\}$ such that $S(t) = S(t_k)$ for all $t \in [t_k, t_{k+1})$, where $\aleph = \{0, 1, ..., \infty\}$,

$$S(t_k) = \begin{bmatrix} 1 & s_{12}(t_k) & \dots & s_{1n}(t_k) \\ s_{21}(t_k) & 1 & \dots & s_{2n}(t_k) \\ \vdots & \vdots & \ddots & \vdots \\ s_{n1}(t_k) & s_{n2}(t_k) & \dots & 1 \end{bmatrix}, \tag{2}$$

$s_{ij}(t) = 1$ if information of $y_j(t)$ is received by the $i$th system, and $s_{ij}(t) = 0$ if otherwise. Extensions to high-order linear systems and nonlinear systems can be found in [2, 13, 17] and references therein.

Although the time sequence of $\{t_k : k \in \aleph\}$ and the topological changes of $S(t)$ are not known a priori, predictable or prescribed/modeled in any specific way, the cooperative control problem is to ensure both boundedness of $y_k$ for all $k$ and consensus of $(y_i - y_j) \to 0$ under the least

 
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requirement on the cumulative information flow within the local communication network. For the systems described by (1), linear cooperative control for the $i$th system can be chosen to be of the form

$$u_i(t) = \sum_{j=1}^{n} \frac{s_{ij}(t)\alpha_{ij}(t)}{\sum_{l=1}^{n} s_{il}(t)\alpha_{il}(t)}[y_j(t) - y_i(t)] \triangleq \sum_{j=1}^{n} d_{ij}(t)[y_j(t) - y_i(t)], \tag{3}$$

where $\alpha_{ij}(t) > 0$ are scalar piecewise-constant control gains (to be selected), and values and changes of $s_{ij}(t)$ for $j = 1, \cdots, n$ are instantaneously detected by the $i$th system according to the information it receives. Substituting (3) into (1) yields the dynamics of the whole networked system

$$\dot{y} = \{[-I_n + D(t)] \otimes I_m\}y \triangleq -[L(t) \otimes I_m]y, \tag{4}$$

where $y^T = \left[y_1^T \; y_2^T \; ... \; y_n^T\right] \in \Re^{n'}$ with $n' = n \times m$, $I_m \in \Re^{m \times m}$ is the $m$-dimensional identity matrix, symbol $\otimes$ denotes the Kronecker product, matrix $D(t) = [d_{ij}(t)] \in \Re^{n \times n}$ defined in (3) is nonnegative, piecewise constant, row stochastic, and diagonally positive, and $L \triangleq I_n - D$ is the Laplacian matrix of the corresponding digraph.

For notational simplicity of subsequent derivations, $m = 1$ is set in the rest of this paper because $m > 1$ can be handled by analogously proceeding with the technical development in the presence of Kronecker product. It is well known [13] that consensus of system (4) is determined qualitatively by the cumulative information flow (which is specified by the sequence of piecewise-constant matrices of $S(t)$) and quantitatively by the changes of both $S(t)$ and $\alpha_{ij}(t)$. Clearly, constant gains $\alpha_{ij}(t)$ are the simplest, but in order to improve performance (i.e., making convergence to the consensus faster), control gains $\alpha_{ij}(t)$ should be adjusted in real time provided that the current network topology and the expected consensus value can be estimated online and distributively.

Let $\gamma(t) \in \Re^n$ denote the first left eigenvector of matrix $D(t)$, that is,

$$D^T(t)\gamma(t) = \gamma(t), \quad \forall\, t \geqslant t_0. \tag{5}$$

Given that Laplacian matrix $L$ and matrix $D$ have the same eigenvalues and eigenvectors, connectivity of Laplacian matrix and the associated directed graph (i.e., digraph) can be captured by the left eigenvector associated with the first (or the smallest-magnitude) eigenvalue of 0. This important property of left eigenvector $\gamma(t)$ is to be stated as Lemma 1 in Section 3. Indeed, unless each of the networked systems can estimate the current network connectivity and record its history, the systems themselves cannot be certain that they could sustain their connectivity, would converge to a common consensus, or meet any other global objective.

If matrix $D(t)$ and its first left eigenvector $\gamma(t)$ were known, both qualitative and quantitative convergence analysis of system (4) could be carried out. Specifically, the expected consensus value is

$$\sigma(t) = \gamma^T(t)y(t), \tag{6}$$

and because $\gamma(t)$ is piecewise constant as does $D(t)$ (because of either topological change or control gain adjustment),

$$\frac{\mathrm{d}\sigma(t)}{\mathrm{d}t} = \gamma^T(t)\dot{y} = \gamma^T(t)[-I_n + D(t)]y = 0 \text{ for all } t \text{ such that } D(t^+) = D(t^-). \tag{7}$$

Furthermore, the entries of the first left eigenvector $\gamma(t)$ provides the cooperative control Lyapunov function [13] for both analysis and design of cooperative systems (see Lemma 3 in Section 3), and their values (of being positive or zero) also capture connectivity and topological structure of the local communication network (as will be stated in Lemmas 1 and 2 in Section 3). Because the $i$th system only knows the $i$th row of time varying matrix $D(t)$ and those output information it receives, the problem of distributively estimating the left eigenvector $\gamma(t)$ and the expected consensus vector $\sigma(t)$ becomes both theoretically interesting and practically useful.



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To meet the three objectives stated earlier (in the Introduction), a systematic and distributed scheme is proposed to improve the network convergence[‡], and the proposed estimation-control algorithm works for any unknown time sequence of $\{t_k : k \in \aleph\}$ and any topological changes of $S(t)$ provided that the following simple assumptions hold.

*Assumption 1*
Time sequence $\{t_k : k \in \aleph\}$ has the property that $(t_{k+1} - t_k) \geqslant T$ for some known constant $T > 0$.[§]

*Assumption 2*
The integer $n$ is the number (or the maximum number) of dynamical systems that would be connected, and it (as an upper bound) is assumed to be known. Communication from the $j$th system to the $i$th system may incur a time delay of $\tau_{ij} \geqslant 0$, but the maximum total propagation delay of any simple path within the corresponding graph is bounded from above by $\overline{\tau}$ for some known constant $\overline{\tau} \in [0,\ T/4)$.

Specifically, the proposed distributed estimation and control scheme includes the following components: (i) a composite observer of state $z_i(t) = \left[\begin{array}{cc} \hat{\gamma}_i^T(t) & \hat{\sigma}_i^T(t) \end{array}\right]^T \in \Re^{n+1}$ is implemented at the $i$th system to use the available information (i.e., $s_{ij}(t)\overline{y}_j(t)$ with $\overline{y}_j^T(t) = \left[y_j(t)\ t\ z_i^T(t)\right]$) and estimate the current status of the communication network, where $\hat{\gamma}_i(t)$ and $\hat{\sigma}_i(t)$ are the local estimates of $\gamma(t)$ and $\sigma(t)$ at the $i$th system and at time $t$, respectively; (ii) cooperative control law (3) is applied to the $i$th system; and (iii) piecewise-constant gains $\alpha_{ij}(t)$ for $j = 1, \cdots, n$ are adjusted online and at the $i$th system to improve the overall network convergence. To implement the proposed design, each channel of the local communication network captured by matrix $S(t)$ is required to transmit the information of $\overline{y}_i \in \Re^{n+3}$. In comparison, each channel of the communication network needs only to transmit $y_i \in \Re$ for standard cooperative control with constant gains. Clearly, this represents the tradeoff between communication overhead and improving situation awareness as well as control performance.

## 3. MATHEMATICAL PRELIMINARIES

Cooperative system (4) is characterized by matrix $D(t)$. Besides being nonnegative, piecewise constant, row stochastic, and diagonally positive, matrix $D(t)$ is either irreducible or reducible. If $D(t)$ is reducible, it has the following 2-by-2 lower triangular canonical form:

$$PDP^T = \left[\begin{array}{cc} E_{11} & 0 \\ E_{21} & E_{22} \end{array}\right] \triangleq E_{\searrow}, \tag{8}$$

where $E_{11}$ is irreducible and $P$ is a permutation matrix. The systems corresponding to block $E_{11}$ form a leader group as they collectively do not receive any information from the rest of the systems. If $E_{21} = 0$, the systems corresponding to block $E_{22}$ form another leader group. If $E_{21} \neq 0$ and if $E_{22}$ is irreducible (otherwise the same argument can recursively be applied to block $E_{22}$), the systems corresponding to block $E_{22}$ form a follower group because they collectively receive some information from the leader group represented by $E_{11}$. In general, matrix $D$ is said to be lower triangularly complete if there is only one leader group or equivalently the graph has one globally reachable node. The following lemma provides the qualitative relationship between topological/physical connectivity property of the systems and the first left eigenvector $\gamma(t)$, and its proof follows from Lemma 4.2 in [13]. In particular, if matrix $D$ is not lower triangularly complete,

[‡]Another (and much simpler) approach to improve convergence is to introduce a gain $K_{ii} > 1$ into cooperative control (3) such that system (4) becomes $\dot{y} = K[-I_n + D(t)]y$ where $K$ is a diagonal matrix with $K_{ii}$ on the diagonal, but that approach requires more control effort (i.e., high-gain control), and its performance could further be improved by applying the results in the paper.

[§]In the event that rapid changes of $S(t)$ are present and that inequality of $(t_{k+1} - t_k) \geqslant T$ is violated occasionally, these changes would be accommodated by the transient of the proposed distributed estimation schedule. Clearly, if $(t_{k+1} - t_k) \geqslant T$ does not hold most of the time, there is little chance for any online estimation scheme of network connectivity to work.



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there are several linearly independent first left eigenvectors, and this nonuniqueness would become manifest to individual systems in the sense that distributed estimation at each of the systems only obtains one of these eigenvectors.

*Lemma 1*
Consider diagonally positive row-stochastic matrix $D \in \Re_+^{n\times n}$ with $D^T\gamma = \gamma$.

(i) If $D$ is irreducible (i.e., its corresponding digraph is strongly connected), then $\gamma$ is positive and unique (except for a positive multiplier).
(ii) Suppose that $D$ is reducible in the form of (8). Then, if $E_{22}$ is irreducible and $E_{\searrow}$ (i.e., $D$) is lower triangularly complete (i.e., $E_{21} \neq 0$), $\gamma^T = \left[\gamma_{E_{11}}^T \; 0\right] P$ is unique, where $\gamma_{E_{11}}$ is positive such that $\gamma_{E_{11}}^T E_{11} = \gamma_{E_{11}}^T$. If $E_{22}$ is irreducible but $E_{21} = 0$, then $\gamma$ is not unique, and its linearly independent choices are $\gamma^T = \left[\gamma_{E_{11}}^T \; 0\right] P$ and $\gamma^T = \left[0 \; \gamma_{E_{22}}^T\right] P$, where $\gamma_{E_{22}}$ is positive such that $\gamma_{E_{22}}^T E_{22} = \gamma_{E_{22}}^T$. If $E_{22}$ is reducible, the properties of the first left eigenvector $\gamma$ can similarly be argued after permutation and lower triangulation of $E_{22}$.

In the simplest case that the network topology does not change, matrix $D$ is constant, and the following lemma illustrates the relationship among connectivity of nonnegative row-stochastic matrix $D$, convergence of system (4), and its first left eigenvector $\gamma$ associated with its eigenvalue $\lambda(D) = 1$. Its proof follows from Corollary 4.13 in [13] and hence is omitted here.

*Lemma 2*
Consider row-stochastic matrix $D \in \Re_+^{n\times n}$ with $\gamma^T D = \gamma^T$ and $\gamma^T \mathbf{1} = 1$, where $\mathbf{1}_n \in \Re^n$ be the vector of 1's. Then, for any $\mu > 0$ and $t > 0$, $\gamma^T e^{\mu(-I+D)t} = \gamma^T$ and $e^{\mu(-I+D)t}\mathbf{1}_n = \mathbf{1}_n$. Furthermore, if $D$ is irreducible or if $D$ is reducible but lower triangularly complete,

$$\lim_{\eta\to+\infty} e^{(-I+D)\eta} = \mathbf{1}_n\gamma^T. \tag{9}$$

If $D$ is reducible and lower triangularly incomplete, there is a permutation matrix $P$ such that $Pe^{\mu(-I+D)t}P^T$ is block diagonal and each of those diagonal blocks has the aforementioned properties.

