Cooperative Control of Dynamical Systems With Application to Autonomous Vehicles

Zhihua Qu, *Senior Member, IEEE*, Jing Wang, *Member, IEEE*, and Richard A. Hull, *Member, IEEE*

***Abstract*—In this paper, a new framework based on matrix theory is proposed to analyze and design cooperative controls for a group of individual dynamical systems whose outputs are sensed by or communicated to others in an intermittent, dynamically changing, and local manner. In the framework, sensing/communication is described mathematically by a time-varying matrix whose dimension is equal to the number of dynamical systems in the group and whose elements assume piecewise-constant and binary values. Dynamical systems are generally heterogeneous and can be transformed into a canonical form of different, arbitrary, but finite relative degrees. Utilizing a set of new results on augmentation of irreducible matrices and on lower triangulation of reducible matrices, the framework allows a designer to study how a general local-and-output-feedback cooperative control can determine group behaviors of the dynamical systems and to see how changes of sensing/communication would impact the group behaviors over time. A necessary and sufficient condition on convergence of a multiplicative sequence of reducible row-stochastic (diagonally positive) matrices is explicitly derived, and through simple choices of a gain matrix in the cooperative control law, the overall closed-loop system is shown to exhibit cooperative behaviors (such as single group behavior, multiple group behaviors, adaptive cooperative behavior for the group, and cooperative formation including individual behaviors). Examples, including formation control of nonholonomic systems in the chained form, are used to illustrate the proposed framework.**

***Index Terms*—Consensus, cooperative control, cooperative controllability, formation control, high-order dynamical systems, matrix theory, networked systems, time-varying sensing/communication.**

I. Introduction

THIS paper proposes a matrix-theory-based framework of analysis and cooperative control designs for a group of individual but heterogeneous dynamical systems and seeks for the least restrictive requirement on sensing and communication among the systems. As an example, a group of unmanned autonomous vehicles are commanded to perform a set of tasks as a group, and individual robots of different capabilities are to exhibit not only certain group behavior but also their individual behaviors. In the general case, the dynamical systems operate in a dynamically changing and uncertain environment. As such, sensing and communication among the systems are intermittent and local, and their changes are not known *a priori* or predictable by either deterministic or probabilistic models. The fundamental questions are as follows: what is the necessary and sufficient condition for sensing/communication and how to design cooperative control to achieve a guaranteed performance.

There have been many earlier results on distributed robotics, and these results are obtained using heuristic approaches. For example, artificial intelligence methods [1] have been extensively used to explore the architecture, task allocation, mapping building, coordination, and control algorithms in multirobot motion systems [2]–[5]. Multirobot localization and exploration are studied in [6] using a probabilistic approach. Path planning and formation control are investigated in [7] using behavior-based control paradigm [8], where the rule-based formation behaviors have been defined and evaluated through simulations. A simple heuristic distributed algorithm is proposed in [9] for identical mobile robots to form a circle of a given radius, where each robot updates its position according to a set of rules. In many cases, cooperative rules are chosen to mimic animal behaviors [10]. The basic cohesion, separation, and alignment rules are extracted by observing the animal flocking and simulated through computer animation [11]. The alignment problem is recently studied in [12], and the so-called nearest-neighbor rule is derived experimentally. That is, all the particles of point mass move in the plane with the same speed, and their headings are updated individually by the same discrete and local rule of averaging its own heading and the headings of its neighbors. The group flocking behaviors such as avoidance, aggregation, and dispersion have also been explored [13].

While heuristic and bioinspired approaches have produced many interesting and very useful results, there was a lack of theoretical frameworks for both analysis and control design. Under the assumption that sensing/communication is time-invariant, analysis and control of multivehicle systems can be done using various standard approaches in control theory, for instance, [14]–[20]. However, cooperative control of dynamical systems often involves intermittent, local, and dynamically changing communication/sensing. Thus, the central and difficult question is twofold: what is the necessary and sufficient condition on sensing/communication to ensure cooperative controllability and how to design cooperative controls for a group of general dynamical systems in the network?

A truly distributed cooperative law based on the nearest-neighbor rule [12] is proposed in [21] for mobile autonomous agents, dynamic changes of communication topology are represented by a undirected graph, performance of the cooperative law is analyzed by graph theory, and convergence is obtained under the assumption that the undirected graph is connected. It is significant and groundbreaking to note that the result in [21]

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Z. Qu and J. Wang are with the School of Electrical Engineering and Computer Science (EECS), University of Central Florida, Orlando, FL 32816 USA (e-mail: qu@mail.ucf.edu; ecejwang@ieee.org).

R. A. Hull is with the Science Applications International Corporation (SAIC), Orlando, FL 32801 USA (e-mail: Rich.A.Hull@saic.com).

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provides a graph-theory-based framework to analyze a group of networked agents. More recently, the result in [21] is extended in [22] to a multiagent system in which communication is represented by a directed graph, and convergence of consensus is ensured under the less restrictive condition that there exists a spanning tree (see the definition in [22]). In parallel, local control strategies for groups of mobile autonomous agents are proposed and analyzed in [23], and formation control is shown in [24] to be convergent for unicycles under the condition that the graph has a globally reachable node (see the definition in [24]). Cooperative controls have also been analyzed by other researchers using the combination of graph theory and Nyquist stability criterion [25], [26], using proximity graphs [27], and using the combination of graph theory, convexity, and discrete-set-valued Lyapunov functions [28]. All of these results use graph theory as the main approach to derive sensor graph conditions and to conclude convergence, and most of them deal with identical agents or linear systems that, using the terminology of control theory, are of relative degree one. Extension of [21] to second-order dynamics has been pursued in [29] and [30]. A more complete account of the convergence of multiagent coordination, consensus, and flocking can be found in a recent paper [31].

