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Smooth time-varying pure feedback control for chained non-holonomic systems with exponential convergent rate

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Abstract: The feedback stabilisation problem of non-holonomic chained systems and a novel feedback design scheme is proposed, which renders a smooth, time-varying, aperiodic, pure feedback control with exponential convergence rates. There are three main advantages with the proposed design. (i) In general, time-varying designs are mostly periodic and render asymptotic stability, whereas the proposed approach is aperiodic and have exponential convergent rates. (ii) A novel state scaling transformation is proposed. It shows that even though u_1 vanishes in regulation problems, intrinsic controllability of chained systems can be regained by judiciously designing the input u_1 and by applying the state scaling transformations. (iii) A class of memory functions is introduced into the control design, the controller dependency on the system's initial conditions in our previous work is removed and the control is a pure feedback. Moreover, the design is shown to be inversely optimal. Simulations and comparisons are conducted to verify the effectiveness of the proposed approach.

1 Introduction

Feedback stabilisation of non-linear systems has been one of the most important subjects in the study of non-linear control problems. As early as the 1980s, feedback linearisation technique has been prevailing, and sufficient and necessary conditions for exact feedback linearisation of large classes of affine non-linear systems were explicitly established with the adoption of differential geometry methods [1, 2]. Later on, the renewed interests on Lyapunov methods become dominant with the invention of the notion of control Lyapunov function and recursive designs such as backstepping [3, 4] in order to deal with more large classes of non-linear systems with unmatched and/or generalised matched uncertainties [5]. While those conventional non-linear control designs are broadly applicable, there exist some classes of inherently non-linear systems, such as non-linear systems with uncontrollable linearisation [6], which do not admit any smooth (or even continuous) pure state feedback controls as observed in the seminal paper [7] and therefore make the standard feedback linearisation technique and Lyapunov direct method no longer straightforwardly applicable. A typical such class of non-linear systems are non-holonomic mechanical systems [8], which are not feedback linearisable and their feedback stabilisation problem is challenging because of Brockett's necessary condition [7].

In recent years, considerable efforts have been devoted to the control and stabilisation of non-holonomic dynamic systems. Because many practical non-holonomic systems can be transformed into the chained form by coordinate and input transformations, the control designs based on chained form ensure their wide applicability. Apparently, chained systems do not satisfy Brockett's necessary condition, discontinuous and/or time-varying feedback controls have to be sought for its stabilisation. During the past decades, extensive studies have been performed and a great deal of solutions have been obtained following the lines of using discontinuous control method and/or time-varying control method [8]. In general, discontinuous controls can render exponential stability [9–13], whereas time-varying controls lead to asymptotic stability [14–17]. More recent studies have also seen the results of ρ-exponential stability of chained systems using periodic time-varying homogeneous feedback controls [18]. Exponential convergence is also reported in [19, 20]; however, the controls are not pure feedback because of the inclusion of systems initial conditions into the controller parameters.

In regulation problems of the chained systems, u_1 has to vanish; hence, the original system loses the controllability over time. However, this controllability may be recovered in a transformed space. An early attempt was the σ-process, which results in discontinuous switching controls. Improvements were made in [21], in which dynamic extension for control component u_1 was introduced to bypass the possible singularity because of singular initial conditions. The proposed methods are quasi-smooth and achieve quasi-exponential stability. Although the existing controls provide elegant solutions, there is still a desire to seek a global singularity-free transformation that maps the chained system into a controllable linear system. In this paper, a feedback design scheme is proposed based on an innovation non-linear design of dynamic control u_1, which is applicable under any initial conditions. Then a global state scaling transformation is applied and the transformed system is composed of a controllable linear nominal system and a time-varying component that vanishes in the sense of L^2. To this end, control u_2 can be synthesised using linear design methods. Both control u_1 and u_2 are shown to be optimal with respect to some performance index. Moreover, by introducing a class of memory function into u_1, the controller is irrelevant of the system's initial conditions, rendering the controls pure feedback.

The paper is organised as follows. In Section 2, the feedback stabilisation problem is formulated. In Section 3, controls are synthesised, convergence is proved and performance is discussed. Section 4 discusses the inverse optimality of the proposed control and Section 5 provides design examples using the proposed scheme. Section 6 gives the simulation results and Section 7 concludes the paper.

The notations used in this paper are standard. For a vector $x \in \Re^n$, $\|x\|$ denotes the euclidean norm. I denotes the identical matrix. For a matrix $A \in \Re^{n\times n}$, A^{T} is its transpose, $\|A\|$ denotes any form of matrix norm. For a scalar function $f(t)$, $\max_{a\le\eta\le b} f(\eta)$ or $\min_{a\le\eta\le b} f(\eta)$ denote the maximum/minimum of f on $[a \quad b]$, and $f(t) \in L^2[a, \infty)$ implies $\int_a^\infty |f(t)|^2\,\mathrm{d}t < \infty$.

2 Problem formulation

The objective of this paper is to present a control design scheme that leads to a smooth pure feedback control, which globally stabilises the chained non-holonomic system with exponential convergence rates. Consider the following chained system with the initial condition $x(t_0)$, where $t_0 \ge 0$ is the initial time.

$$\begin{cases} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= x_3 u_1 \\ & \vdots \\ \dot{x}_{n-1} &= x_n u_1 \\ \dot{x}_n &= u_2 \end{cases} \tag{1}$$

where $x = [x_1, \ldots, x_n]^{\mathrm{T}} \in \Re^n$ is the state.