In general, system (4) has a piecewise continuous matrix $D(t)$ for any topological changes of $s_{ij}(t)$ and discrete adjustments of control gains $\alpha_{ij}(t)$, and the discontinuity of $D(t)$ occurs only at countable instants of times. As a result, state $x(t)$ has a unique solution that is continuous and in terms of multiplicative sequence of matrices $e^{[-I+D(t_k)]\eta_k}$. It is shown in [13] that the networked system is convergent to a consensus if matrices $D(t_k)$ are sequentially complete or the corresponding composite digraphs have a global reachable node.

Convergence to consensus can be analyzed using graph method [1], or matrix technique [2], or Lyapunov direct method [13]. In particular, the Lyapunov direct method provides analytical expressions that quantitatively describe convergence of a networked system. The cooperative control Lyapunov function for system (4) is provided in the following lemma, and its detailed proof can be found in [13] (specifically, Theorems 5.19 and 5.20 in [13]).

*Lemma 3*
Consider networked system (4), whose nonnegative row-stochastic matrix $D(t)$, the first left eigenvector $\gamma(t)$, and expected consensus value $\sigma(t)$ are piecewise constant with respect to some time sequence $\{t_l', \, l \in \aleph^+\}$. That is, over each time interval $t \in [t_l', t_{l+1}')$, $\gamma(t) = \gamma(t_l')$, $\sigma(t) = \sigma(t_l')$, $D(t) = D(t_l')$, and $D(t_l')$ has its lower triangular canonical form in the form of (8). Then, there exist piecewise-constant nonnegative time functions $\beta_{ij}(t) = \beta_{ij}(t_l')$ for $i, j = 1, \cdots, n$ such that the quadratic function

$$V_c(\beta(t), y(t)) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\beta_{ij}(t)[y_i(t) - y_j(t)]^2 \tag{10}$$



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is a cooperative control Lyapunov function in the sense that $V_c$ is positive definite and $\dot{V}_c$ is negative semi-definite, both with respect to the consensus of $y_i = y_j$ for all $i, j = 1, \cdots, n$ and over time intervals of $\left[t'_l, t'_{l+1}\right)$. Furthermore, the following statements are true:

(i) If $D\left(t'_l\right)$ is irreducible, cooperative control Lyapunov function (10) can be chosen as $\beta_{ij}(t) = \gamma^T(t)e_i\gamma^T(t)e_j > 0$ for all pairs of $\{i, j\}$, and $\dot{V}_c$ is negative definite, where $e_i \in \Re^n$ is a vector of zeros except its $i$th entry being 1.
(ii) If $D\left(t'_l\right)$ is reducible but lower triangularly complete, cooperative control Lyapunov function (10) can be chosen as $\beta_{ij}(t) = 0$ for all $j \notin \mathscr{L}$ and $\dot{V}_c$ is negative definite, where leader set $\mathscr{L}$ consists of the system indices corresponding to those of $E_{11}$ in (8), where $\mathscr{L} = \{i \in \{1, ..., n\} : \gamma^T(t)e_i > 0\}$ is the index set of leader systems.
(iii) If $D(t'_l)$ is reducible and lower triangularly incomplete, the lower triangular canonical form of $D(t'_l)$ is block diagonal, and as stated in (i), cooperative control Lyapunov functions in the form of (10) can be found for each of those irreducible diagonal blocks.

Distributed estimation needs to be robust with respect to not only potential changes of topology but also delays. To study the impact of time delay on distributed estimation (as well as networked consensus), consider the following linear delayed vector differential equation:

$$\dot{z}(t) = A_k z\left(t - \overline{\tau}\right), \quad \overline{\tau} \geqslant 0, \quad t \in [t_k,\ t_{k+1}), \tag{11}$$

$$z(t) = \Phi(t_k, \overline{\tau}), \quad t \in [t_k,\ t_k + \overline{\tau}], \tag{12}$$

where $z \in \Re^l$ is the state, $A_k$ is a constant matrix of appropriate dimension and over time interval $[t_k, t_{k+1})$, interval $[t_k,\ t_k + \overline{\tau}]$ is the so-called pre-interval, and $\Phi\left(t_k, \overline{\tau}\right)$ in (12) is the so-called preshape function or simply the initial value of $z(t)$ on the pre-interval. Solution $z(t)$ to system (11) and its properties are summarized into the following lemma, and its proof is omitted here because it combines the results in [30, 31]. Although exact solution to Lambert function (15) is a function of infinite branches [32], it is quite convenient to use $W(\cdot)$ as an operator in analysis.

*Lemma 4*
Delayed differential equation (11) has the following characteristic equation in matrix form: for $t \in [t_k, t_{k+1})$,

$$se^{\overline{\tau}s}I_l = A_k, \tag{13}$$

whose eigenvalue solutions can be expressed by using the inverse function of $F(q) = qe^q$ and as

$$sI_l = \frac{1}{\overline{\tau}}W(A_k\overline{\tau}), \tag{14}$$

where $W(\cdot)$ is the so-called Lambert matrix function defined by

$$W(A)e^{W(A)} = A, \quad w(0) = 0, \quad \lim_{\overline{\tau}\to 0}\frac{1}{\overline{\tau}}W\left(\overline{\tau}A\right) = A. \tag{15}$$

And, the solution to delayed differential equation (11) is

$$z(t) = e^{W(\overline{\tau}A_k)\frac{(t-t_k)}{\overline{\tau}}}\Phi\left(t_k, \overline{\tau}\right).$$

## 4. DISTRIBUTED ESTIMATION OF NETWORK CONNECTIVITY

In this section, the first left eigenvector and the expected consensus vector of piecewise-constant networked system (4) are estimated online and distributively by each of the individual systems. The basic idea is that because the same network is used to transmit information for both estimation and control, a distributed estimation algorithm can be designed in a way parallel to the distributed cooperative control except that its convergence does not involve any motion of physical systems and hence could be made much faster by using a larger gain.



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### 4.1. Distributed estimation of the first left eigenvector

For distributed estimation, the interesting and difficult case is that $T$ in Assumption 1 is small. Changes of matrix $D(t)$ over time are due to primarily the changes of the network topology (i.e., matrix $S(t)$) as well as potentially the changes of control gain $\alpha_{ij}(t)$. For the purpose of both distributed estimation and distributed control, it will be imposed that $\alpha_{ij}(t)$ are piecewise constant and, if any, their changes (i.e., online gain updating) are made only after distributed estimates $\hat{\gamma}_i$ and $\hat{\sigma}_i$ have already converged (within certain small bounds of error) and also occur under a frequency no faster than $2/T$. The former calls for an online algorithm of estimating the left eigenvector $\gamma$ and expected consensus vector $\sigma$ so that their convergence is achieved within a period of $T/4$. For the $i$th system, the latter implies that adjustments of $\alpha_{ij}(t)$ do not occur more than once within any interval of length equal to $T/2$. This combination ensures that changes of $d_{ij}(t)$ do not happen more than twice within any interval of length $T$ and that, under Assumptions 1 and 2, convergence of online distributed estimation becomes possible.

Topological changes in $S(t)$ are not known either a priori or real time by all the systems, nor is the corresponding time sequence $\{t_k : k \in \aleph\}$. The $i$th system adjusts $\alpha_{ij}(t)$ or their weighted average $d_{ij}(t)$, and it can also detect whether individual link connectivity $s_{ij}(t)$ and/or received information $s_{ij}(t)\hat{\gamma}_i(t-\tau_{ij})$ experience any discontinuous jump(s) at any time $t$. The $i$th system knows that if $s_{ij}(t)$ has a binary change, the corresponding communication link to itself has changed its status at time $t$ (such a change is generally not known to any other system) and that if the received information of $s_{ij}(t)\hat{\gamma}_j(t-\tau_{ij})$ contains jump(s), the topology or some gain elsewhere in the network must have experienced a change within the interval $[t-\overline{\tau},\ t]$ (should $\hat{\gamma}_j(\cdot)$ react properly to the topology changes detected locally). With the limited information, a distributed estimator is constructed by using the following algorithm: for the $i$th system and at any time instant $t$,

(i) If $d_{ij}(t)$ does not experience any jump (i.e., $d_{ij}(t^+) = d_{ij}(t^-)$) for all $j$ and if, for all $j$ with $d_{ij}(t^+) > 0$, received information $\hat{\gamma}_j(t-\overline{\tau})$ does not experience any resetting (i.e., $\hat{\gamma}_j\left(t^+ - \overline{\tau}\right) = \hat{\gamma}_j\left(t^- - \overline{\tau}\right)$), then local estimate $\hat{\gamma}_i(t)$ evolves according to continuous differential equation

$$\dot{\hat{\gamma}}_i(t) = \mu \sum_{j=1}^{n} d_{ij}(t)\left[\hat{\gamma}_j\left(t-\overline{\tau}\right) - \hat{\gamma}_i\left(t-\overline{\tau}\right)\right], \quad i = 1, \cdots, n; \tag{16}$$

(ii) If otherwise, that is, either the piecewise-constant value of $d_{ij}(t)$ is adjusted locally or a reset is detected from received information $\hat{\gamma}_j\left(t-\overline{\tau}\right)$ (i.e., $\hat{\gamma}_j\left(t^+ - \overline{\tau}\right) = e_j \neq \hat{\gamma}_j\left(t^- - \overline{\tau}\right)$) or both, local estimate $\hat{\gamma}_j(t)$ is reset as

$$\hat{\gamma}_i(\tau) = e_i, \quad \forall \tau \in [t,\ t+\overline{\tau}], \tag{17}$$

where $e_i$ is the unit vector defined in Lemma 3, $\hat{\gamma}_i(t) = e_i$ for $t \in [t_0,\ t_0+\overline{\tau}]$, and $\mu > 1$ is the gain¶ to be chosen.

Resetting law (17) is to accommodate switching topologies or discontinuous adjustments of gains. Specifically, should the topology is experiencing a binary change of $s_{ij}$, the corresponding jump of $d_{ij}$ can be detected locally, and resetting of $\hat{\gamma}_i$ is carried out by the $i$th system; by comparing with the past data, those systems receiving information from the $i$th system would in turn become aware of the topological change through the resetting of $\hat{\gamma}_i$, and then they would also proceed with their resetting to ensure that a new round of estimation is properly initiated. Property of this proposed left eigenvector estimation algorithm is given by the following theorem.

*Theorem 1*
Consider cooperative system (4) whose first left eigenvector is denoted by $\gamma^T(t)$ with $\gamma^T \mathbf{1}_n = 1$. Then, for any time sequence $\{t_k : k \in \aleph\}$ and topological changes $S(t_k)$ satisfying Assumptions 1 and 2, output $\hat{\gamma}_i(t)$ of distributed estimator (16) and (17) over time interval $[t_k,\ t_{k+1})$ converges

¶Design parameter $\mu$ in (16) can be replaced by $\mu_i$ so the design also becomes distributed.



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to either the network's unique left eigenvector $\gamma(t_k)$ or one of its linearly independent components (as specified by Lemma 1) provided that $\mu$ is sufficiently large.

*Proof*

It follows that whenever switching laws (17) are not active, the combined dynamics of all the distributed estimators can be expressed in a matrix form as

$$\dot{\hat{\overline{\gamma}}}(t) = \mu\{[-I_n + D(t)] \otimes I_n\}\hat{\overline{\gamma}}(t - \overline{\tau}), \tag{18}$$

where $\hat{\overline{\gamma}}(t) = [\hat{\gamma}_1^T(t)\ \hat{\gamma}_2^T(t)\ ...\ \hat{\gamma}_n^T(t)]^T \in \Re^{n^2}$.