Different from and complementary to graph theory, matrix theory and control theory are used as the means in this paper to develop a new framework for analyzing a group of dynamical systems and for designing cooperative controls (and so are the preliminary results in [32]–[36]). Specifically, by using a piecewise-constant and binary-valued matrix to capture the changes of sensing/communication, two sets of new results on matrix reducibility and irreducibility have been developed. The first set is on augmentation of irreducible and reducible matrices, which enables us to analyze not only identical agents, but also heterogeneous dynamical systems of arbitrary, but finite relative, degree in the presence of dynamically changing communication/sensing among them. The second set is reducible matrices, and by utilizing the canonical form of lower triangulation of reducible matrices, a necessary and sufficient condition is obtained under which any multiplicative sequence of row-stochastic matrices is convergent, and the condition is stated and easily explained in terms of changing communication/sensing patterns. Specifically, a multiplicative sequence of solution matrices (obtained from reducible system matrices) is convergent to a matrix of identical rows if and only if it consists of a nonvanishing, lower triangularly complete subsequence (that is, a subsequence whose lower triangulations have at least one nonvanishing element in each lower triangular matrix block row). Through development and adoption of lower triangulation as the canonical form, all the existing graph theory results (such as a strongly connected graph, a spanning tree, and a globally reachable node) have their counterparts in algebraic matrix theory. Besides admitting high-order dynamical systems, the algebraic matrix approach also has the advantage that it allows us to explore different convergence, obtain explicit expression of convergence rate for the matrix sequence, and provide a new more intuitive concept and also a new simpler convergence test. Specifically, it is shown that systems can be cooperatively controlled if they have over time only one sensing/communication group that can be tested by simply calculating the binary product of communication/sensing matrices over an interval.

In the new framework, a canonical form of vehicle-level dynamics is proposed for separating the designs of vehicle-level control and cooperative control. Using this canonical form, a designer can embed different control objectives and handle different vehicle dynamics. As such, designs of single-objective cooperative control, multiobjective cooperative control, formation control, and adaptive cooperative control are unified in this paper, and their differences boil down to simple choices of a constant gain matrix in the design. These new results and their matrix-theory-based framework of analysis and design show how existing control theories can be enriched to handle networked dynamical systems, and they also enable us to solve more complicated problems such as cooperative control of nonlinear systems such as nonholonomic systems in the chained form.

II. Problem Formulation

Consider a group of q vehicles and suppose that dynamics of the ith vehicle are described by

$$\dot{\phi}_i = f_i(\phi_i, v_i), \qquad \psi_i = h_i(\phi_i) \tag{1}$$

where $i \in \{1, \ldots, q\}$, $\phi_i \in \Re^{n_i}$ is the original state, $\psi_i(t) \in \Re^m$ is the output, and $v_i(t) \in \Re^m$ is the control input. The proposed cooperative control design consists of the following two-level control hierarchy.

1) *Local Cooperative Strategy:* A local vehicle-level command $u_i = u_i(t, s_{i1}(t)\psi_1, \ldots, s_{iq}(t)\psi_q)$ is synthesized by taking into account all the information available to the ith vehicle about outputs of other vehicles, where $s_{ij}(t)$ are binary time functions, $s_{ii} \equiv 1$; $s_{ij}(t) = 1$ if $\psi_j(t)$ (or its equivalence) is known to the ith vehicle at time t, and $s_{ij} = 0$ if otherwise.
2) *Vehicle-Level Control:* Vehicle control $v_i = v_i(t, \phi_i, u_i)$ implements the local cooperative strategy of u_i at the ith vehicle, and for the ease of designing u_i, it transforms vehicle dynamics into a canonical form (which is introduced in Section II-A).

As the focus of this paper, cooperative control $u = [\, u_1^T \ \cdots \ u_q^T \,]^T$ will be synthesized in Sections II-B and IV, and its objective is to ensure that all the state variables of the systems are uniformly bounded and that, for any $i \in \{1, \ldots, q\}$, the steady-state error $e^{ss} \triangleq \lim_{t\to\infty}[\psi_i(t) - \psi_i^d(t) - y^d]$ exists and is independent of i, where constant vector y^d describes the desired cooperative behaviors for the whole group of vehicles while vector $\psi_i^d(t)$ describes the desired individual behavior(s) in addition to the group behavior and is uniformly bounded and smooth. In other words, the cooperative control objective is to guarantee that, for all i, $[\psi_i(t) - \psi_i^d(t)]$ converges to the consensus steady state $y^{ss} \triangleq y^d + e^{ss}$. Furthermore, an adaptive cooperative control law is to be designed to ensure $e^{ss} = 0$, i.e., $y^{ss} = y^d$. It will be shown in Section IV that the value of vector y^d can be used to characterize either single cooperative behavior or multiple cooperative behaviors.