It follows that system (1) can be reorganised into the following two subsystems

$$\dot{x}_1 = u_1 \tag{2}$$

And

$$\dot{z} = u_1 A z + B u_2 \tag{3}$$

where $z = [z_1 \ z_2 \ \cdots \ z_{n-1}]^{\mathrm{T}} \triangleq [x_2 \ x_3 \ \cdots \ x_n]^{\mathrm{T}}$, and

$$A \triangleq \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & & 1 \\ 0 & 0 & \cdots & & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Obviously, subsystem (2) is linear and can be easily stabilised. Subsystem (3) is a linear time-varying system, it is a chain of integrators weighted by u_1. Although chained system (1) is non-linearly controllable, the stabilisation of chained systems is difficult because of the following technical issues:

1. Chained systems are not linearly controllable around the origin.

2. Topologically, the chained systems cannot be stabilised by any continuous static feedback control $u = u(x)$ because of its non-linear characteristics.

3. Chained systems are not globally feedback linearisable (local feedback incarnation is possible such as the σ-process, but singularity manifold remains in all the neighbourhoods around the origin).

A straightforward thinking to overcome these difficulties is to search for a global-singularity free transformation that maps the chained system into a controllable linear system, then obtain controls in the transformed domain. We propose such a state scaling transformation, and based on this transformation, a feedback control design scheme is derived.

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3 Global state scaling transformation and control design scheme

In this section, the feedback control design of a component of u_1 is proposed. Based on the design, a global state-scaling transformation is introduced to overcome the singularity problem of the existing scaling transformations. This new transformation enables the designer to regain uniform complete controllability for the chained system and to design a class of smooth, time-varying, aperiodic, pure feedback and optimal controls which make the system states converge to the origin exponentially.

3.1 Design of control component u_1

Before giving the design of u_1, let us first define a set of memory functions.

Definition 1: For a time set:

$$\mathcal{T} = [t_0, \ t], \quad t \geq t_0 \geq 0$$

a set of memory function is defined to be

$$\mathcal{M}_{\mathcal{F}} = \{f : \Re^n \times \Re \rightarrow \Re^m \,|\, f(x(\eta), \eta), \ \forall \eta \in \mathcal{T}; \, n, m \in \aleph\}$$

From the definition, it is clear that the output of a memory function not only relates to its current variables, but also relates to the history of its variables.

The proposed control for component $u_1(t)$ is

$$u_1(t) = -\alpha x_1 + g(z, t)e^{-\beta t} \tag{4}$$

where $\alpha > \beta > 0$. To be a pure state feedback and non-switching control, $g(z, t)$ is required to have the following two properties:

1. $g(z, t)$ is smooth, uniformly bounded by $c > g(z, t) \geq \underline{g} \geq 0$ for some constants $c > \underline{g} \geq 0$.

2. In case of $\|z(t_0)\| = 0$, there should be $\underline{g} = 0$ and $g(z, t) \equiv 0$ for all $t > t_0$, that is, if the subsystem (3) is initially at the origin, control u_1 reduces to a regular negative state feedback. In case of $\|z(t_0)\| \neq 0$, $g(z, t)$ should monotone converge to c from $\underline{g}$ and $(c - g(z, t)) \in L^2[t_0 \ \infty)$.

Property 2 requires that if $\|z(t_0)\| = 0$, then $g(z, t) \equiv 0$. However, in case of $\|z(t_0)\| \neq 0$, there is $\lim_{t\rightarrow\infty} \|x(t)\| = 0$, which implies $\lim_{t\rightarrow\infty} \|z(t)\| = 0$, but now $\lim_{t\rightarrow\infty} g(z, t) = c \neq 0$. From this contradiction, one can conclude that if $g(z, t)$ is to meet the requirements for both cases, it can only be a memory function, that is, $g(z, t) \in \mathcal{M}_{\mathcal{F}}$. The second property also implies that if $x(t_0)$ is in the singular manifold $\{x | x_1 = 0, \|z\| \neq 0\}$, then $g(z, t)$ is able to yield a non-zero number so that x_1 and u_1 will deviate from zero. Then the controllability of subsystem (3) can be recovered in the subsequent design through state scaling transformations.

Remark 1: Although the first property sets $c > g(z, t) \geq \underline{g} \geq 0$, the design scheme is also valid if $c < g(z, t) \leq \overline{\overline{g}} \leq 0$, with $\overline{g} \leq 0$ to be some constant and corresponding changes are made in property 2.

3.2 Global state transformation

For subsystem (3), the following novel state scaling transformation is proposed: for $i = 1, \ldots, n-1$

$$\xi_i = \begin{cases} 0 & \text{if } \|z(t_0)\| = 0 \\ \dfrac{z_i}{e^{-(n-1-i)\beta t}} & \text{if else} \end{cases} \tag{5}$$

In the case that $\|z(t_0)\| = 0$, the ξ-system would not move. In the case that $\|z(t_0)\| \neq 0$, for $i = 1, \ldots, n-2$, the new dynamic equations are

$$\begin{aligned} \dot{\xi}_i &= \frac{\dot{z}_i}{e^{-(n-1-i)\beta t}} - \frac{-\beta(n-1-i)e^{-\beta t}}{e^{-(n-i)\beta t}} z_i \\ &= \frac{u_1}{e^{-\beta t}} \xi_{i+1} + \beta(n-1-i)\xi_i \end{aligned} \tag{6}$$

For $i = n-1$, since $\xi_i = z_i$, it follows that

$$\dot{\xi}_{n-1} = u_2 \tag{7}$$

Combine (6) and (7) into a matrix form and put together with the case that $\|z(t_0)\| = 0$, then the following dynamic model in transformed space is established

$$\dot{\xi} = \begin{cases} 0 & \text{if } \|z(t_0)\| = 0 \\ F(z, t)\xi + Bu_2 & \text{if else} \end{cases} \tag{8}$$

where

$$\begin{aligned} F(z, t) &= \mathrm{diag}\{\beta(n-2), \beta(n-3), \ldots, \beta, 0\} \\ &\quad + \left[g(z, t) - \alpha \frac{x_1}{e^{-\beta t}}\right] A \end{aligned}$$

The uniform complete controllability of the transformed system $\{F(z, t), B\}$ is established in the following theorem.