It follows from Lemma 4 that delayed differential equation (16) is piecewise constant and has the following solution: over the interval $t \in [t_k + \overline{\tau},\ t_{k+1})$,

$$\hat{\overline{\gamma}}(t) = e^{W(\overline{\tau}\mu[-I_n+D(t_k)]\otimes I_n)\frac{(t-t_k)}{\overline{\tau}}}\Phi(t_k, \overline{\tau}), \tag{19}$$

where $\Phi(t_k, \overline{\tau})$ is the preshape function of $\hat{\overline{\gamma}}(\tau) = \left[e_1^T\ e_2^T\ \cdots\ e_n^T\right]^T$ for $\tau \in [t_k,\ t_k + \overline{\tau}]$. If $\overline{\tau} = 0$, the solution reduces to the following: for $t \in [t_k,\ t_{k+1})$,

$$\hat{\overline{\gamma}}(t) = e^{\mu\{[-I_n+D(t_k)]\otimes I_n\}(t-t_k)}\hat{\overline{\gamma}}(t_k) = [e^{\mu[-I_n+D(t_k)](t-t_k)} \otimes I_n]\hat{\overline{\gamma}}(t_k), \tag{20}$$

where $\hat{\overline{\gamma}}(t_k) = \left[e_1^T\ e_2^T\ \cdots\ e_n^T\right]^T$.

The proof is performed consecutively for each interval corresponding to time sequence $\{t_k : k \in \aleph\}$. Assume without loss of any generality that $D$ is irreducible or reducible but lower triangularly complete (otherwise, $PDP^T$ will be block diagonal, and each of its diagonal sub-blocks can be studied accordingly). Let $\gamma(t_k)$ denote (one of) the first left eigenvector(s) defined by $\gamma^T(t_k)D(t_k) = \gamma^T(t_k)$ and normalized to $\gamma^T(t_k)\mathbf{1}_n = 1$. For the case that $\overline{\tau} = 0$, it follows from (20) and Lemma 2 that for any $t \in (t_k,\ t_{k+1})$,

$$\lim_{\mu\to\infty}\hat{\overline{\gamma}}(t) = \lim_{\eta=\mu(t-t_k)\to\infty}[e^{[-I_n+D(t_k)]\eta}\otimes I_n]\hat{\overline{\gamma}}(t_k) = \{[\mathbf{1}_n\gamma^T(t_k)]\otimes I_n\}\hat{\overline{\gamma}}(t_k) = \mathbf{1}_n\otimes\gamma(t_k). \tag{21}$$

For the case that $\overline{\tau} > 0$, it follows from (13) and (14) that

$$[\mathbf{1}_n\otimes\gamma(t_k)]^T W(\overline{\tau}\mu[-I_n+D(t_k)]\otimes I_n) = 0 \implies [\mathbf{1}_n\otimes\gamma(t_k)]^T e^{W(\overline{\tau}\mu[-I_n+D(t_k)]\otimes I_n)} = I_{n^2}. \tag{22}$$

Applying the argument of Lemma 2 and using the property in (22) yield

$$\lim_{\eta\to+\infty} e^{\left\{e^{W(\overline{\tau}\mu[-I_n+D(t_k)]\otimes I_n)}([-I_n + D(t_k)] \otimes I_n)\eta\right\}} = \frac{1}{n}\left[\mathbf{1}_n\gamma^T(t_k)\right]\otimes\left(\mathbf{1}_n\mathbf{1}_n^T\right). \tag{23}$$

On the other hand, it follows from (15) that solution (19) can be expressed as

$$\hat{\overline{\gamma}}(t) = e^{\left\{e^{W(\overline{\tau}\mu[-I_n+D(t_k)]\otimes I_n)}\mu([-I_n + D(t_k)] \otimes I_n)(t - t_k)\right\}}\Phi(t_k),$$

which together with (23) implies that for any $t \in (t_k + \overline{\tau},\ t_{k+1})$,

$$\lim_{\mu\to\infty}\hat{\overline{\gamma}}(t) = \lim_{\eta=\mu(t-t_k)\to\infty} e^{\{e^{W(\overline{\tau}\mu[-I_n+D(t_k)]\otimes I_n)}([-I_n + D(t_k)] \otimes I_n)\eta\}}\Phi(t_k) = \mathbf{1}_n \otimes \gamma(t_k). \tag{24}$$

The limits of (21) and (24) show that the observer (16) of (sufficiently) high gain would work and its steady state is independent of $\overline{\tau}$. □

For practical applications, gain $\mu$ needs to be selected to implement observer (16). The following corollary provides a conservative estimate on how high the gain should be to ensure separation of time scales (between the topology changes and the observer) and in turn convergence. The condition of $s_{ij} = 1$ implying $d_{ij} \geqslant \underline{d}$ can easily be satisfied distributively by the choices of control gains $\alpha_{ij}$.



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*Corollary 1*
Consider the setup in Theorem 1. If individual system of (4) ensures that if $s_{ij} \neq 0$, $d_{ij}(t_k) \geqslant \underline{d}$ for some small constant $\underline{d} > 0$, then any choice of $\mu$ satisfying the following inequality guarantees the convergence stated in Theorem 1:

$$\mu \geqslant \frac{16(n-1)}{(T-4\overline{\tau})\underline{d}^{n-1}}. \tag{25}$$

*Proof*
In order to ensure that distributed estimation and subsequent gain adaptation work, it is sufficient to require that

$$\overline{\tau} + T_s \leqslant \frac{1}{4}T, \tag{26}$$

where $T_s$ is the settling time of observer (16). Although $T_s$ is unknown, conservative estimation of $T_s$ will be performed for the cases that $D(t_k)$ is either irreducible or reducible but lower triangularly complete (or, if otherwise, $D(t_k)$ is equivalent to diagonal blocks to which the same argument applies). It follows from (21) or (24) that

$$e^{[-I_n + D(t_k)]\eta} = e^{-\eta} e^{F^{\frac{1}{n-1}}(t_k)\eta}, \tag{27}$$

where $F(t_k) = D^{n-1}(t_k)$ is a row-stochastic matrix whose elements are denoted by $F_{ij}(t_k)$ and whose spectrum is $\rho(F) = 1$.

Consider first that $D(t_k)$ is irreducible. In this case, because $D(t_k)$ is also diagonally positive, matrix $F(t_k)$ is strictly positive. It is obvious that

$$\overline{f} \triangleq \max_{i,j} F_{ij} \leqslant 1, \quad \underline{f} \triangleq \min_{i,j} F_{ij} \geqslant \underline{d}^{n-1}.$$

Recalling that the following inequality [33, 34] holds for the $(n-1)$ eigenvalues $\lambda(F)$ with $\lambda(F) \neq \rho(F)$:

$$|\lambda(F)| \leqslant \frac{\overline{f} - \underline{f}}{\overline{f} + \underline{f}} \rho(F),$$

we know that

$$1 - \max|\lambda(F)|^{\frac{1}{n-1}} = \frac{1 - \max|\lambda(F)|}{1 + \max|\lambda(F)|^{\frac{1}{n-1}} + \cdots + \max|\lambda(F)|^{\frac{n-2}{n-1}}} \geqslant \frac{1 - \max|\lambda(F)|}{n-1}$$
$$\geqslant \frac{2\underline{f}}{(n-1)\left(\overline{f} + \underline{f}\right)} \geqslant \frac{\underline{f}}{(n-1)\overline{f}} \geqslant \frac{1}{n-1}\underline{d}^{n-1}.$$

It follows from (27) that if $D(t_k)$ is irreducible,

$$T_s = \frac{4}{\mu(1 - \max|\lambda(F)|^{\frac{1}{n-1}})} \leqslant \frac{4(n-1)}{\mu \underline{d}^{n-1}}, \tag{28}$$

in which the last inequality follows directly from the preceding inequality. Hence, inequality (26) is guaranteed under the choice of $\mu$ in (25).

If $D(t_k)$ is reducible but lower triangularly complete, it follows from lower triangular canonical form of (8) that for any $l > 1$,

$$PD^l(t_k)P^T = \begin{bmatrix} E_{11}^{l-1} & 0 & \cdots & 0 \\ W_{21} & E_{22}^{l-1} & \ddots & 0 \\ \vdots & & \ddots & \vdots \\ W_{p1} & \cdots & W_{p(p-1)} & E_{pp}^{l-1} \end{bmatrix} \triangleq E_{\searrow}^{l-1},$$



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where $E_{jj} \in \Re^{n_j \times n_j}$ are irreducible and lower triangular blocks such as $W_{21}$ can be found by inductive computation. The preceding argument can recursively be applied to the leader group represented by $E_{11}$ and then to each of subsequent row blocks of irreducible matrix $E_{jj}$. Because the settling time of each subsystem is of form (28) except that $n$ is replaced by $n_i$ and because $\sum_{j=1}^{p} n_j = n$, the settling time of the overall system is conservatively the sum of individual settling times, that is, the one in (28). Hence, choice (25) of $\mu$ is also valid for the case that $D(t_k)$ is reducible. □

The proposed distributed estimation algorithm of network's first left eigenvector enables each system to identify qualitative topological properties of the whole network, or more specifically, distributively figure out connectivity of the network. That is, given $\hat{\gamma}_i(t) \to \gamma_i^*$, the $i$th system knows that the network is irreducible if $\gamma_i^*$ is positive (and $\gamma_i^* = \gamma^*$ for all $i$) and that the network is reducible if $\gamma_i^*$ contains at least one zero entry. Furthermore, if the $i$th element in $\gamma_i^*$ is positive, the $i$th system knows that itself belongs to (one of) the leader group(s); and if the $i$th element in $\gamma_i^*$ is zero, the $i$th system knows that itself is a member in (one of) the follower group(s). This information would enable each of the systems to take certain corrective measure in a higher-level control of the communication network. In terms of cooperative control design, distributed estimation of network's first left eigenvector means that every system knows its (or the whole) cooperative control Lyapunov function and, as will be shown in Section 4, a distributed scheme of updating control gains can be designed to improve convergence of the overall networked system without making any change in the network topology.

The following simple example is used to illustrate the performance of the proposed distributed estimation algorithm of $\gamma(t)$. Should $D(t)$ be reducible and lower triangularly incomplete, the corresponding graph would consist of several decoupled subgraphs, the first left eigenvector $\gamma$ would not be unique, and the distributed estimate by a specific system would converge to the left eigenvector corresponding to its particular subgraph.

*Example 1*
Consider time subsequence $\{t_k : k = 1, 2, 3\}$ and suppose that $D(t) = D_k$ for $t \in [5(k-1), 5k)$, where

$$D_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0.5 & 0.5 \end{bmatrix}, \quad D_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \end{bmatrix}. \tag{29}$$

It is straightforward to show that the first left eigenvectors associated with $D_k$ are

$$\gamma(D_1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \quad \gamma(D_2) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; \quad \text{and } \gamma(D_3) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{30}$$

The distributed observer given by (16) and (17) is implemented with $T = 1$, $\overline{\tau} = 0.1$, and $\mu = 40$. Performance of the observer is illustrated in Figure 1, where $\gamma_{ij}^*$ is the $j$th component of the first eigenvector estimated by the $i$th system. It follows from Figure 1 that the distributed estimates are convergent as $\hat{\gamma}_i(t)|_{D(t)=D_k} \to \gamma_i^*(D_k)$, where

$$\begin{cases} \gamma_1^*(D_1) = \gamma_1^*(D_2) = \gamma_1^*(D_3) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T; \\ \gamma_2^*(D_1) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T, \ \gamma_2^*(D_2) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T, \ \gamma_2^*(D_3) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T; \\ \gamma_3^*(D_1) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T, \ \gamma_3^*(D_2) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T, \ \gamma_3^*(D_3) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T; \end{cases}$$

which are consistent with the results in Theorem 1 and Lemma 1. In fact, convergence of distributed estimation is also independent of $\overline{\tau}$ used in the observer. In addition, as indicated in Figure 1, coordination of estimators are accomplished in a successive manner.