The proposed cooperative control requires little but reacts to information about sensing/communications among the vehicles. In this paper, we consider the general cases that vehicles operate by themselves most of the time and that exchange of output information among the vehicles occurs only intermittently and locally. To capture this nature of information flow, let us define the following sensing/communication matrix and its corresponding time sequence $\{t_k^s : k = 0, 1, \ldots\}$ as $S(t) \in \{0,1\}^{q\times q} = S(k) \stackrel{\triangle}{=} S(t_k^s), \forall t \in [t_k^s, t_{k+1}^s)$,

$$S(t) = \begin{bmatrix} s_{11} & s_{12}(t) & \cdots & s_{1q}(t) \\ s_{21}(t) & s_{22} & \cdots & s_{2q}(t) \\ \vdots & \vdots & \vdots & \vdots \\ s_{q1}(t) & s_{q2}(t) & \cdots & s_{qq} \end{bmatrix} \tag{2}$$

where elements $s_{ij}(t)$ are those mentioned before, and $t_0^s \stackrel{\triangle}{=} t_0$. Time sequence $\{t_k^s\}$ and the corresponding changes in the row $S_i(t)$ of matrix $S(t)$ are detectable instantaneously by and locally at the ith vehicle, but they are *not* predictable or prescribed or known *a priori* or modeled in any way. Nonetheless, it can be assumed that $0 < \underline{c}_t \leq t_{k+1}^s - t_k^s \leq \overline{c}_t < \infty$, where $\underline{c}_t$ and $\overline{c}_t$ are constant bounds.[1]

The goal of this paper is twofold: 1) determine cooperative controllability (i.e., the necessary and sufficient condition on the inherent changes of $S(t)$ under which cooperative behavior(s) can be achieved) and 2) develop a systematic way of synthesizing cooperative controls. In what follows, the mathematical problem of achieving a cooperative behavior is formulated in Section II-C, and a necessary and sufficient condition on its solvability is explicitly found in Section III. The condition is then used in Section IV to design cooperative controls and ensure desired cooperative behaviors.

A. Vehicle-Level Canonical Form for Designing Cooperative Control

In order to focus on cooperative control design, the following assumption is introduced. In essence, the assumption says that, although vehicle systems are heterogeneous, vehicle-level controls can be designed to make their I/O relationship canonical. That is, for the purpose of achieving cooperative behavior in the m-dimensional output subspace, their I/O relationship is represented by triplet $\{A_i, B_i, C_i\}$.

Assumption 1: There exist a diffeomorphic state transformation $[x_i^T, \varphi_i^T]^T = \mathcal{X}_i(t, \phi_i)$ and a decentralized control law $v_i = \mathcal{V}_i(t, \phi_i, u_i)$ such that vehicle dynamics in (1) are transformed into

$$\dot{x}_i = A_i x_i + B_i u_i, \qquad y_i = C_i x_i, \qquad \dot{\varphi}_i = g_i(t, \varphi_i, x_i) \tag{3}$$

where $x_i \in \Re^{l_i m}$ with integer $l_i \geq 1$ contains output-related variables, $\varphi_i \in \Re^{n_i - l_i m}$ is the vector containing internal state variables, $y_i = \psi_i - \psi_i^d$, $I_{m\times m}$ is the m-dimensional identity matrix, $\otimes$ denotes the Kronecker product (defined as $D \otimes E = [d_{ij}E]$), J_k is the kth-order Jordan canonical form given by

$$J_k = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & -1 \end{bmatrix} \in \Re^{k\times k}$$

and

$$A_i = J_{l_i} \otimes I_{m\times m} \in \Re^{(l_i m)\times(l_i m)} \quad B_i = \begin{bmatrix} 0 \\ I_{m\times m} \end{bmatrix} \in \Re^{(l_i m)\times m},$$

$$C_i = [\, I_{m\times m} \quad 0 \,] \in \Re^{m\times(l_i m)}$$

u_i is the cooperative control to be designed and internal dynamics $\dot{\varphi}_i = g_i(t, \varphi_i, x_i)$ are input-to-state stable [37]. $\diamond$

Canonical form (3) has the structure that, if u_i converges to any given constant vector, so does output y_i. In other words, through the transformation into (3), the resulting controls $v_i(\cdot)$ are decentralized at individual vehicles and capable of following the cooperative strategy of u_i. It is straightforward to verify that canonical form (3) holds if tracking dynamics of the vehicles are I/O feedback linearizable with stable internal dynamics and that l_i is the relative degree for the ith vehicle. Hence, technical conditions equivalent to Assumption 1 can be found in standard texts, such as [37] and [38], and are omitted here for length. Instead, examples are included in Section V-B to illustrate Assumption 1.

B. General Class of Cooperative Controls

Cooperative controls proposed in this paper are the class of linear, piecewise constant, locally feedback controls

$$u_i \stackrel{\triangle}{=} \sum_{j=1}^{q} G_{ij}(t)[s_{ij}(t)y_j] = \sum_{j=1}^{q} G_{ij}(t)y_j = G_i(t)y \tag{4}$$

where $i = 1, \ldots, q$, $y = [y_1^T \ \cdots \ y_q^T]^T$ is the overall output, $G_i(t)$ are feedback gain matrices defined by

$$G_i(t) \stackrel{\triangle}{=} [\, G_{i1}(t) \quad \cdots \quad G_{iq}(t) \,]$$
$$G_{ij}(t) = G_{ij}(k) \stackrel{\triangle}{=} G_{ij}(t_k^s) \qquad \forall\, t \in [t_k^s, t_{k+1}^s)$$
$$G_{ij}(t_k^s) \stackrel{\triangle}{=} \frac{s_{ij}(t_k^s)}{\sum_{\eta=1}^{q} s_{i\eta}(t_k^s)} K_c, \qquad j = 1, \ldots, q \tag{5}$$

$s_{ij}(t)$ are piecewise constant as defined in (2) and $K_c \in \Re^{m\times m}$ is a constant, nonnegative, and row-stochastic matrix (see the definitions in Appendix A) to be selected in Section IV.