Theorem 1: If $g(z, t)$ has the properties given in Section 3.1, then the transformed system $\{F(z, t), B\}$ is uniformly completely controllable.

Proof: Simple derivation shows that

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{x_1(t)}{e^{-\beta t}} = -(\alpha - \beta) \frac{x_1}{e^{-\beta t}} + g(z, t)$$

Therefore $x_1(t)/e^{-\beta t}$ can be solved as

$$\frac{x_1(t)}{e^{-\beta t}} = \frac{x_1(t_0)}{e^{-\beta t_0}} e^{-(\alpha-\beta)(t-t_0)} + \int_{t_0}^{t} e^{-(\alpha-\beta)(t-\tau)} g(z, \tau)\mathrm{d}\tau \quad (9)$$

Since $\lim_{t\to\infty} g(z, t) = c$, there is $\lim_{t\to\infty} (x_1(t)/e^{-\beta t}) = (c/(\alpha-\beta))$. Therefore we can obtain

$$\lim_{t\to\infty}\left[g(z, t) - \alpha\frac{x_1}{e^{-\beta t}}\right] = \lim_{t\to\infty} g(z, t) - \alpha \lim_{t\to\infty}\frac{x_1}{e^{-\beta t}} = -\frac{c\beta}{\alpha-\beta}$$

It follows that the time-varying system $\{F(z, t), B\}$ can be partitioned into a nominal component and a time-varying component

$$F(z, t) = F_n + F_t(z, t)$$

where

$$F_n = \mathrm{diag}\{\beta(n-2), \ldots, \beta, 0\} - \frac{c\beta}{\alpha-\beta}A$$

and

$$F_t(z, t) = \left[g(z, t) - \alpha\frac{x_1}{e^{-\beta t}} + \frac{c\beta}{\alpha-\beta}\right]A \quad (10)$$

It is clear that the time-varying component $F_t(z, t)$ vanishes, hence the transformed system $\{F(z, t), B\}$ converges to its nominal system $\{F_n, B\}$. By the design properties of $g(z, t)$, there is $c \neq 0$. Hence the pair $\{F_n, B\}$ is completely controllable, which implies the time-varying system $\{F(z, t), B\}$ is uniformly completely controllable. □

3.3 Design of control component u_2

The dynamic control component u_2 is designed to be

$$u_2(t) = -R_2^{-1}B^{\mathrm{T}}\hat{P}(t)\xi \quad (11)$$

where $\hat{P}(t) > 0$ is symmetric, uniformly bounded and satisfies the following matrix differential Riccati equation with $\hat{P}(\infty) > 0$

$$\dot{\hat{P}}(t) + \hat{P}(t)\hat{F}(t) + \hat{F}^{\mathrm{T}}(t)\hat{P}(t) + Q_2 - \hat{P}(t)BR_2^{-1}B^{\mathrm{T}}\hat{P}(t) = 0 \quad (12)$$

where

$$\hat{F}(t) = F_n + \left[\underline{g} + \frac{\beta c}{\alpha-\beta} + \alpha(\underline{g} - c)(t - t_0)\right]e^{-(\alpha-\beta)(t-t_0)}A$$

and $Q_2 \in \Re^{n-1\times n-1}$, $R_2 \in \Re$ are constant and positive-definite matrices. By a procedure similar to Theorem 1, the uniform complete controllability of the pair $\{\hat{F}(t), B\}$ can be verified, hence such a $\hat{P}(t)$ can always be found.

Lemma 1: Let

$$\hat{F}_t(t) = \left[\underline{g} + \frac{\beta c}{\alpha-\beta} + \alpha(\underline{g} - c)(t - t_0)\right]e^{-(\alpha-\beta)(t-t_0)}A \quad (13)$$

then the norm of difference $\|F_t(z, t) - \hat{F}_t(t)\| \in L^2[t_0\ \infty)$.

Proof: It follows from (9) and (10) that

$$\begin{aligned} F_t(z, t) &= \left[g(z, t) - \alpha\frac{x_1(t)}{e^{-\beta t}} + \frac{c\beta}{\alpha-\beta}\right]A \\ &= \left\{[g(z, t) - c] - \gamma e^{-(\alpha-\beta)(t-t_0)} \right. \\ &\quad \left. -\alpha\int_{t_0}^{t} e^{-(\alpha-\beta)(t-\tau)}[g(z, \tau) - c]\,\mathrm{d}\tau\right\}A \end{aligned} \quad (14)$$

where $\gamma = \alpha[[x_1(t_0)/e^{-\beta t_0}] - [c\beta/(\alpha-\beta)]]$. In the above equation, by the design properties of $g(z, t)$, $(g(z, t) - c)$ is L^2, and the second term is also L^2. For the third term, it can be treated as the input response of an exponential stable, linear time-invariant dynamic system with a L^2 input; hence, this term has to be L^2 [3]. Therefore $\|F_t(z, t)\|$ is L^2. Moreover, from (13), $\|\hat{F}_t(t)\|$ is L^2. Since

$$\|F_t(z, t) - \hat{F}_t(t)\| \le \|F_t(z, t)\| + \|\hat{F}_t(t)\|$$

there must be $\|F_t(z, t) - \hat{F}_t(t)\| \in L^2[t_0\ \infty)$. □

The closed-loop convergence of system (1) under control (4) and (11) is stated in the following theorem.