Theorem 1 and Corollary 1 show that, in spite of the presence of data transmission delays $\tau_{ij}$, the first left eigenvector(s) characterizing the network topology can be estimated distributively by

 
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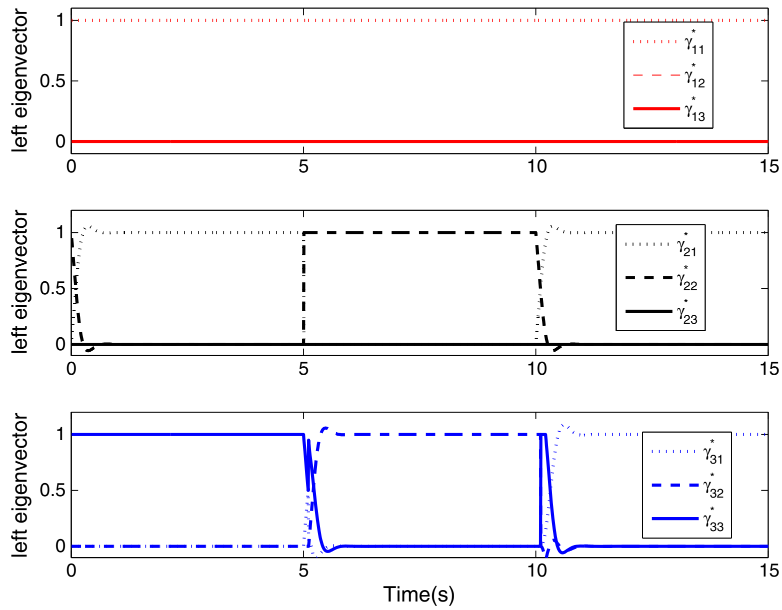


Figure 1. Real-time estimation of the first left eigenvector.

knowing the upper bound $\overline{\tau}$ and that the steady state solution of observer (16) and (17) is independent of $\overline{\tau}$. Accordingly, the delays are no longer included in the subsequent analysis and design in order to avoid unnecessary complication.

### *4.2. Distributed estimation of consensus vector*

In this section, expected consensus vector $\sigma(t)$ defined in (6) and for system (4) is estimated using a distributed strategy parallel to that of the first left eigenvector. That is, consensus estimate $\hat{\sigma}_i(t)$ at the $i$th system evolves according to the locally available information, and its value also needs to be reset properly in order to accommodate for topological changes in the network. Specifically, current estimate $\hat{\sigma}_i(t)$ by the $i$th system will be reset to its current local output $y_i(t)$ if $d_{ij}(t)$ exhibits a discontinuity at time $t$ (i.e., $d_{ij}(t^-) \neq d_{ij}(t^+)$) for some $j$ or if $d_{ij}(t^+) > 0$ and $\hat{\sigma}_j(t)$ experiences a discontinuity (due to the reset of $\hat{\sigma}_j(t^+) = y_j(t^+)$ from $\hat{\sigma}_j(t^-)$),

$$\hat{\sigma}_i(t_k) = y_i(t_k); \tag{31}$$

and if there is no discontinuity or subsequent reset needed,

$$\dot{\hat{\sigma}}_i(t) = \mu \sum_{j=1}^{n} d_{ij}(t)[\hat{\sigma}_j(t) - \hat{\sigma}_i(t)]; \tag{32}$$

where $\hat{\sigma}_i(t_0) = y_i(t_0) \in \Re$ and $\mu \gg 1$ is the gain (whose value is conservatively chosen in Corollary 1). Properties of the proposed distributed consensus estimator are provided in the following theorem.

*Theorem 2*
Consider cooperative control system (4) whose expected consensus vector is given by (6). Then, under Assumptions 1 and 2 and for $\mu$ satisfying Corollary 1, $\sigma(t)$ can be estimated distributively by the estimator of (32) and (31). In particular, for any time sequence $\{t_k : k \in \aleph\}$ and for any piecewise-constant topological changes of $S(t)$, distributed estimate $\hat{\sigma}_i(t)$ at the $i$th system converges to the system consensus vector $\sigma(t)$, where $\sigma(t)$ is unique and identical everywhere if $D(t)$ is either irreducible or reducible but lower triangularly complete and $\sigma(t)$ is not unique (because $\gamma(t)$ is not unique as stated in Lemma 1) if $D(t)$ is lower triangularly incomplete.



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*Proof*
Whenever resetting law (31) is not triggered, the closed-loop consensus estimator is

$$\dot{\hat{\overline{\sigma}}}(t) = \mu[-I_n + D(t)]\hat{\overline{\sigma}}(t), \tag{33}$$

where $\hat{\overline{\sigma}}(t) = \left[\hat{\sigma}_1^T(t)\, \hat{\sigma}_2^T(t) \ldots \hat{\sigma}_n^T(t)\right]^T \in \Re^n$ is the overall vector of distributed consensus estimation.

The proof can be performed consecutively for each interval corresponding to time sequence $\{t_k : k \in \aleph\}$ as carried out in the proof of Theorem 1. In particular, within time interval $t \in [t_0,\ t_1)$, the solution to (33) is

$$\hat{\overline{\sigma}}(t) = e^{\mu[-I_n + D(t_0)](t-t_0)}\hat{\overline{\sigma}}(t_0),$$

where $\hat{\overline{\sigma}}(t_0) = \left[y_1^T(t_0) \ldots y_n^T(t_0)\right]^T = y(t_0) \in \Re^n$. It follows from Lemma 2 that for any $t \in (t_0,\ t_1)$,

$$\lim_{\mu\to\infty} \hat{\overline{\sigma}}(t) = \left[\mathbf{1}_n \gamma^T(t_0)\right]\hat{\overline{\sigma}}(t_0) = \left[\mathbf{1}_n \gamma^T(t_0)\right] y(t_0).$$

Recalling (6) and (7), we can rewrite the preceding limit as

$$\lim_{\mu\to\infty} \hat{\overline{\sigma}}(t) = \mathbf{1}_n \otimes \sigma(t_0) = \mathbf{1}_n \otimes \sigma(t),$$

where $t \in (t_0, t_1)$. That is, $\hat{\sigma}_i(t) \to \sigma(t)$ for all $i = 1, \cdots, n$.

The value of consensus vector $\sigma(t)$ and its uniqueness depend on connectivity of matrix $D(t_0)$. In particular, $\gamma(t_0)$ is unique, and hence $\sigma(t)$ is unique if $D(t_0)$ is either irreducible or reducible but lower triangularly complete. It follows from decomposition (8) and from Lemma 1 that if $D(t)$ is lower triangularly incomplete (say $E_{21} = 0$), $\hat{\sigma}_i(t)$ converge to $\left[\gamma_{E_{11}}^T\ \ 0\right] y(t_0)$ if the $i$th system corresponds to block $E_{11}$ or to $\left[0\ \ \gamma_{E_{22}}^T\right] y(t_0)$ if otherwise.

The proof is performed by repeating the same argument for each of time intervals $[t_k,\ t_{k+1})$ and by noting that updating law (31) appropriately resets all the initial conditions for $\hat{\overline{\sigma}}(t_k)$. □

In what follows, Example 1 is carried out further to illustrate the performance of consensus estimator (32) and (31).

*Example 1 (Continued)*
Consider again $D(t) = D_k$ over time intervals $t \in [5(k-1),\ 5k)$ for $k = 1, 2, 3$. Suppose that $y(0) = [10\ \ 0\ \ 3]^T$ is the initial state vector. It follows from networked system (4) and from matrices $D_k$ in (29) that

$$y(0) = [10\ \ 0\ \ 3]^T, \quad y(5) = [10\ \ 9.2\ \ 3]^T, \quad y(10) = [10\ \ 9.2\ \ 8.6]^T.$$

It follows from the first left eigenvector $\gamma(D_k)$ in (30) that

$$\sigma(D_1) = \gamma(D_1)y(0) = 10 \text{ or } 3; \quad \sigma(D_2) = 10 \text{ or } 9.2; \quad \sigma(D_3) = 10.$$

The distributed consensus observer given by (32) and (31) is implemented with $T = 1$, and $\mu = 40$. It follows from Theorem 2 and Lemma 1 that distributed estimates are convergent as $\hat{\sigma}_i(t)|_{D(t)=D_k} \to \sigma_i^*(D_k)$, where

$$\begin{cases} \sigma_1^*(D_1) = 10,\ \sigma_1^*(D_2) = 10,\ \sigma_1^*(D_3) = 10; \\ \sigma_2^*(D_1) = 10,\ \sigma_2^*(D_2) = 9.2,\ \sigma_2^*(D_3) = 10; \\ \sigma_3^*(D_1) = 3,\ \sigma_3^*(D_2) = 9.2,\ \sigma_3^*(D_3) = 10; \end{cases}$$

which can also be observed in Figure 2. And, the convergence is prompt.



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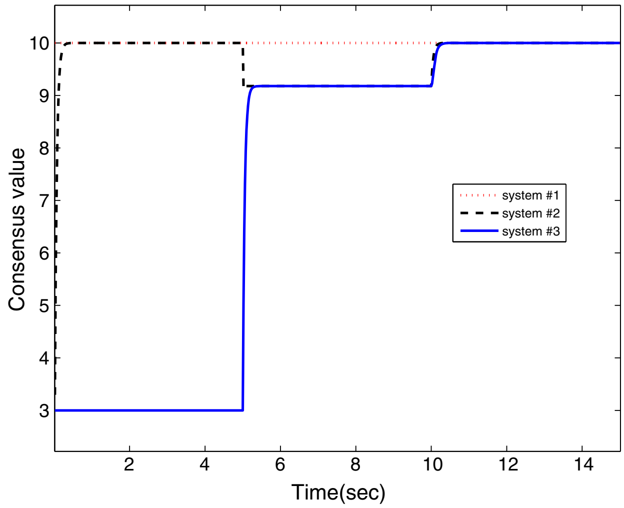


Figure 2. Real-time estimation of consensus vector.

### 4.3. *Discrete estimation of network topology*

Rather than designing continuous-time observer and control, estimation and control can also be carried out in the discrete-time domain. It follows from (1) that, within interval $t \in [t_k,\ t_{k+1})$, its discretized version is

$$y_i'(k'+1) = y_i'(k') + u_i'(k'), \quad y_i'(0) = y_i(t_k), \quad k' = 0, 1, \cdots, N(k),$$

where $0 < \Delta T \ll 1$ is the sampling period, $y_i'(k') = y_i(k'\Delta T + t_k)$, $u_i'(k') = u_i(k'\Delta T + t_k)$, mod$(\cdot)$ is the modulo operation, $N(k) = (t_{k+1} - t_k)/\Delta T - \text{mod}((t_{k+1} - t_k)/\Delta T)$. Using the zero-order hold, the discrete-time cooperative control is chosen as

$$u_i'(k') = \sum_{j=1}^{n} d_{ij}(k)\left[y_j'(k') - y_i'(k')\right],$$

where $d_{ij}(k)$ are the gains to be distributively chosen such that $\sum_j d_{ij}(k) = 1$, $d_{ij}(k) \geqslant \underline{d} > 0$ if $s_{ij}(t_k) \neq 0$ and $d_{ij}(k) = 0$ if $s_{ij}(t_k) = 0$. Hence, the overall closed-loop discrete-time networked system is

$$y'(k'+1) = D(k)y'(k'), \quad k' = 0, 1, \cdots, N(k), \tag{34}$$

Discrete-time system (34) is analogous to continuous-time system (4), and their corresponding graphs are identical.

For system (34), its first left eigenvector $\gamma(k)$ can be estimated locally at the $i$th system using the observer

$$\hat{\gamma}_i(k'+1) = \sum_{j=1}^{n} d_{ij}(k)\hat{\gamma}_j(k'), \quad \hat{\gamma}_i(0) = e_i, \tag{35}$$

and its consensus value can be estimated by the estimator

$$\hat{\sigma}_i(k'+1) = \sum_{j=1}^{n} d_{ij}(k)\hat{\sigma}_j(k'). \tag{36}$$

It is straightforward to show [13] that $\lim_{N(k)\to\infty} D^{N(k)}(k) = \mathbf{1}_n \otimes \gamma^T(k)$ with $\gamma^T(k)\mathbf{1} = 1$. By repeating the argument in Corollary 1, we can find a conservative estimate on how small $T_s$ (i.e., how large $N(k)$) needs to be in order to ensure convergence.



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## 5. COOPERATIVE CONTROL WITH ADAPTIVE GAINS

In this section, cooperative control with adaptive gains is to be designed to improve convergence. Convergence of networked system (4) is determined qualitatively by the cumulative property of varying topologies and quantitatively by gain changes with respect to the system topology and the state at any specific time, and Lyapunov direct method is the approach chosen in this paper to carry out analytical analysis and synthesis. As shown in Lemmas 1 and 2, the first left eigenvector captures critical topological properties of networked system (4), and hence it (or its estimate) could be used to find cooperative control Lyapunov function by applying Lemma 3 and consequently determine the subsequent gain selections in the proposed cooperative control design.