Although $S(t)$ is not known *a priori* nor can it be modeled, $S(t)$ is piecewise constant, diagonally positive, and binary, and the value of row $S_i(t)$ is known at time t to the ith vehicle. The aforementioned choice of feedback gain matrix block $G_{ij}(t)$ in terms of $s_{ij}(t)$ ensures that matrices $G_i(t)$ are row-stochastic

[1] If $S(t)$ becomes a constant matrix after some finite time, an infinite time sequence $\{t_k^s\}$ can always be chosen to yield a finite $\overline{c}_t$ except that $S(t_k^s)$ remains constant. On the other hand, a requirement of $\underline{c}_t$ not being too small is required for implementation but is not needed mathematically. Specifically, in the case that $\underline{c}_t = 0$, the proposed results remain valid upon excluding the intervals of infinitesimal or zero length from the test of sequential completeness.

and that control (4) is always local and implementable with all and only available information.

Remark 1: It is apparent that choice (5) can be generalized to $G_{ij}(t_k^s) = s_{ij}(t_k^s)w_{ij}K_{ij}/[\sum_{\eta=1}^q w_{i\eta}s_{i\eta}(t_k^s)]$, where $w_{ij} > 0$ are weighting coefficients, K_{ij} are constant and nonnegative, K_{ii} are either irreducible or diagonally positive, and the row-block sum $\sum_{j=1}^q G_{ij}(t_k^s)$ is a row-stochastic matrix. Then, all the subsequent results can be similarly developed. As will be used in Section V-C.2, adjusting weighting w_{ij} can improve convergence performance. $\diamond$

Remark 2: It follows from Definition (5) that $G_i(t)$ is row-stochastic and that control (4) has the following alternative expression:

$$u_i = G_i(t)y = K_c y_i + G_i(t)[y - \mathbf{1}_q \otimes y_i] \tag{6}$$

where vector $\mathbf{1}$ is that defined in Appendix A. Thus, implementation of control (4) only requires measurements of other vehicles' outputs relative to the subject vehicle. In general, for vehicle i of relative degree l_i higher than one, absolute measurement of its own output y_i may be needed for controlling its motion. In the case that $l_i = 1$, the ith subsystem becomes $\dot{y}_i = -y_i + u_i \stackrel{\triangle}{=} u_i'$, and it follows from (6) that $u_i' = (K_c - I_{m\times m})y_i + G_i(t)[y - \mathbf{1}_q \otimes y_i]$. Hence, upon setting $K_c = I_{m\times m}$, vehicle i is cooperatively controlled without any absolute measurement. $\diamond$

C. Mathematical Analysis of Cooperative Control Design

By simply combining (3) and (4) for all i and by exploring the special structures of system matrices, one can express dynamics of the overall closed-loop system as

$$\dot{x} = [A + BG(t)C]x = [-I_{N_q\times N_q} + \overline{G}(t)]x \tag{7}$$

where $N_q = mL_q$, $L_q = \sum_{i=1}^q l_i$,

$$A = \mathrm{diag}\{A_1, \ldots, A_q\} \in \Re^{N_q\times N_q}$$
$$B = \mathrm{diag}\{B_1, \ldots, B_q\} \in \Re^{N_q\times (mq)}$$
$$C = \mathrm{diag}\{C_1, \ldots, C_q\} \in \Re^{(mq)\times N_q}$$
$$G(t) = \left[G_1^T(t) \quad \cdots \quad G_q^T(t)\right]^T \in \Re^{(mq)\times(mq)} \tag{8}$$

and matrices $\overline{G}(t) \in \Re^{N_q\times N_q}$, $\overline{G}_{ii} \in \Re^{(l_i m)\times(l_i m)}$, $\overline{G}_{ij}(t) \in \Re^{(l_i m)\times(l_j m)}$ are defined as[2]

$$\overline{G}(t) = \begin{bmatrix} \overline{G}_{11}(t) & \overline{G}_{12}(t) & \cdots & \overline{G}_{1q}(t) \\ \overline{G}_{21}(t) & \overline{G}_{22}(t) & \cdots & \overline{G}_{2q}(t) \\ \vdots & \vdots & \vdots & \vdots \\ \overline{G}_{q1}(t) & \overline{G}_{q2}(t) & \cdots & \overline{G}_{qq}(t) \end{bmatrix}$$
$$\overline{G}_{ii}(t) = \begin{bmatrix} 0 & I_{(l_i-1)\times(l_i-1)} \otimes I_{m\times m} \\ G_{ii}(t) & 0 \end{bmatrix}$$
$$\overline{G}_{ij}(t) = \begin{bmatrix} 0 & 0 \\ G_{ij}(t) & 0 \end{bmatrix}, \qquad \text{if i} \neq \text{j}. \tag{9}$$

[2]Whenever $l_i - 1 = 0$, the corresponding rows and columns of $I_{(l_i-1)\times(l_i-1)} \otimes I_{m\times m}$ are empty, i.e., removed from $\overline{G}_{ii}$.