Theorem 2: For any $g(z, t)$ that has the properties presented in Section 3.1, the control (4) and (11) globally asymptotic stabilise system (1) with exponential convergence rates.

Proof: It is clear from (4) and (11) that if $\|z(t_0)\| = 0$, then $u_2 \equiv 0$ and u_1 reduces to $u_1 = -\alpha x_1$, therefore system (1) is exponentially stabilised. Consider the case that $\|z(t_0)\| \neq 0$. For subsystem (2), construct the following Lyapunov function candidates $V_1(x_1) = 1/2x_1^2$, and $V_2(\xi) = \xi^{\mathrm{T}}\hat{P}(t)\xi$. It follows that

$$\begin{aligned} \dot{V}_1(x_1) &= x_1\dot{x}_1 = -\alpha x_1^2 + x_1 g(z, t)e^{-\beta t} \\ &\le -\alpha x_1^2 + ce^{-\beta t_0}|x_1| \end{aligned} \quad (15)$$

Equation (15) shows that x_1 is uniformly ultimately bounded by the set

$$\Omega \triangleq \left\{x_1 : |x_1| \le \frac{ce^{-\beta t_0}}{\alpha}\right\}$$

Since if $x_1(t_0) \notin \Omega$, there will be $\dot{V}_1 < 0$, hence $|x_1|$ monotonically decreases to Ω. If $x_1(t_0) \in \Omega$, $x_1(t)$ cannot get out of Ω where $\dot{V}_1 < 0$. Therefore a uniform bound

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for $x_1(t)$ is

$$\delta \triangleq \max\left\{|x_1(t_0)|, \frac{ce^{-\beta t_0}}{\alpha}\right\}$$

It follows that

$$\dot{V}_1(x_1) \leq -2\alpha V_1 + \delta c e^{-\beta t}$$

Hence subsystem (2) is globally exponentially attractive by [5, Lemma 2.19]. Therefore subsystem (2) is asymptotic stable with exponential convergence.

The closed-loop system of (8) is

$$\begin{aligned}\dot{\xi} &= F(z, t)\xi - BR_2^{-1}B^{\mathrm{T}}\hat{P}(t)\xi \\ &= [F_n - BR_2^{-1}B^{\mathrm{T}}\hat{P}(t) + F_t(z, t)]\xi \\ &= [\hat{F}(t) - BR_2^{-1}B^{\mathrm{T}}\hat{P}(t) + F_t(z, t) - \hat{F}_t(t)]\xi\end{aligned}$$

where $F_t(z, t)$ is defined in (10). It follows that

$$\begin{aligned}\dot{V}_2(\xi) &= \xi^{\mathrm{T}}\{\dot{\hat{P}}(t) + [\hat{F}(t) - BR_2^{-1}B^{\mathrm{T}}\hat{P}(t) + F_t(z, t) \\ &\quad - \hat{F}_t(t)]^{\mathrm{T}}\hat{P}(t) + \hat{P}(t)[\hat{F}(t) - BR_2^{-1}B^{\mathrm{T}}\hat{P}(t) \\ &\quad + F_t(z, t) - \hat{F}_t(t)]\}\xi \\ &= \xi^{\mathrm{T}}[\dot{\hat{P}}(t) + \hat{F}(t)^{\mathrm{T}}\hat{P}(t) + \hat{P}(t)\hat{F}(t) \\ &\quad - 2\hat{P}(t)BR_2^{-1}B^{\mathrm{T}}\hat{P}(t) + N(z, t)]\xi \\ &= -\xi^{\mathrm{T}}[Q_2 + \hat{P}(t)BR_2^{-1}B^{\mathrm{T}}\hat{P}(t) - N(z, t)]\xi \\ &\leq \left[-\frac{c_2}{c_3} + \frac{\sum_{i=1}^{n-1}|\lambda_i(N(z, t))|}{c_4}\right]V_2 \end{aligned} \tag{16}$$

where $\lambda_i(\cdot)$ denotes the ith eigenvalue of a square matrix, c_2, c_3, c_4 are constants that satisfy

$$\begin{aligned}&c_1 I > Q_2 + \hat{P}(t)BR_2^{-1}B^{\mathrm{T}}\hat{P}(t) > c_2 I > 0, \\ &c_3 I > \hat{P}(t) > c_4 I > 0\end{aligned}$$

and

$$\begin{aligned}N(z, t) &= [\hat{P}(t)(F_t(z, t) - \hat{F}_t(t)) + (F_t(z, t) - \hat{F}_t(t))^T\hat{P}(t)] \\ &\in \Re^{n-1\times n-1}\end{aligned} \tag{17}$$

Since Q_2, R_2 are constant matrices, hence $\hat{P}(t)$ is uniformly bounded and constants c_1, c_2, c_3, c_4 can be found.