### *5.1. Gain adaptation based on the knowledge of first left eigenvector and expected consensus value*

The following lemma illustrates how nonzero entries $d_{ij}(t)$ (i.e., gains $\alpha_{ij}(t)$) can be updated on the basis of the local knowledge of the first left eigenvector $\gamma(t)$, the expected consensus value $\sigma(t)$, and the locally available state variables (i.e., terms $s_{ij}(t)y_j(t)$ and for the $i$th system). Such a gain adjustment, if made at time $t = t_*$, is to make $\dot{V}_c(\beta, y(t_*^+))$ more negative than $\dot{V}_c(\beta, y(t_*^-))$ so that convergence to the expected consensus becomes faster. That is, gain adaptation laws (37) and (38) are to make $\dot{V}_c(\beta, y(t))$ become more negative over consecutive intervals so that convergence is gradually improved.

*Lemma 5*

Consider networked control system (4) and whose connectivity matrix $S(t)$ is constant over intervals of $[t_k,\ t_{k+1})$ for $k \in \aleph^+$. Accordingly, within the interval $t \in [t_k,\ t_{k+1})$, nonnegative and row-stochastic matrix $D(t)$ given by (3) and (4) depends solely upon the choices of control gains $\alpha_{ij}(t)$. Suppose that, for the $i$th system, $d_{ij}(t)$ remain to be constant until $d_{ii}(t)$ and another nonzero entry $d_{i\ell_i^*}(t)$ are adjusted from $d_{ii}(t_*^-)$ and $d_{i\ell_i^*}(t_*^-)$ to $d_{ii}(t_*^+) = d_{ii}(t_*^-) - \varepsilon_i$ and $d_{i\ell_i^*}(t_*^+) = d_{i\ell_i^*}(t_*^-) + \varepsilon_i$, respectively, and at some $t_* \in (t_k,\ t_{k+1})$. Then, convergence of the overall system is improved provided that $\ell_i^*$ and $\varepsilon_i$ are chosen as follows: for the $i$th system with $i \in \mathscr{L}$,

$$\ell_i^* \in \mathscr{N}_i \implies |y_i(t_*) - y_{\ell_i^*}(t_*)| = \max_{j\in\mathscr{N}_i} |y_i(t_*) - y_j(t_*)|,$$

$$\varepsilon_i = \begin{cases} 0 & \text{if } [y_i(t_*) - y_{\ell_i^*}(t_*)]\left[y_i(t_*) - \sigma(t_*^-)\right] = 0 \\ K_a d_{ii}(t_*^-) & \text{if } \left[y_i(t_*) - y_{\ell_i^*}(t_*)\right]\left[y_i(t_*) - \sigma(t_*^-)\right] > 0 \\ -K_a d_{i\ell_i^*}(t_*^-) & \text{if } \left[y_i(t_*) - y_{\ell_i^*}(t_*)\right]\left[y_i(t_*) - \sigma(t_*^-)\right] < 0 \end{cases}; \tag{37}$$

for the $i$th system with $i \notin \mathscr{L}$,

$$\ell_i^* \in \mathscr{N}_i \implies |y_{\ell_i^*}(t_*) - \sigma(t_*^-)| = \min_{j\in\mathscr{N}_i} |y_j(t_*) - \sigma(t_*^-)|,$$

$$\varepsilon_i = \begin{cases} 0 & \text{if } |y_{\ell_i^*}(t_*) - \sigma(t_*^-)| \geqslant |y_i(t_*) - \sigma(t_*^-)| \\ K_a d_{ii}(t_*^-) & \text{if } |y_{\ell_i^*}(t_*) - \sigma(t_*^-)| < |y_i(t_*) - \sigma(t_*^-)| \end{cases}; \tag{38}$$

where $\mathscr{L}$ is the leader set defined in Lemma 3, $\mathscr{N}_i = \{j : j \in \{1, \cdots, n\},\ j \neq i,\ d_{ij}(t_*^-) > 0\}$ is the neighboring set for the $i$th system, $0 < \underline{d} \ll 1$ is a given small constant, and $K_a$ is a nonlinear adaptation gain[‖]:

$$K_a = \begin{cases} K_a \in (0, 1) & \text{if } \min\{d_{ii}(t_*^+), d_{i\ell_i^*}(t_*^+)\} \geqslant \underline{d} \\ 0 & \text{if otherwise} \end{cases}.$$

[‖]The choice of $K_a$ is constant except for the deadzone to ensure that if $s_{ij} = 1$, $d_{ij} \geqslant \underline{d} > 0$.



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*Proof*

It follows from (4) and Lemma 3 that although $d_{ij}(t)$ are piecewise continuous, solutions of $y_i(t)$ are uniformly continuous and uniformly bounded for all $t$ and for all $i$.

In the case that only the control gains of the $i$th system are adjusted according to (37), it follows from (37) that at most two elements in the $i$th row of $D(t_*^+)$ may have jumps from those of $D(t_*^-)$ as

$$D(t_*^+)=\begin{bmatrix} d_{11}(t_*) & \cdots & d_{1i}(t_*) & \cdots & d_{1\ell_i^*}(t_*) & \cdots & d_{1n}(t_*) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ d_{i1}(t_*) & \cdots & d_{ii}(t_*^-)-\varepsilon_i & \cdots & d_{i\ell_i^*}(t_*^-)+\varepsilon_i & \cdots & d_{in}(t_*) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ d_{n1}(t_*) & \cdots & d_{ni}(t_*) & \cdots & d_{n\ell_i^*}(t_*) & \cdots & d_{nn}(t_*) \end{bmatrix} = D(t_*^-)-\varepsilon_i e_i e_i^T+\varepsilon_i e_i e_{\ell_i^*}^T,$$

where $e_i, e_j \in \Re^n$ are the standard unit vectors. In general, all the systems could adjust control gains at time $t = t_*$ and hence we have

$$D(t_*^+) = D(t_*^-) - \sum_{i=1}^n \varepsilon_i e_i e_i^T + \sum_{i=1}^n \varepsilon_i e_i e_{\ell_i^*}^T. \tag{39}$$

Now, consider cooperative control Lyapunov function (10) over time interval $[t_k,\ t_*)$. By direct computation, we know that its time derivative along system (4) and at time $t = t_*^-$ is

$$\begin{aligned} \dot{V}_c(\beta(t_k), y(t_*^-)) = \sum_{i=1}^n \sum_{j=1}^n \beta_{ij}(t_k) \Bigg\{ & -[2-d_{ii}(t_*^-)-d_{jj}(t_*^-)][y_i(t_*)-y_j(t_*)]^2 \\ & + \sum_{l\neq i,\, l=1}^n d_{il}(t_*^-)[y_i(t_*)-y_j(t_*)][y_l(t_*)-y_j(t_*)] \\ & + \sum_{l\neq i,\, l=1}^n d_{jl}(t_*^-)[y_i(t_*)-y_j(t_*)][y_i(t_*)-y_l(t_*)] \Bigg\}. \end{aligned}$$

Similarly, by using matrix expression (39), the time derivative of the same cooperative control Lyapunov function in (10) (i.e., without changing $\beta_{ij}$) can be calculated at time $t = t_*^+$. Defining the total change in the time derivative of $\dot{V}_c$ and because of the gain adjustments as

$$\delta \dot{V}_c(t_*) \triangleq \dot{V}_c\left(\beta(t_k), y(t_*^+)\right) - \dot{V}_c\left(\beta(t_k), y(t_*^-)\right),$$

we know from continuity of $y_l(t)$ that

$$\delta \dot{V}_c(t_*) = \sum_{i=1}^n \delta \dot{V}_{c_i}(t_*),$$

where

$$\delta \dot{V}_{c_i}(t_*) = -\varepsilon_i \sum_{j=1}^n [\beta_{ij}(t_k)+\beta_{ji}(t_k)][y_i(t_*)-y_j(t_*)][y_i - y_{\ell_i^*}(t_*)]. \tag{40}$$

Hence, control gains of the $i$th system should be adjusted such that its resulting jump $\varepsilon_i$ ensures $\delta \dot{V}_{c_i}(t_*) \leqslant 0$ and makes the inequality strict whenever possible.

Let us study first the case that $\gamma(t_k)$ is a positive vector, namely, matrix $D(t)$ is (constant and) irreducible for $t \in [t_k,\ t_*)$. It follows from (i) of Lemma 3, from $\mathbf{1}^T\gamma(t_k) = 1$, from $\sigma(t) = \gamma^T(t)y(t) = \sigma(t_k)$, and from (40) that $\beta_{ij}(t_k) = \gamma^T(t_k)e_i\gamma^T(t_k)e_j$, and hence

$$\delta \dot{V}_{c_i}(t_*) = -2\varepsilon_i \gamma^T(t_k) e_i \left[y_i(t_*) - y_{\ell_i^*}(t_*)\right][y_i(t_*) - \sigma(t_k)]. \tag{41}$$



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Therefore, we have $\delta \dot{V}_{c_i} < 0$ or equivalently $\dot{V}_c(\beta(t_k), y(t_*^+)) < \dot{V}_c(\beta(t_k), y(t_*^-)) < 0$ if and only if

$$\varepsilon_i \left[ y_i(t_*) - y_{\ell_i^*}(t_*) \right] [y_i(t_*) - \sigma(t_k)] > 0. \tag{42}$$

Because $\mathscr{N}_i$ is the neighboring set for the $i$th system, $\mathscr{N}_i = \{j \,:\, j \in \{1, \cdots, n\},\ j \neq i,\ d_{ij}(t_*^-) > 0\}$. Because $D(t_*^-)$ is irreducible, $\mathscr{N}_i$ is nonempty, and inequality (42) can always be satisfied for some $i$ unless $y_i(t_*^-)$ has already reached the expected consensus $\sigma(t_*)$. It is straightforward to verify that the best choices of $\ell_i^*$ and $\varepsilon_i$ in the sense of making $\delta \dot{V}_{c_i}$ most negative are those given in (37).

In the case that $\gamma(t_k)$ is unique but contains zero entries, matrix $D(t)$ is reducible but lower triangularly complete. In this case, coefficients $\beta_{ij}$ in cooperative control Lyapunov function (10) are too involved to be analytically solved, and the time derivative (40) of cooperative control Lyapunov function does not yield a condition as simple as (42). Nonetheless, by the lower triangular canonical form, we know that all the systems can be classified into two groups: leader group $\mathscr{L}$ and follower group $\mathscr{L}^c$. For the $i$th system belonging to the leader group, it is associated with an irreducible submatrix of $D(t)$, and hence adaptation law (37) can be invoked. For the $i$th system belonging to the follower group, its state should track the expected consensus $\sigma(t_k)$, which is determined solely by the leader group, and hence its gain adaptation law can be designed as follows. It follows from (4) that, for $t \in [t_k, t_*)$,

$$\begin{aligned}\frac{\mathrm{d}[y_i(t) - \sigma(t_k)]}{\mathrm{d}t} &= \sum_{j \neq i,\, j=1}^{n} d_{ij}(t)[y_j(t) - \sigma(t_k)] - \sum_{j \neq i,\, j=1}^{n} d_{ij}(t)[y_i(t) - \sigma(t_k)] \\ &= \sum_{j \in \mathscr{N}_i}^{n} d_{ij}(t)[y_j(t) - \sigma(t_k)] - [1 - d_{ii}(t)][y_i(t) - \sigma(t_k)].\end{aligned}$$

Because set $\mathscr{N}_i$ may contain mostly the systems in the follower group, either the terms of $[y_j(t) - \sigma(t_k)]$ or their maximum relative difference $\max_{j \in \mathscr{N}_i} |y_j(t) - y_i(t)|$ may not be consistent with the goal of making $y_i(t)$ track $\sigma(t_k)$. Nonetheless, the last expression suggests that $d_{ii}(t_*^+)$ can be adjusted to make the term of $[1 - d_{ii}(t)][y_i(t) - \sigma(t_k)]$ become more prominent and in turn make $y_i(t)$ track $\sigma(t_k)$ faster. Accordingly, the adaptation law in (38) is selected.