It is obvious that matrix $\overline{G}(t)$ is also piecewise constant and row-stochastic at any given time instant t. Hence, we can obtain the state solution to system (7) and the following steady state: letting $x^{\mathrm{ss}} \stackrel{\triangle}{=} \lim_{t\to\infty} x(t)$,

$$x^{\mathrm{ss}} = \lim_{t\geq t_{k+1}^s,\, k\to\infty} \left\{ e^{[-I+\overline{G}(t_{k+1}^s)](t-t_{k+1}^s)} Q(k) \right\} x(t_0) \tag{10}$$

provided that the above limit exists, where

$$Q(k) \stackrel{\triangle}{=} \prod_{\eta=0}^{k} P(\eta) \stackrel{\triangle}{=} P(k)P(k-1)\cdots P(2)P(1)P(0) \tag{11}$$

with $\aleph \stackrel{\triangle}{=} \{0, 1, 2, \ldots, \infty\}$, and

$$P(k) = e^{[-I+\overline{G}(t_k^s)](t_{k+1}^s - t_k^s)}, \qquad k \in \aleph. \tag{12}$$

Recalling that cooperative control design is successful if $x^{\mathrm{ss}} = \mathbf{1}_{L_q} \otimes y^{\mathrm{ss}}$ for some constant vector $y^{\mathrm{ss}} \in \Re^m$, we know that premultiplying matrix products (11) is mathematically of central importance. In Section III, a necessary and sufficient condition is found to ensure convergence of $\lim_{k\to\infty} \prod_{\eta=0}^k P(\eta) = \mathbf{1}_{N_q} c$ for some $c \in \Re^{1\times N_q}$. In Section IV, the convergence condition is used to choose K_c in order to achieve single cooperative behavior, multiple cooperative behaviors, and adaptive cooperative behaviors.

Remark 3: The proposed framework of cooperative control design also applies to discrete time systems and sampled data systems. Specifically, the vehicle-level canonical form in the discrete time domain can be chosen to be

$$x_i(k+1) = A_{d_i} x_i(k) + B_i u_{d_i}(k), \qquad y_i(k) = C_i x_i(k)$$
$$\varphi_i(k+1) = g_i(k, \varphi_i(k), x_i(k))$$

where A_i, B_i, C_i are the same as those in (3), $0 \leq c_d < 1$ is a design constant, and $A_{d_i} \stackrel{\triangle}{=} I_{(l_i m)\times(l_i m)} + (1-c_d)A_i$. Then, under cooperative control, $u_{d_i}(k) = (1-c_d)u_i(k)$ where $u_i(k)$ is given by (4) (except that t is replaced by k), the closed-loop system consisting of all the vehicles is

$$x(k+1) = [c_d I_{N_q\times N_q} + (1-c_d)\overline{G}(k)]x(k) \stackrel{\triangle}{=} P_d(k)x(k)$$

where matrix $\overline{G}(k)$ is the same as $\overline{G}(t)$ in (7). In this case, stability and convergence can again be analyzed by the product sequence[3] of $\prod_{\eta=0}^k P_d(\eta)$, and all the subsequent results can be similarly developed except for two differences. First, Lemma III.1 is no longer needed; instead, since diagonal blocks in the lower triangulation of $P_d(k)$ are all irreducible and diagonally positive, certain finite-length products of these diagonal blocks all become positive (according to Corollary B.3 in Appendix B). Second, if $\lambda(P_d(\eta)) = 0$ for some finite η, cooperative behavior(s) can be achieved in finite steps for discrete time systems [but impossible for continuous time systems as $P(\eta)$ in (11) are invertible, and hence, $\lambda(P(\eta)) > 0$ for all η], where matrix function $\lambda(\cdot)$ is defined by (31) in the Appendix. $\diamond$

[3]In the sequence, matrices $P_d(k)$ all have positive diagonal elements, which is guaranteed physically and mathematically. If $c_d = 0$ and $s_{ii} = 0$ were imposed, $P_d(k)$ would have zero diagonal, and it is straightforward to show by counterexample (e.g., the case of $q = 2$) that their multiplicative sequence is not convergent even if all the matrices are row-stochastic and irreducible.

III. Convergence of Matrix Products

Necessary and sufficient condition is found in this section by developing the canonical forms for matrices $S(t)$ and $P(k)$, by analyzing convergence of a matrix sequence in the canonical form, and by extending the convergence result to sequences in the general form.

A. Lower Triangular Form of the Closed-Loop System

Let us consider the general case that matrix $S(t)$ is reducible (defined in Appendix B). It is shown in the following lemma that, whatever irreducibility properties matrix $S(t)$ has, those properties are retained after matrix augmentations/transformation from $S(t)$ to $G(t)$, then from $G(t)$ to $\overline{G}(t)$, and finally, from $\overline{G}(t)$ to $P(k)$. In fact, Corollary C.4, Corollary C.2, and Lemma D.1 can all be viewed as the special case of $p = 1$ in Lemma III.1. Additionally, the so-called uniformly nonvanishing property of a submatrix sequence is defined in Definition III.1, and in Lemma III.1, this property is shown to be invariant from $S(t)$ to $P(k)$.