Note that

$$|\lambda_i(N(z, t))| \leq \|N(z, t)\| \leq 2\|\hat{P}(t)\|\|F_t(z, t) - \hat{F}_t(t)\|$$

Since $\hat{P}(t)$ is uniformly bounded, and by Lemma 1, $\|F_t(z, t) - \hat{F}_t(t)\| \in L^2[t_0\ \infty)$, both $\|N(z, t)\|$ and $|\lambda_i(N(z, t))|$ are L^2. Then treating (16) as a scalar dynamic system, V_2 is exponentially stabilised by invoking [22, Lemma 2.2] and comparison principle. It follows that the ξ-systems is exponentially stabilised, which implies that the z-system is exponentially stabilised according to the transformation (5). After combining the results for subsystems (2) and (3), it is concluded that the overall system has asymptotic stability with exponential convergence rates. Since the argument is globally valid, the stability results is global. □

The control u_2 in (11) shows that the underlying idea is that using the pure time function $\hat{F}_t(t)$ in (13) to approximate the time-varying component $F_t(z, t)$ of $F(z, t)$, which is given in (14). The goal is to remove the state variable z from the system matrix so that the control u_2 can be derived from the linear time-varying system $\{\hat{F}(t), B\}$. This approximation assumes that $g(z, t)$ converges to c exponentially, that is

$$g(z, t) - c \simeq (\underline{g} - c)e^{-(\alpha-\beta)(t-t_0)}$$

In this case, the model difference $\|F_t(z, t) - \hat{F}_t(t)\|$ is L^2 by Lemma 1, which guarantees the exponential stability.

Note that in limit, both $F(z, t)$ and $\hat{F}(t)$ reduce to their nominal system F_n. Hence by solving $P > 0$ from the following algebraic Riccati equation

$$F_n^{\mathrm{T}}P + PF_n + Q_2 - PBR_2^{-1}B^{\mathrm{T}}P = 0$$

the control

$$u_2'(t) = -R_2^{-1}B^{\mathrm{T}}P\xi \tag{18}$$

is also a stabilising control, since this case is equivalent to take $\hat{F}_t(t) \equiv 0$, $\hat{F}(t) = F_n$, and the model difference is $\|F_t(z, t)\|$, which by itself is L^2 as shown in Lemma 1. In simulations, we compared control effects for both u_2 and u_2'. It shows that the performance of u_2 with $\hat{F}_t(t)$ in (13) is much better.

4 Optimal performance

The following theorem shows that the proposed control (4) and (11) is optimal with respect to certain performance index.

Theorem 3: For system (1), the feedback control (4) and (11) is optimal with respect to performance index $J = J_1 + J_2$, where

$$J_1(t, u_1(t)) = \int_t^\infty \left\{\begin{bmatrix} x_1 & y\end{bmatrix} Q_1(t)\begin{bmatrix} x_1 \\ y\end{bmatrix} + u_1^2\right\}\mathrm{d}t$$

and

$$J_2(t, u_2(t)) = \int_t^\infty [\xi^{\mathrm{T}}\hat{Q}_2(t)\xi + u_2 R_2 u_2]\mathrm{d}t$$

where $y = e^{-\beta t}$ is the augmented state

$$Q_1(t) = \begin{bmatrix} \alpha^2 & \dot{g} - (\alpha+\beta)g \\ \dot{g} - (\alpha+\beta)g & 2k\beta + g^2 \end{bmatrix}$$

with k chosen to satisfy

$$k > \max\left\{\frac{c^2}{\alpha}, \frac{\dot{g}^2 + g^2\beta(4\alpha+\beta) - 2g\dot{g}(\beta+2\alpha)}{4\alpha^2\beta}, \frac{\dot{g}^2 + g^2\beta(2\alpha+\beta) - 2g\dot{g}(\beta+\alpha)}{2\alpha^2\beta}\right\}$$

and $\hat{Q}_2(t) = Q_2 - N(z, t)$, with $N(z, t)$ defined in (17).

Proof: By design properties of $g(z, t)$, g is monotone and uniformly bounded, therefore g must be uniformly continuous, hence $\dot{g}$ is uniformly bounded. Therefore such a k can always be found and by the specified choice of k, $Q_1(t)$ is positive definite.

Under control (4), the closed-loop system of subsystem (2) is

$$\dot{x}_1 = u_1 = -\alpha x_1 + g(z, t)y \tag{19}$$

We first show that

$$V_1'(x_1, y) \triangleq \alpha x_1^2 - 2gx_1y + ky^2$$

is a Lyapunov function of the augmented system (19). It is straightforward that by the specified choice of k, V_1' is positive definite. It follows that

$$\begin{aligned}\dot{V}_1' &= 2\alpha x_1\dot{x}_1 + 2ky\dot{y} - 2\dot{g}x_1y - 2g\dot{x}_1y - 2gx_1\dot{y} \\ &= -2\alpha^2x_1^2 - 2(k\beta + g^2)y^2 + (4\alpha g + 2g\beta - 2\dot{g})x_1y \\ &= -\begin{bmatrix} x_1 & y \end{bmatrix}\begin{bmatrix} 2\alpha^2 & \dot{g} - g(2\alpha+\beta) \\ \dot{g} - g(2\alpha+\beta) & 2k\beta + 2g^2 \end{bmatrix}\begin{bmatrix} x_1 \\ y \end{bmatrix}\end{aligned}$$

$\dot{V}_1'$ is negative definite, hence V_1' is a Lyapunov function of the augmented system. To show the optimality of u_1 w.r.t. J_1, substitute control u_1 in (4) with an incremental term Δu_1 into J_1, that is, $u_1(t) = -\alpha x_1 + g(z, t)y + \Delta u_1$. Evaluate $\dot{V}_1'$ along the system's new trajectory with the perturbed control, we have

$$\dot{V}_1' = -\begin{bmatrix} x_1 & y \end{bmatrix}\begin{bmatrix} 2\alpha^2 & \dot{g} - g(2\alpha+\beta) \\ \dot{g} - g(2\alpha+\beta) & 2k\beta + 2g^2 \end{bmatrix}\begin{bmatrix} x_1 \\ y \end{bmatrix} - 2u_1\Delta u_1$$