If the first left eigenvector $\gamma(t_k)$ is not unique, it follows from Lemma 1 that matrix $D(t)$ consists of block diagonal submatrices in its canonical form, and the expected consensus value $\sigma(t_k)$ is different for each of those decoupled groups. Nonetheless, gain adaptation laws (37) and (38) can still be applied to each and every group because the aforementioned analysis holds. □

In Lemma 5, control gains are adjusted to make the time derivative of $V_c$ more negative anywhere in the state space (and at any time instant that the first left eigenvector and expected consensus value are identified). The proposed method of control gain adjustment not only is intuitive and well rooted in control theory but also includes the graph theoretical concept of algebraic connectivity [22] as a special case. The following lemma shows that if the graph corresponding to the networked system is undirected, the proposed method improves convergence through increasing the Fiedler eigenvalue (i.e., algebraic connectivity of the graph), and gain adaptation can be implemented (differently from that in Lemma 5) such that symmetry of the resulting graph can be maintained. Indeed, by applying the proposed method, the algebraic connectivity is improved in the subspace of the Fiedler eigenvector, and so are the rest of graph eigenvalues in their associated subspaces. In other words, the proposed gain adaptation approach generalizes the algebraic connectivity of undirected network (and its subspace in $\Re^n$) to the cooperative control Lyapunov function of directed network (and the whole state space).



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*Lemma 6*
Consider networked control system (4), whose system matrix $D(t)$ is constant over a time interval $[t_k,\ t_{k+1})$ for some $k \in \aleph^+$. If $D$ is symmetric, the corresponding graph is undirected, and its algebraic connectivity defined by

$$\lambda_2 \triangleq \min_{y^T\mathbf{1}=0,\, y\neq 0} \frac{y^T(I-D)y}{y^Ty} \quad \text{or} \quad \lambda_2 \triangleq \min_{y^T\mathbf{1}=0,\, y\neq 0} \frac{\sum_{i=1}^n \sum_{j=1}^n d_{ij}(y_i-y_j)^2}{2\sum_{i=1}^n y_i^2} \tag{43}$$

is related to the cooperative control Lyapunov function as

$$\lambda_2 = \min_{y^T\mathbf{1}=0,\, y\neq 0} \frac{-n\dot{V}_c}{2y^Ty}. \tag{44}$$

Moreover, gain adaptation that improves convergence and maintains symmetry can be achieved as follows: the $i$th system first selects $\ell_i^*$ according to the criterion that

$$\ell_i^* \in \mathscr{N}_i \implies |y_i(t_*) - y_{\ell_i^*}(t_*)| = \max_{j\in\mathscr{N}_i} |y_i(t_*) - y_j(t_*)|, \tag{45}$$

then it negotiates with the $\ell_i^*$th system (and may also be called to negotiate with other systems) and chooses $\varepsilon_{i\ell_i^*} > 0$ and $\varepsilon_{k\ell_k^*} > 0$ with $\ell_k^* = i$ and $k \neq i$ such that

$$\varepsilon_{i\ell_i^*} + \sum_{k\neq i:\, \ell_k^*=i} \varepsilon_{k\ell_k^*} = K_a d_{ii}(t_*^-), \tag{46}$$

and finally, at time $t_*$, it resets $d_{ii}(t)$, $d_{i\ell_i^*}(t)$, and $d_{k\ell_k^*}(t)$ with $\ell_k^* = i$ as

$$d_{ii}(t_*^+) = (1-K_a)d_{ii}(t_*^-), \quad d_{i\ell_i^*}(t_*^+) = d_{i\ell_i^*}(t_*^-)+\varepsilon_{i\ell_i^*}, \quad \text{and} \quad d_{k\ell_k^*}(t_*^+) = d_{k\ell_k^*}(t_*^-)+\varepsilon_{k\ell_k^*}, \tag{47}$$

respectively, where $\mathscr{N}_i$ and $K_a$ are those defined in Lemma 5.

*Proof*
Matrix $D$ being symmetric implies that the first left eigenvector is $\gamma = \mathbf{1}/n$. It follows from (i) of Lemma 3 that $\beta_{ij}(t) \equiv 1/n$ and

$$V_c(\beta, y(t)) = \frac{1}{2n^2}\sum_{i=1}^n \sum_{j=1}^n (y_i - y_j)^2 \tag{48}$$

$$= \frac{1}{n}y^Ty - \rho^2, \tag{49}$$

where $\rho = y^T\mathbf{1}/n$. It follows from (49) and (4) and from $\rho$ being invariant over the time interval that

$$\dot{V}_c(y) = \frac{1}{n}y^T(-I+D)y.$$

Alternatively, the expression of

$$\dot{V}_c(y) = -\sum_{i=1}^n \sum_{j=1}^n \frac{d_{ij}}{n}(y_i-y_j)^2 \tag{50}$$

can be established by directly differentiating (48) and invoking $d_{ij} = d_{ji}$. Comparing (43) and (50) yields (44).

It follows from (50) that, under gain adaptation from (45) and (46), the resulting matrix $D(t_*^+)$ remains symmetric and

$$\delta\dot{V}_c(t_*) \triangleq \dot{V}_c(\beta, y(t_*^+)) - \dot{V}_c(\beta, y(t_*^-)) = -\frac{2}{n}\sum_{i=1}^n \varepsilon_{i\ell_i^*}[y_i(t_*) - y_{\ell_i^*}(t_*)]^2, \tag{51}$$

which demonstrates faster convergence. □



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It is clear that Lemma 6 reveals the intuitive connection between algebraic connectivity and Lyapunov function for undirected networks, whose convergence rate is determined by the Fiedler value and can be improved by increasing off-diagonal gains. In order to maintain symmetry in implementation, any pair of connected systems needs to negotiate and synchronize their adjustments according to (46). Moreover, expression (44) does not mean that any optimization is needed in real time. Rather, it implies that, because inequality (51) holds everywhere in the state space, the Fiedler eigenvalue as well as all other eigenvalues (except for the first) are improved as a result under gain adaptation (45) and (46).

Note that Lemma 6 can be treated as a special case of Lemma 5; the difference between them is that the algorithm in Lemma 6 keeps the consensus value invariant (unless the network topology is no longer undirected) but not the adjustment scheme of Lemma 5. This extra freedom of indirectly changing the consensus value (as the consequence of $D(t)$ being unsymmetric and its first left eigenvector being changed) makes it possible for the state to converge to the consensus faster. In addition, while the adjustment scheme of Lemma 6 improves all eigenvalues of $D(t)$, only the relevant ones are improved under Lemma 5. The following simple example illustrates the observations made earlier about directed networks.

*Example 2*

Consider networked control system (4) and assume that $S(t)$ is directed and constant for $t \in [0,\ 5]$ and

$$D(0) = \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.2 & 0.8 & 0 \\ 0.3 & 0 & 0.7 \end{bmatrix}. \tag{52}$$

The first eigenvalue/left eigenvector and the nonzero eigenvalues/right eigenvectors of $(I_3 - D(0))$ are

$$\begin{cases} \lambda_1(0) = 0 \\ \gamma_1(0) = [0.55\ 0.27\ 0.18]^T \end{cases}; \quad \begin{cases} \lambda_2(0) = 0.24 \\ \upsilon_2(0) = [0.13\ \ -0.73\ 0.67]^T \end{cases}; \quad \begin{cases} \lambda_3(0) = 0.46 \\ \upsilon_3(0) = [-0.45\ 0.34\ 0.83]^T \end{cases};$$

where $\gamma_i$ and $\upsilon_i$ are the left and right eigenvectors associated with $\lambda_i$, respectively. The adjustment algorithm of Lemma 5 is applied to matrix $D(t)$ with $K_a = 0.9$ and at time instants of $t_* = 0.5l$ where $l \in \aleph^+$. In what follows, performance of the algorithm is evaluated for different initial conditions $y_0$.

Let us consider the initial condition of $y_0 = [1.4\ \ -7.3\ 6.7]^T$. The nonzero eigenvalues of $(I_3 - D(t))$ and the expected consensus value are shown in Figures 3(a) and 3(b), respectively. For instance, the first adjustment occurs at time $t = 0.5$, and the corresponding changes are

$$y(0.5) = [1.2\ \ -6.4\ 5.9]^T,\ \ D(0.5^-) = D(0),\ \ D(0.5^+) = \begin{bmatrix} 0.08 & 0.82 & 0.1 \\ 0.92 & 0.08 & 0 \\ 0.93 & 0 & 0.07 \end{bmatrix}. \tag{53}$$

Matrix $(I_3 - D(0.5^+))$ has the following pairs of the first eigenvalue/left eigenvector and nonzero eigenvalues/right eigenvectors:

$$\begin{cases} \lambda_1(0.5^+){=}0 \\ \gamma_1(0.5^+){=}[0.50\ 0.45\ 0.05]^T \end{cases}; \quad \begin{cases} \lambda_2(0.5^+){=}0.93 \\ \upsilon_2(0.5^+){=}[0\ \ -0.12\ 0.99]^T \end{cases}; \quad \begin{cases} \lambda_3(0.5^+){=}1.84 \\ \upsilon_3(0.5^+){=}[-0.57\ 0.57\ 0.59]^T \end{cases} \tag{54}$$

It follows that

$$y(0.5) = -3.36\frac{\mathbf{1}}{\sqrt{3}} + 11.22\upsilon_2(0.5^+) - 5.5\upsilon_3(0.5^+),$$

where $\mathbf{1} = [1\ \ 1\ \ 1]^T$. Because both $\lambda_2(t)$ and $\lambda_3(t)$ are improved at $t = 0.5$ (as shown in Figure 3(a)), convergence of $y(t)$ for $t > 0.5$ becomes better than that of $t < 0.5$. Meantime,



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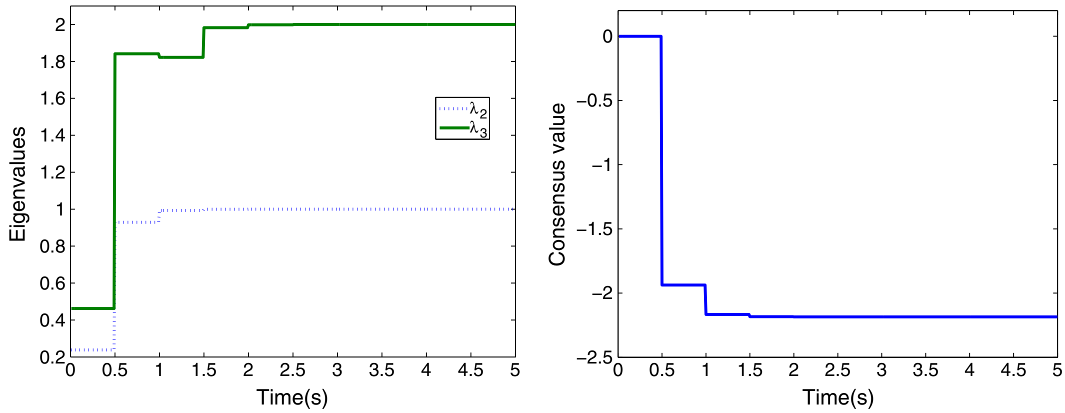


(a) Non-zero eigenvalues

(b) Expected consensus value

Figure 3. Improvement of nonzero eigenvalues and expected consensus value: Example 2.

the expected consensus value is adjusted from $\sigma(0.5^-) = \sigma(0) = \gamma_1^T(0)y(0) = 0$ to $\sigma(0.5^+) = \gamma_1^T(0.5^+)y(0.5) = -1.98$, and accordingly, it is closer for $y(0.5)$ (i.e., its first component in the previous decomposition) to converge to $\sigma(0.5^+)$ than to $\sigma(0.5^-)$.

In essence, the scheme of Lemma 5 enhances convergence by directly adjusting nonzero entries of piecewise constant matrix $D(t)$ in order to make $\dot{V}_c(t)$ more negative; equivalently, the scheme makes convergence faster by indirectly improving eigenvalues and expected consensus value (but without requiring the knowledge of either nonzero eigenvalues or their corresponding right eigenvectors).