Definition III.1: Consider a sequence of nonnegative matrix blocks $\{E_{ij}(k) : k \in \aleph\}$. The sequence $\{E_{ij}(k)\}$ is said to be *uniformly nonvanishing*, denoted by $\{E_{ij}(k)\} \succ 0$, if there are a constant $\epsilon > 0$ and an infinite subsequence $\{l_v, v \in \aleph\}$ of $\aleph$ such that $\lim_{v\to\infty} l_v = +\infty$ and that, for every choice of $v \in \aleph$, at least one element of $E_{ij}(l_v)$ is greater than or equal to ϵ. $\diamond$

Lemma III.1: Consider three matrices: $G(t)$ defined by (5) and (8); matrix $\overline{G}(t)$ in (9); and matrix $P(k)$ in (12). Suppose that K_c is an irreducible and row-stochastic matrix and that $S'_{\triangle}(k)$ is the lower triangular canonical form of matrix $S(k)$ on irreducibility, that is, for an integer p with $1 < p \leq q$ and a permutation matrix $T_s(k)$, $T_s^T(k)S(k)T_s(k) = S'_{\triangle}(k)$ where

$$S'_{\triangle}(k) \triangleq \begin{bmatrix} S'_{11}(k) & 0 & \cdots & 0 \\ S'_{21}(k) & S'_{22}(k) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ S'_{p1}(k) & S'_{p2}(k) & \cdots & S'_{pp}(k) \end{bmatrix}. \tag{13}$$

Then, matrices $G(k)$, $\overline{G}(k)$, and $P(k)$ have identical block structure in their lower triangular forms. That is, $T^T(k)G(k)T(k) = G'_{\triangle}(k)$, $\overline{T}^T(k)\overline{G}(k)\overline{T}(k) = \overline{G}'_{\triangle}(k)$, and $\overline{T}^T(k)P(k)\overline{T}(k) = P'_{\triangle}(k)$, where

$$G'_{\triangle}(k) \triangleq \begin{bmatrix} G'_{11}(k) & 0 & \cdots & 0 \\ G'_{21}(k) & G'_{22}(k) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ G'_{p1}(k) & G'_{p2}(k) & \cdots & G'_{pp}(k) \end{bmatrix}$$

$$\overline{G}'_{\triangle}(k) \triangleq \begin{bmatrix} \overline{G}'_{11}(k) & 0 & \cdots & 0 \\ \overline{G}'_{21}(k) & \overline{G}'_{22}(k) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \overline{G}'_{p1}(k) & \overline{G}'_{p2}(k) & \cdots & \overline{G}'_{pp}(k) \end{bmatrix} \tag{14}$$

$$P'_{\triangle}(k) \triangleq \begin{bmatrix} P'_{11}(k) & 0 & \cdots & 0 \\ P'_{21}(k) & P'_{22}(k) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ P'_{p1}(k) & P'_{p2}(k) & \cdots & P'_{pp}(k) \end{bmatrix} \tag{15}$$

and $T(k)$ and $\overline{T}(k)$ are permutation matrices properly augmented from $T_s(k)$. Furthermore, both $P(k)$ and $P'_{\triangle}(k)$ are row-stochastic; $P'_{ii}(k) \in \Re^{r_i \times r_i}$ are square and uniformly positive for all k; and $\{P'_{ij}(k)\} \succ 0$ if $\{S'_{ij}(k)\} \succ 0$.

Proof: The augmentation from $S(k)$ to $G(k)$ is defined by (5) and (8), and the lower triangular canonical form of $G(k)$ can be obtained by constructing its permutation matrix as $T(k) = T_s(k) \otimes I_{m\times m}$, by applying Lemma B.1 in Appendix B, and then by invoking Corollary C.4 in Appendix C.

Similarly, augmentation from $G(k)$ to $\overline{G}(k)$ is defined by (9), and the lower triangular canonical form of $\overline{G}(k)$ can be obtained by constructing its permutation matrix $\overline{T}(k)$ (through appropriately adding rows and columns with $I_{l_i\times l_i} \otimes I_{m\times m}$ and zero), by applying Lemma B.1, and then by invoking Corollary C.2. If $l_i = l$ for $i = 1, \ldots, q$, $\overline{T}(k) = T(k) \otimes I_{l\times l}$.

It follows from (14) that

$$\begin{aligned} P'_{\triangle}(k) &= e^{-2\tau_k}\overline{T}^T(k)e^{[I+\overline{G}(t_k^s)]\tau_k}\overline{T}(k) \\ &= e^{-2\tau_k}\left\{I + \tau_k[I + \overline{G}'_{\triangle}(k)] + \frac{\tau_k^2}{2!}[I + \overline{G}'_{\triangle}(k)]^2 \right. \\ &\qquad \left. + \cdots + \frac{\tau_k^{n-1}}{(n-1)!}[I + \overline{G}'_{\triangle}(k)]^{n-1} + \cdots\right\}. \end{aligned}$$

Since $\overline{G}(k)$ is row-stochastic, it follows that $P(k)$ in (12) is row-stochastic, and so is $P'_{\triangle}(k) = \overline{T}^T(k)P(k)\overline{T}(k)$. By lower triangular structure of $\overline{G}'_{\triangle}(k)$, one can conclude that $P'_{ii}(k) = e^{-2\tau_k}e^{[I+\overline{G}'_{ii}(k)]\tau_k}$ from which $P'_{ii}(k) > 0$ is shown by applying Lemma D.1 in Appendix D.