It follows that the performance index J_1 for the perturbed system is

$$\begin{aligned}J_1 &= \int_t^\infty \left\{\begin{bmatrix} x_1 & y \end{bmatrix} Q_1(t)\begin{bmatrix} x_1 \\ y \end{bmatrix} + (u_1 + \Delta u_1)^2\right\}\mathrm{d}t \\ &= \int_t^\infty \left\{\begin{bmatrix} x_1 & y \end{bmatrix} Q_1(t)\begin{bmatrix} x_1 \\ y \end{bmatrix} + u_1^2 + 2u_1\Delta u_1 + \Delta u_1^2\right\}\mathrm{d}t \\ &= -\int_t^\infty \mathrm{d}V_1' + \int_t^\infty \Delta u_1^2\,\mathrm{d}t \\ &= V_1'(x_1(t), y(t)) + \int_t^\infty \Delta u_1^2\mathrm{d}t\end{aligned}$$

which is minimised by $\Delta u_1 = 0$, hence u_1 is optimal with respect to J_1.

For system (8), it is straightforward to verify that the following matrix differential equation holds

$$\begin{aligned}&\dot{\hat{P}}(t) + \hat{P}(t)F(z, t) + F(z, t)\hat{P}(t) + \hat{Q}_2(t) \\ &\quad - \hat{P}(t)BR_2^{-1}B^{\mathrm{T}}\hat{P}(t) = 0\end{aligned} \tag{20}$$

To show the optimality of system (8) with respect to J_2, substituting u_2 in (11) with an incremental term Δu_2 (i.e. $u_2 = -R_2^{-1}B^{\mathrm{T}}\hat{P}\xi + \Delta u_2$) into J_2, and evaluating $\dot{V}_2$ along the new state trajectory and control

$$\dot{V}_2 = -\xi^{\mathrm{T}}[\hat{Q}_2(t) + \hat{P}(t)BR_2^{-1}B^{\mathrm{T}}\hat{P}(t)]\xi + 2\xi^{\mathrm{T}}\hat{P}(t)B\Delta u_2$$

It follows that the performance index J_2 for the perturbed system is

$$\begin{aligned}J_2 &= \int_t^\infty [\xi^{\mathrm{T}}\hat{Q}_2(t)\xi + \xi^{\mathrm{T}}\hat{P}(t)BR_2^{-1}B^{\mathrm{T}}\hat{P}(t)\xi \\ &\quad - 2\xi^{\mathrm{T}}\hat{P}(t)B\Delta u_2 + \Delta u_2^{\mathrm{T}}R_2\Delta u_2]\mathrm{d}t \\ &= -\int_t^\infty \mathrm{d}V_2 + \int_t^\infty \Delta u_2^{\mathrm{T}}R_2\Delta u_2\,\mathrm{d}t \\ &= V_2(\xi(t)) + \int_t^\infty \Delta u_2^{\mathrm{T}}R_2\Delta u_2\,\mathrm{d}t\end{aligned}$$

Table 1 Summary of various control approaches

	Discontinuous control	Ordinary time-varying	ρ-exponential	Our approach
continuity	discontinuous	smooth	smooth	smooth
convergence	exponential	asymptotic	exponential	exponential
oscillation	aperiodic	periodic	periodic	aperiodic
stability region	global	global	global	global

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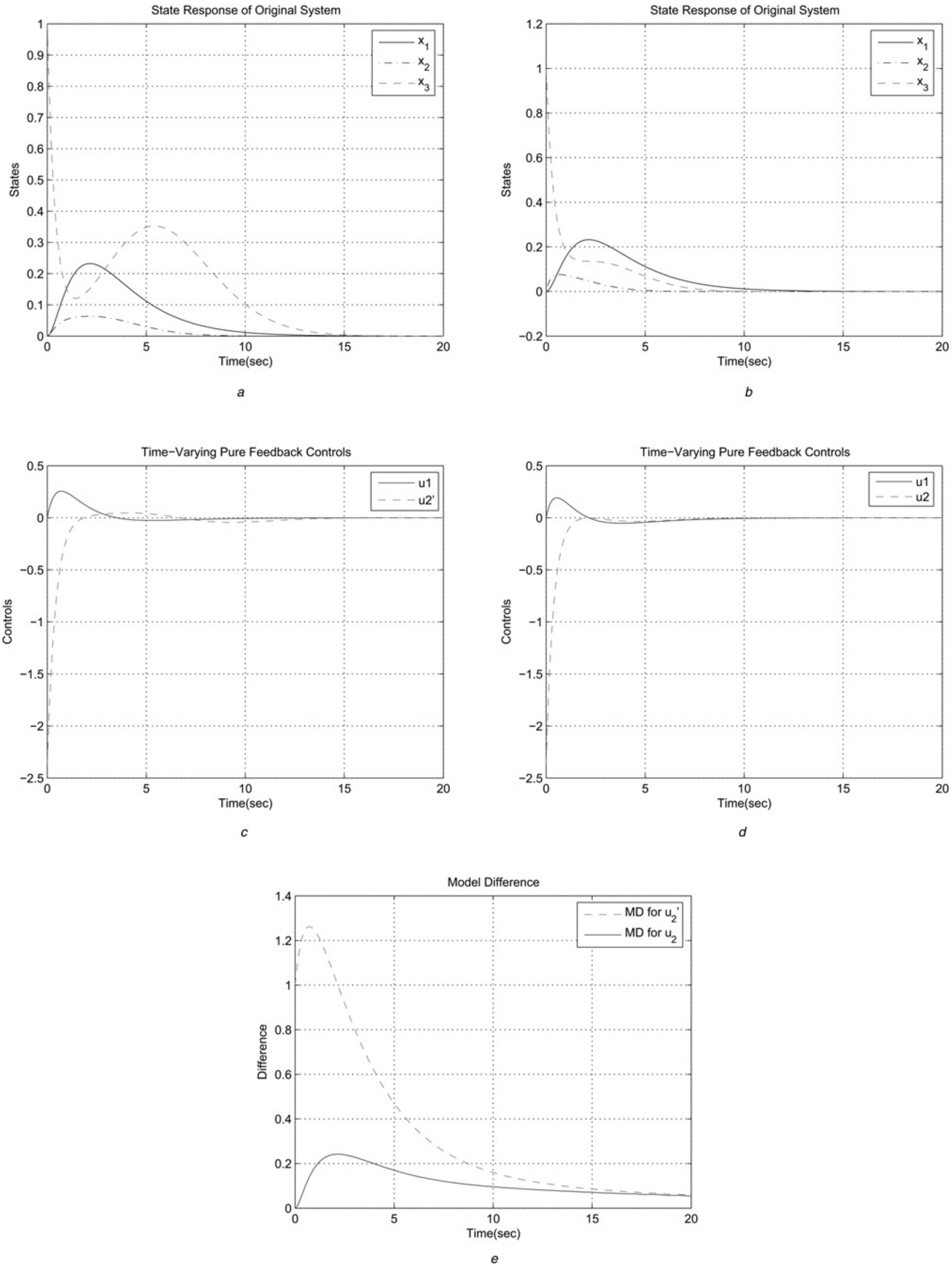