### 5.2. *Distributed gain adaptation based on distributed estimation of network connectivity*

In essence, under gain modification (37) or (38), matrix $D(t_*^+)$ in (39) remains to be nonnegative and row stochastic and has the same reducible/irreducible properties as $D(t_*^-)$. The piecewise-constant gain design in Lemma 5 would be distributed except that the knowledge of $\gamma(t)$ and $\sigma(t)$ is required. Fortunately, it has been shown in Theorems 1 and 2 that both $\gamma(t)$ and $\sigma(t)$ can be estimated distributively and promptly. Combining distributed estimation and piecewise-constant gain adjustment yields the following result on adaptive cooperative control design. Similarly, estimation-based gain adaptation for undirected networked control systems can be derived using incremental changes in (46) and (47).

*Theorem 3*

Consider the networked systems given by (1) and under cooperative control (3). Suppose that Assumptions 1 and 2 hold and that distributed estimators (16)–(17) and (32)–(31) are implemented to generate distributed estimates $\hat{\gamma}_i(t)$ and $\hat{\sigma}_i(t)$, respectively. Then, convergence of the networked systems to their expected consensus value(s) becomes faster if piecewise-constant cooperative control gains $\alpha_{ij}(t)$ are adjusted (asynchronously) as follows: for the $i$th system

$$\alpha_{ij}(t^+) = \begin{cases} \alpha_{ij}(t^-) & \text{if } t \neq \hat{t}_k^i + 0.5lT \text{ for some } l \in \aleph^+ \\ \frac{1}{\kappa_i}\alpha_{ij}(t^-) & \text{if } t = \hat{t}_k^i + 0.5lT \text{ for some } l \in \aleph^+ \text{ but } j \neq i, \ell_i^* \\ \dfrac{1}{\kappa_i}\left[\alpha_{i\ell_i^*}(t^-) + \dfrac{\alpha_{i\ell_i^*}(t^-)}{d_{i\ell_i^*}(t^-)}\varepsilon_i\right] & \text{if } t = \hat{t}_k^i + 0.5lT \text{ for some } l \in \aleph^+ \text{ and } j = \ell_i^* \\ \dfrac{1}{\kappa_i}\left[\alpha_{ii}(t^-) - \dfrac{\alpha_{ii}(t^-)}{d_{ii}(t^-)}\varepsilon_i\right] & \text{if } t = \hat{t}_k^i + 0.5lT \text{ for some } l \in \aleph^+ \text{ and } j = i \end{cases} \quad (55)$$



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where $\kappa_i = \min\left\{1,\ \alpha_{ii}(t^-) + \frac{\alpha_{ii}(t^-)}{d_{ii}(t^-)}\varepsilon_i,\ \alpha_{i\ell_i^*}(t^-) - \frac{\alpha_{i\ell_i^*}(t^-)}{d_{i\ell_i^*}(t^-)}\varepsilon_i\right\} > 0$ is the scaling gain, $\alpha_{ij}(t_0) = 1$ for all $j$, $\aleph^+$ is the set of positive integers, $t_k$ is the most recent time instant when the network topology has changed, $\hat{t}_k^i$ is the estimate** of $t_k$ by the $i$th system, and index $\ell_i^*$ and incremental change $\varepsilon_i$ are selected according to (37) and (38) after replacing $t_k$ by $\hat{t}_k^i$, $t^*$ by $\left(\hat{t}_k^i + 0.5lT\right)$, $\mathscr{L}$ by $\hat{\mathscr{L}}_i$ where $\hat{\mathscr{L}}_i \triangleq \{i \in \{1, \cdots, n\} :\ e_i^T \hat{\gamma}_i \left(\hat{t}_k^i + 0.5lT\right) > 0\}$, and $\sigma(t)$ by $\hat{\sigma}_i(t)$.

*Proof*

It follows from Theorems 1 and 2 as well as their proofs that time instants $t_k$, first left eigenvector $\gamma(t)$, and expected consensus $\sigma(t)$ can all be estimated in a distributed manner.

First, consider the time interval $t \in \left(\hat{t}_k^i + 0.5lT,\ \hat{t}_k^i + 0.5lT + 0.5T\right]$. It is apparent from (55) that, for the $i$th system and at any time $t$, there are at most two of the piecewise-constant gains (i.e., $\alpha_{ii}(t)$ and $\alpha_{i\ell_i^*}(t)$) adjusted (and hence have discontinuity). It follows from (3) and from $\ell_i^*(t) \in \mathscr{N}_i$ that, $j = 1, \cdots, n$,

$$
\begin{aligned}
d_{ij}(t) &= \frac{s_{ij}(t)\alpha_{ij}(t)}{\sum_{\nu=1}^{n} \alpha_{i\nu}(t)s_{i\nu}(t)} = \frac{s_{ij}(t)\alpha_{ij}(t)}{\alpha_{ii}(t) + \alpha_{i\ell_i^*}(t) + \sum_{\nu \neq i,\, \ell_i^*;\, \nu=1}^{n} \alpha_{i\nu}(t)s_{i\nu}(t)} \\
&= \frac{\kappa_i s_{ij}(t)\alpha_{ij}(t)}{\kappa_i\alpha_{ii}(t) + \kappa_i\alpha_{i\ell_i^*}(t) + C_i(t)},
\end{aligned}
\tag{56}
$$

where

$$
C_i(t) = \kappa_i \sum_{\nu \neq i,\, \ell_i^*;\, \nu=1}^{n} \alpha_{i\nu}(t)s_{i\nu}(t)
$$

is continuous for all the time in the interval. Using (56) with $j = i, \ell_i^*$, we can find the following solutions: at any time instant $t$,

$$
\begin{bmatrix} \alpha_{ii}(t) \\ \alpha_{i\ell_i^*}(t) \end{bmatrix} = \frac{C_i(t)}{[1 - d_{ii}(t) - d_{i\ell_i^*}(t)]\kappa_i} \begin{bmatrix} d_{ii}(t) \\ d_{i\ell_i^*}(t) \end{bmatrix}, \text{ if } C_i(t) \neq 0; \quad \frac{\alpha_{ii}(t)}{\alpha_{i\ell_i^*}(t)} = \frac{d_{ii}(t)}{d_{i\ell_i^*}(t)}, \text{ if } C_i(t) = 0.
\tag{57}
$$

To achieve adjustments from $d_{ii}(t^-)$ and $d_{i\ell_i^*}(t^-)$ to $d_{ii}(t^+) = d_{ii}(t^-) - \varepsilon_i$ and $d_{i\ell_i^*}(t^+) = d_{i\ell_i^*}(t^-) + \varepsilon_i$ at certain time instants, respectively, we note that the sum of $\left[d_{ii}(t) + d_{i\ell_i^*}(t)\right]$ is continuous and that hence, by (57), the corresponding changes from $\alpha_{ii}(t^-)$ and $\alpha_{i\ell_i^*}(t^-)$ to $\alpha_{ii}(t^+)$ and $\alpha_{i\ell_i^*}(t^+)$ are given by the regression equations in (55). Specifically, (57) still applies with inclusion of gain $\kappa_i$, with which $\alpha_{ij}(t)$ will be scaled at each row and $\alpha_{ij}(t) \geqslant 1$ is ensured anywhere. Hence, by (56), all the $d_{ij}(t)$ except for $d_{ii}(t)$ and $d_{i\ell_i^*}(t)$ are continuous.

Next, it again follows from Theorems 1 and 2 that within time interval $t \in \left(\hat{t}_k^i + 0.5lT,\ \hat{t}_k^i + 0.5lT + 0.5T\right)$, estimates $\hat{\gamma}_i(t)$ and $\hat{\sigma}_i(t)$ should have already converged (very closely) to their appropriate steady state value(s) (as specified by Lemma 1). As such, Lemma 5 can be applied to improve the state convergence to its consensus. □

The proposed cooperative control with adaptive gains would improve transient response within each of time intervals $t \in [t_k,\ t_{k+1})$. If the corresponding matrix $D(t)$ is irreducible or reducible but lower triangularly complete, the outputs of all the systems would converge faster to the expected consensus values. Upon each of the gain updates (which occur at a period of $0.5T$ and whose number of time depends upon the ratio of $2(t_{k+1} - t_k)/T$), both the first left eigenvector and the expected

**The $i$th system has observed the most recent time instant, $\hat{t}_k^i$ (being the estimate of $t_k$), at which either a binary change of $s_{ij}(t)$ for some $j$ occurred or a reset of $\hat{\gamma}_j(t)$ (for some $j$ with $s_{ij}(t) = 1$) was detected.



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consensus vector have been changed, and they have to be estimated again distributively and before the next update can be made. In essence, improvement of transient response by gain adjustments is twofold: the gain adjustments amplify the error signals in the control law and they also modify the expected consensus value to match the current states of all the systems. In the event that the corresponding matrix $D(t)$ is reducible and lower triangularly incomplete, the overall networked system consists of several decoupled subgroups, and the proposed cooperative control would only enable the systems converge faster to their own consensus values. In other words, because the cooperative control is memoryless, it can only improve the performance on the basis of the current topological connectivity.

The following two examples are used to illustrate the proposed adaptation algorithm. In particular, Example 3 demonstrates that the proposed distributed estimation and adaptive gain adjustment are asynchronously performed and that the proposed scheme is robust even in the absence of the assumption (i.e., there are occasionally fast topology changes for which the requirement of $t_{k+1} - t_k \geqslant T$ does not hold).

*Example 1 (Continued)*

Consider system (1) with $n = 3$, with initial state vector $y(0) = [10 \ \ 0 \ \ 3]^T$, and with $S(t) = S(5(k-1))$ over time intervals $t \in [5(k-1),\ 5k)$ for $k = 1, 2, 3$, where $S(\cdot)$ are defined by

$$S(0) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad S(5) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad S(10) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{58}$$

It follows from (3) that if constant control gains $\alpha_{ij} = 1$ are used, the resulting system (4) has $D(t) = D_k$ over the time intervals, where $D_k$ are defined in (29). Alternatively, the adaptation scheme in Theorem 3 can be applied. Figure 4(a) shows the comparison between performance under constant gains and that under the proposed gain adaptation scheme, where the trajectories under gain adaptation are decorated with diamond markers (versus those without), and design choices are $T = 1$, $K_a = 0.9$, and $\mu = 40$. It follows that the state values at the times of topology changes are as follows: with constant gains,

$$y(5) = [10 \ \ 9.18 \ \ 7.37]^T, \quad y(10) = [8.69 \ \ 9.18 \ \ 8.68], \quad y(15) = [8.76 \ \ 8.81 \ \ 8.68]^T;$$

and with adaptive gains,

$$y(5) = [10 \ \ 9.86 \ \ 9.37]^T, \quad y(10) = [9.69 \ \ 9.86 \ \ 9.69]^T, \quad y(15) = [9.69 \ \ 9.69 \ \ 9.69]^T.$$

The time histories of varying gains $\alpha_{ii}$ and their corresponding matrix diagonal elements $d_{ii}$ are provided in Figures 4(b) and 4(c), which indicate clearly that estimation convergence and subsequent gain adaptations occur consecutively after each of the topology changes. And, convergence is improved under the proposed gain adaptation scheme.