Finally, it follows from the augmentations that $\{S'_{ij}(k)\} \succ 0$ implies $\{G'_{ij}(k)\} \succ 0$ and in turn $\{\overline{G}'_{ij}(k)\} \succ 0$. The proof can now be completed by noting that $P'_{ij}(k) \geq e^{-2\tau_k}\tau_k\overline{G}'_{ij}(k)$. $\square$

It is worth noting that, by the same process, $S'_{\triangle}(k)$ can be augmented to $G'_{\triangle}(k)$ and then to $\overline{G}'_{\triangle}(k)$ from which $P'_{\triangle}(k)$ can be found. Generally, index p and dimensions r_i of diagonal blocks in (13) and (15) are functions of k.

B. Convergence of Lower Triangular Matrix Sequences

The lower triangular canonical form of (13) and (15) provides an avenue to systematically study convergence of multiplicative sequence (11). Specifically, the following theorem provides a necessary and sufficient condition on convergence by focusing upon the case that the sequence has a constant permutation matrix $\overline{T}(k) = \overline{T}$, i.e., the whole sequence itself is lower triangular as $\overline{T}^T Q(k)\overline{T} = \prod_{\eta=0}^{k} P'_{\triangle}(\eta)$. In comparison, the existing result on sequence convergence, Lemma D.2 in Appendix D, is only sufficient and in fact corresponds to the simplest case of $p = 1$ (that is, $P'_{\triangle}(k) = P(k)$ is positive in the lemma).

Definition III.2: A nonnegative matrix sequence $\{E(k) : k \in \aleph\}$ is said to be *sequentially lower triangular* if the associated permutation matrix of $E(k)$ on irreducibility is independent of k. ◇

Definition III.3: Nonnegative matrix sequence $\{E(k) : k \in \aleph\}$ is said to be *sequentially lower triangularly complete* if it is sequentially lower triangular, and if in every row i of its lower triangular canonical form $E'_{\triangle}(k)$, there is at least one $j < i$ such that the corresponding block[4] is uniformly nonvanishing as $\{E'_{ij}(k)\} \succ 0$. ◇

Theorem III.2: Consider the nonnegative and row-stochastic sequence $\{P(k) : k \in \aleph\}$ with $P(k)$ defined in (12). Suppose that the sequence is sequentially lower triangular. Then, the multiplicative sequence (11) is exponentially convergent as

$$\left|Q(k) - \mathbf{1}_{N_q} c\right| \leq \sigma^k \mathbf{J}_{N_q \times N_q}$$

if and only if the sequence is sequentially lower triangularly complete, where $c \in \Re^{1 \times N_q}$ may contain up to $\max_k r_1(k)$ nonzero elements, and $0 < \sigma < 1$ is a constant.

Proof: Sufficiency is shown by considering two cases. In what follows, permutation matrix $\overline{T}$ is neglected for notational simplicity, i.e., simply consider $Q(k) = \prod_{\eta=0}^{k} P'_{\triangle}(\eta)$. The ith block row and (i, j)th block of $Q(k)$ are represented by $Q_i(k)$ and $Q_{ij}(k)$, respectively; the limit of $Q(k)$ is denoted by Q^{ss} whose rows and block elements are Q_i^{ss} and Q_{ij}^{ss}, respectively.

Let us first consider the case that all the blocks in $P'_{\triangle}(k)$ are of fixed dimension, that is, the corresponding p and r_i in (15) are all independent of k. It is obvious from the lower triangular structure that $Q_{11}(k) = \prod_{\eta=0}^{k} P'_{11}(\eta)$. It follows from $P'_{11}(k) \geq \epsilon_1 \mathbf{J}_{r_1 \times r_1}$ that $0 \leq \lambda(P'_{11}(k)) \leq (1 - r_1\epsilon_1) \stackrel{\triangle}{=} \sigma_1 < 1$. Invoking inequality (34) in Appendix D yields $\delta\left(Q_{11}(k)\right) < \sigma_1^k$. On the other hand, by Lemma D.2 in Appendix D, we know that $Q_{11}^{\text{ss}} = \mathbf{1}_{r_1} c_1$ for some $c_1 \in \Re^{1 \times r_1}$ and that $Q_{1j}^{\text{ss}} = 0$ for $j > 1$. As a result of convergence, matrix $Q_{11}^c(k) \stackrel{\triangle}{=} \prod_{\eta=k+1}^{\infty} P'_{11}(\eta)$ must exist. Therefore, it follows from Lemma E.2 that

$$\begin{aligned} |Q_{11}(k) - \mathbf{1}_{r_1} c_1| &= |Q_{11}(k) - Q_{11}^c(k) Q_{11}(k)| \\ &\leq \delta\left(Q_{11}(k)\right) \mathbf{J}_{r_1 \times r_1} \\ &\leq \sigma_1^k \mathbf{J}_{r_1 \times r_1}. \end{aligned}$$