Figure 1 *Simulation results of the proposed control*

a, *c* State and control with u_2' in (18)
b, *d* State and control with u_2 in (11)
e Model difference for u_2' and u_2

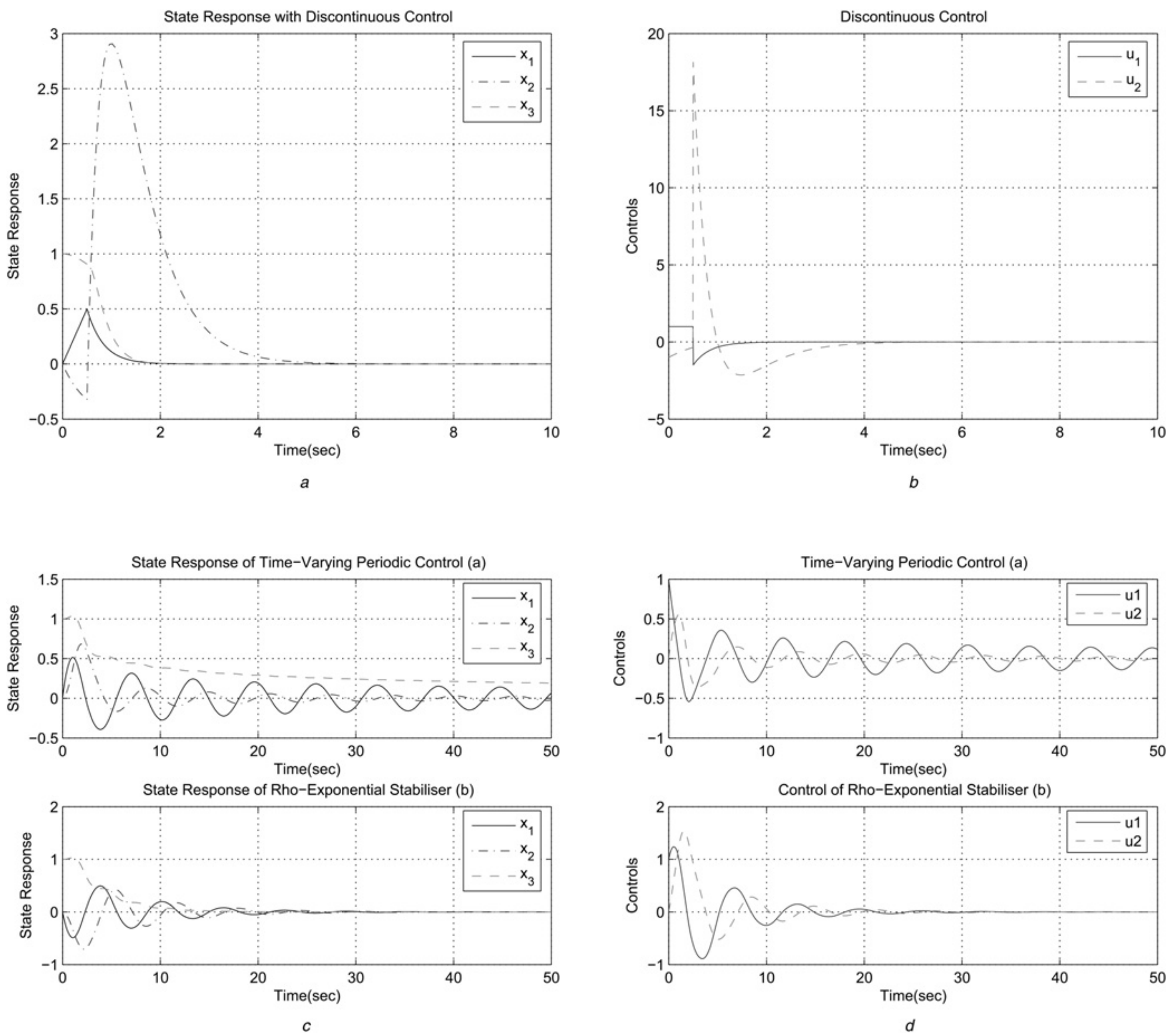

Figure 2 *Control effects for various other control designs*

a, *b* State and control of discontinuous control

c, *b* States and controls for an ordinary periodic time-varying feedback design and ρ-exponential stabiliser