In particular, the second and third eigenvalues of $(I_3 - D(t))$ are provided in Figure 4(d), in which the two eigenvalues under gain adaptation are plotted with diamond markers. It is obvious that either $\lambda_2$ or $\lambda_3$ has been improved during each of the time intervals and in a state-dependent way. Note that at time $t = 0.5$, the corresponding state, system matrix, and expected consensus are $y(0.5) = [10\ 2.21\ 2.60]^T$, $D(0.5^-) = D(0)$, and $\sigma(0.5^-) = 10$, respectively. It is straightforward to verify using (38) that no adjustment is needed for system 2 or 3. This means that the first adaptation occurs at $t = 1$, and the corresponding state and changes are

$$y(1) = [10\ 3.93\ 2.72]^T, \quad D(1^-) = D(0), \quad D(1^+) = \begin{bmatrix} 1 & 0 & 0 \\ 0.95 & 0.05 & 0 \\ 0 & 0.95 & 0.05 \end{bmatrix}.$$

It follows that the nonzero eigenvalues/right eigenvectors of $(I - D(1^+))$ are

$$\begin{cases} \lambda_2(1^+) = 0.95 \\ \upsilon_2(1^+) = [0\ 0\ 1]^T \end{cases} \qquad \begin{cases} \lambda_3(1^+) = 0.95 \\ \upsilon_3(1^+) = [0\ 0\ 1]^T \end{cases}$$



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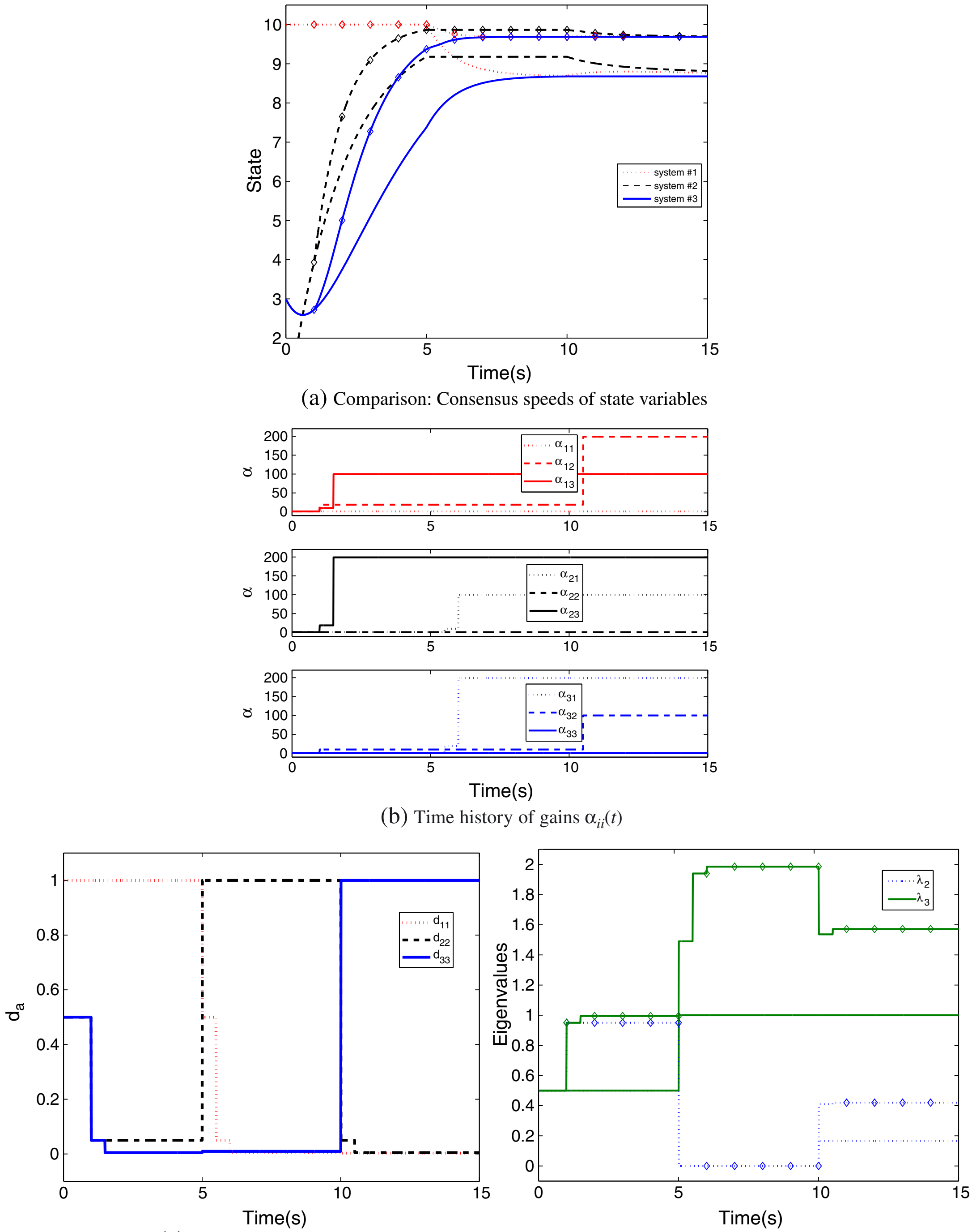



(a) Comparison: Consensus speeds of state variables

(b) Time history of gains $\alpha_{ij}(t)$

(c) Real time adaptation of $d_{ii}$

(d) Comparison of eigenvalues

Figure 4. Performance with gain adaptations.

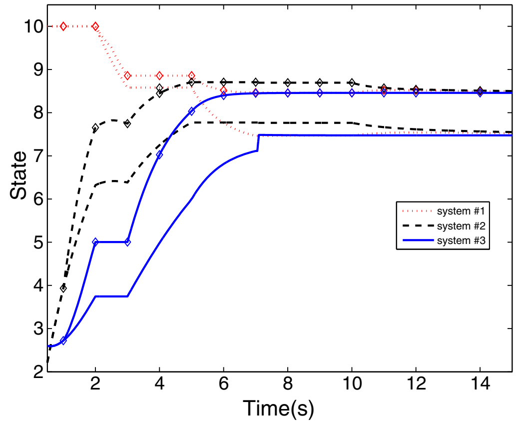
and that the current state can be expressed as

$$y(1) = 12.06\frac{\mathbf{1}}{\sqrt{3}} - 2.12\upsilon_2(1^+) - 2.12\upsilon_3(1^+).$$

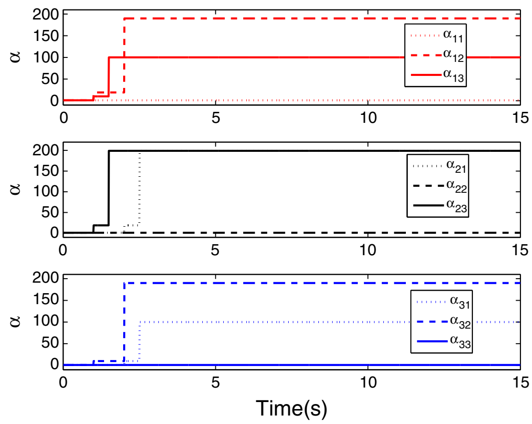
The decomposition shows that $\lambda_2(1^+)$ and $\lambda_3(1^+)$ are of equal importance on convergence and as shown in Figure 4(d), both $\lambda_2$ and $\lambda_3$ have been improved under gain adaptation.

*Example 3*

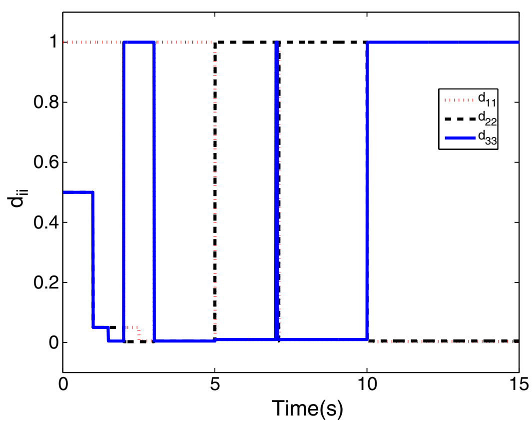
Consider the same systems with the same initial conditions as those in Example 1, and apply the proposed estimation and adaptation scheme with the same design parameters in Example 1. Suppose that topological changes of the network include not only those in (58) but also



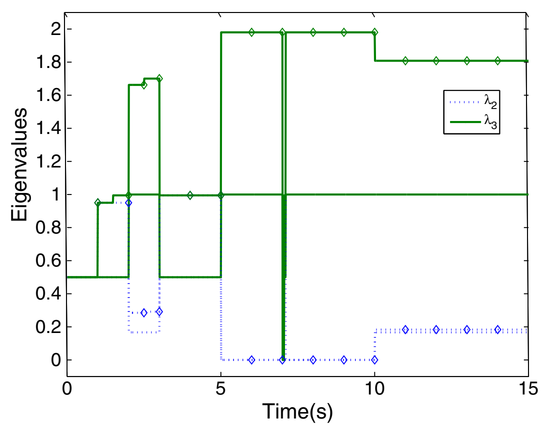

(a) Trajectories of state variables




(b) Asynchronous gain adjustment




(c) Real time adaptation of $d_{ii}$




(d) Comparison of eigenvalues

Figure 5. Robust performance under very fast topology changes.



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$$S(2)=\begin{bmatrix}1&1&0\\1&1&1\\0&0&1\end{bmatrix},\ S(3)=S(0),\ S(7)=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\ S(7.05)=\begin{bmatrix}1&0&0\\1&1&0\\1&0&1\end{bmatrix},\ S(7.1)=S(5).$$

In Figure 5(b), state trajectories under gain adaptation (i.e., the colored curves with diamond marks) are compared with those under constant gains (i.e., the curves without any marks). It is apparent that superior performance is achieved under the proposed estimation and gain adaptation scheme. In particular, at $t = 15$, the maximum discrepancy among the states is 0.03 under adaptive gains and 0.08 under constant gains. Figure 5(a) shows that adaptation of the gains is both asynchronous and distributed while being reactive to the topological changes of the communication/sensing network among the networked systems. Figure 5(c) shows the changes of $d_{ii}(t)$ due to both topological changes and gain adaptations. Because $T = 1$ is set, the rapid topological changes within interval $t \in [7,\ 7.1]$ do not satisfy the assumption. Nonetheless, as shown in Figures 5(a) and 5(c), the scheme of gain adaptation is robust with respect to unexpected and rapid changes in topology, and so is the system performance.

In Figure 5(d), evolution of the nonzero eigenvalues of $(I_3 - D(t))$ is plotted, in which the results under gain adaptation are decorated with diamond markers. As explained in the preceding examples, the proposed adaptation scheme improves (selectively some of) these eigenvalues.

## 6. CONCLUSION

This paper addresses the problem of designing cooperative control with distributively adaptive gains for networked systems whose communication network has intermittently changing and generally directed topologies. The proposed design is based on two novel solutions: real-time distributed estimation of connectivity of the network topology and asynchronous gain adaptation.

Connectivity of a directed network (such as irreducibility, reducibility, and leader–follower configurations) is captured by nonzero components of the first left eigenvector(s) associated with the first eigenvalue of 1 for piecewise-constant system matrix $D(t)$ (or the Laplacian of the corresponding graph). By utilizing the existing communication network, distributed high-gain estimators can be implemented at each system. The estimators are hybrid dynamic systems themselves in the sense that they have both continuous dynamics and discontinuous switches (i.e., reset of their state variables). It is shown that under the mild assumptions that the topological changes are not too rapid (than expected by the designer) and that transmission delays are bounded, distributed estimators all converge to the first left eigenvector(s). The success of estimating the first left eigenvector also enables each of the systems to distributively estimate the expected consensus value(s) of the overall system. Separation of the time scales between the networked system and its eigenvector observer is rigorously demonstrated, and a conservative estimate for choosing the observer gain is explicitly found by developing a lower bound on the magnitude of the Fiedler eigenvalue (or all nonzero eigenvalues of $(I - D)$) for digraphs in general. The proposed asynchronous and distributed gain adaptation scheme is developed using the Lyapunov direct method. Specifically, the time derivative of a cooperative control Lyapunov function naturally provides an expression in which contributions of distributed gain adjustments can be identified. Accordingly, control gains can be adjusted in a distributive and asynchronous way such that the time derivative of the cooperative control Lyapunov function becomes more negative for the purpose of enhancing convergence. It is also shown that for undirected networks, the proposed Lyapunov analysis reduces to the well-known concept of algebraic connectivity (i.e., the Fiedler eigenvalue) should the state enters the subspace spanned by the Fiedler eigenvector. In contrast, the proposed synthesis process applies to general networks and improves performance anywhere in the state space. Compared with cooperative control of constant gains, the proposed design of adaptive-gain cooperative control provides better performance. And, the resulting cooperative control is shown to be robust with respect to unexpectedly rapid topological changes.

Future work includes the development of distributed estimation algorithms without the use of resetting and their applications to meet other control objectives such as optimal performance, and so on.



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