Here, (20) is used. It is clear that J_2 is minimised by $\Delta u_2 = 0$. Note that Q_2 is positive definite and in Theorem 2, we have shown $\|N(z, t)\|$ is L^2, therefore $N(z, t)$ vanishes. Hence in some cases, $\hat{Q}_2(t)$ might need a finite period to be positive definite. But by the above Lyapunov argument, the performance index J_2 would be always positive. To this end, the overall system has been shown to be optimal with respect to J. □

5 Design examples

In this section, design examples are provided by applying the proposed scheme. Examples of non-trivial memory functions in $\mathcal{M}_{\mathcal{F}}$ include, for instance

$$\int_{t_0}^{t} l(\|z(\tau)\|)\,\mathrm{d}\tau, \quad \min_{t_0 \le \eta \le t} l(\|z(\eta)\|), \quad \max_{t_0 \le \eta \le t} l(\|z(\eta)\|)$$

where $l(\cdot)$ is a function. For example, we design $g(z, t)$ to be

$$g(z, t) = \frac{t\int_{t_0}^{t} \|z(\tau)\|\,\mathrm{d}\tau}{1 + t\int_{t_0}^{t} \|z(\tau)\|\,\mathrm{d}\tau} \tag{21}$$

According to Theorem 2, to show the stability, one only needs to show that $g(z, t)$ in (21) has the two properties given in Section 3.1.

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It is straightforward to verify that the closed-loop systems of (2) and (8) under control (4) and (11) are globally Lipschitz. Therefore the solution x_1 and ξ exist and is unique, hence by transformation (5), solution z exists. For property 1, clearly $g(z, t)$ is differentiable everywhere for $t \geq t_0$ and uniformly bounded by $\underline{g} = 0$ and $c = 1$. For property 2, if $\|z(t_0)\| = 0$, then $u_2(\bar{t}_0) = 0$. Subsystem (3) would not move, hence $z(t) \equiv 0$, which in turn yields $g(z, t) \equiv 0$. In case of $\|z(t_0)\| \neq 0$, there is $\lim_{t\to\infty} g(z(t), t) = c = 1$. Moreover

$$c - g(z, t) = \frac{1}{1 + t\int_{t_0}^{t} \|z(\tau)\| \mathrm{d}\tau} > 0$$

It is clear that whether or not z is exponential convergent, $(c - g(z, t)) \in L^2[t_0\ \infty)$.

6 Simulations and comparisons

In this section, simulation results are provided to illustrate the effectiveness of the proposed control. Comparisons are made with existing controls and their features are summarised into Table 1.

In the simulation, a third-order chained system is studied. $g(z, t)$ in (21) is used. The design parameters are set to be $\alpha = 1$, $\beta = 0.5$, $Q_2 = I$ and $R_2 = 1$. To verify the effectiveness of avoiding singularity, initial condition of the state is set to be $x(t_0) = [0\ 0\ 1]^{\mathrm{T}}$.

The results in Fig. 1 verify that the proposed stabilising control is successful. Figs. 1*a* and *c* illustrate the state and control for u_2' in (18). Figs. 1*b* and *d* illustrate the control effects for u_2 in (11). Fig. 1*e* shows the model difference for the two cases. Clearly, in both cases, despite $x_1(t_0) = 0$, asymptotic stability and exponential convergence rates are achieved and both states and controls are aperiodic. When u_2' is used, $\hat{F}_t(t) \equiv 0$, the model difference is $\|F_t(z, t)\|$, its transient is larger and converges slower. Fig. 1*e* shows that by applying (13), the model difference $\|F_t(z, t) - \hat{F}_t(t)\|$ is smaller, hence the transient response is improved.

For the same system with the same initial condition, simulations for discontinuous controls [13], ordinary periodic time-varying feedback controls [14] and ρ-exponential stabiliser [18] are also conducted. Figs. 2*a* and *b* show the state and control of the discontinuous control. Fig. 2*c* shows the states for an ordinary periodic time-varying feedback control and the ρ-exponential stabiliser. Fig. 2*d* shows the controls for an ordinary periodic time-varying feedback control and the ρ-exponential stabiliser. Although both the proposed control and the discontinuous control have exponential convergence rates, the discontinuous control easily achieves a faster rate as a trade-off for the continuity. Ordinary periodic time-varying state feedback is well-known for its low convergence rates. The ρ-exponential stabiliser is improved from an ordinary periodic time-varying feedback control, which could achieve exponential convergence rates, hence would be much faster than an ordinary one. From the simulations, the proposed control with aperiodic feature seems a bit faster than ρ-exponential stabiliser. The characteristics of the aforementioned controls and our proposed control are summarised in Table 1, and their differences are easily seen.

7 Conclusion

In this paper, we presented a novel design scheme to synthesise a pure feedback, exponential convergent regulators for chained non-holonomic systems. It is known that if u_1 vanishes, the original chained system is not controllable. However, based on an innovative design of dynamic control u_1, which includes a regular feedback term and an additive disturbance composed of a bounded memory function and an exponential decaying term, and by applying a global singularity-free state scaling transformation, subsystem (3) is cast into a new linear time-varying system, which is guaranteed to be uniformly completely controllable, therefore the controllability can be recovered, and optimal controls can be derived. Moreover, the controller is irrelevant of the system's initial condition. The design approach is systematic and straightforward and simulation results verify the effectiveness of the proposed control.

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