Upon having shown convergence of the first block row, consider the second block row. Since $\{P_{21}(k)\} \succ 0$, it follows from Lemma E.1 that index subsequence $\{k_l : l \in \aleph\}$ exists such that $Q(k_l) = \prod_{\eta=0}^{k_l} P'_{\triangle}(\eta) = \prod_{\mu=0}^{l} E_{\triangle}(\mu) \stackrel{\triangle}{=} Q'(l)$, where E_{11}, $E_{22}(\mu) > 0$, and more importantly, $E_{21}(\mu) > 0$. Therefore, the second-row matrix equation is $Q'(l+1) = E_{\triangle}(l+1) Q'(l)$, that is,

$$Q'_2(l+1) = E_{22}(l+1) Q'_2(l) + E_{21}(l+1) Q'_1(l) \qquad (16)$$

where $Q'_i(l) \in \Re^{r_i \times N_q}$ is the ith block row of $Q'(l)$. Recall that $Q'_1(l)$ has been shown to converge to $\mathbf{1}_{r_1} c$, where $c \in \Re^{1 \times N_q} = [c_1, 0, \ldots, 0]$. Applying Lemma E.3 in Appendix E to (16) yields $|Q'_2(l+1) - \mathbf{1}_{r_2} c| \leq (\sigma'_2)^l \mathbf{J}_{r_i \times N_q}$ for some positive constant $\sigma'_2 < 1$. Hence, $|Q_2(k_l) - \mathbf{1}_{r_2} c| \leq \sigma_2^{k_l} \mathbf{J}_{r_i \times N_q}$ where $\sigma_2 = (\sigma'_2)^{l/k_l} < 1$.

Convergence of $Q(k)$ as a whole can be done inductively by repeating the previous argument. That is, for the subject row block of $Q(k)$ (say, the ith block row), we know from the lower triangular structure that the preceding rows are convergent to the same limit, then from Lemma E.1 that $\{P'_{ij}(k)\} \succ 0$ for any $j < i$ implies $\{E'_{ij}(l)\} > 0$, and finally, from Lemma E.3 that the ith block row is exponentially convergent.

As the second case of sufficiency proof, consider that $P'_{ii}(k)$ are of different dimensions with respect to k. In this case, the product $\prod_{\eta=k_1}^{k_2} P'_{\triangle}(\eta)$ contains diagonal blocks that are irreducible and diagonally positive and whose sizes are not smaller than those of $P'_{\triangle}(\eta)$. Hence, by Corollary B.3 in Appendix B, productive sequence $\prod_{\eta=0}^{\infty} P'_{\triangle}(\eta)$ can be regrouped into a new sequence $\prod_{\eta'=0}^{\infty} P''_{\triangle}(\eta')$ such that diagonal blocks of $P''_{\triangle}(\eta')$ are all positive and of their largest dimensions, which implies that new sequence $\{P''_{\triangle}(\eta')\}$ must again have fixed block dimensions. Upon realizing this property, proof of the first case can be repeated for the second case.

To show necessity, assume that $\{P(k) : k \in \aleph\}$ is sequentially lower triangular but not sequentially lower triangularly complete. Thus, there exist integers i and $\overline{k}_i > 0$ such that $P'_{ij}(k) = 0$ for all $k > \overline{k}_i$ and all $j < i$. Consider the infinite product $R^{ss} \stackrel{\triangle}{=} \prod_{k=\overline{k}_i+1}^{\infty} P'_{\triangle}(k)$ and let R_i^{ss} and R_{ij}^{ss} be its ith block row and its (i, j)th block, respectively. Since $P'_{ij}(k) = 0$ for all $k > \overline{k}_i$ and all j except for $P'_{ii}(k) > 0$, we can repeat the analysis of Q_{11}^{ss} and show that $R_{ij}^{ss} = 0$ for all j except that $R_{ii}^{ss} = \mathbf{1}_{r_i} c'_i$ for some $c'_i \in \Re^{1 \times r_i}$. Postmultiplying the finite matrix product $[P'_{\triangle}(\overline{k}_i) \cdots P'_{\triangle}(0)]$ yields $Q_i^{ss} = R_i^{ss} P'_{\triangle}(\overline{k}_i) \cdots P'_{\triangle}(0)$, and hence, $Q_i^{ss} \neq Q_1^{ss}$, from which necessity is shown. □

As will be illustrated further in Section IV, while the limit of $\lim_{k \to \infty} Q(k) = \mathbf{1}_{N_q} c$ says convergence to a single cooperative behavior, $\lim_{k \to \infty} Q_i(k) = \mathbf{1}_{r_i} c'_i$ with $c'_i \neq c'_j$ for $i \neq j$ implies multiple cooperative behaviors, and in a cooperative control design, multiple behaviors can be intentionally generated by decoupling output channels [in which case Theorem III.1 can also be applied diagonally according to the block diagonal structure of $\overline{G}'_{\triangle}(k)$]. The following two remarks further elaborate the concept of a matrix productive sequence being sequentially lower triangularly complete.

Remark 4: It follows from Lemma III.1 that, if sequence $\{S(t_k^s) : k \in \aleph\}$ is sequentially lower triangularly complete, so is sequence $\{P(k) : k \in \aleph\}$, and vice versa. Whenever the property holds, all those nodes associated with $S'_{11}(k)$ are globally reachable (see the definition in [24]), and the directed graph corresponding to $S(t)$ has a spanning tree beginning at any one of those nodes (according to the definition in [22]). Thus, the proposed matrix-theory-based framework is not only general (in terms of system dynamics and of convergence rate) but also complementary (as it admits the best results obtained using the graph theory). ◇

[4] If sizes of block rows of $E'_{\triangle}(k)$ vary, the larger one or their union of overlapping block rows should be considered.

Remark 5: The proof of Theorem III.2 says that convergence rate σ is determined by the lower triangularly nonvanishing subsequence contained in $\{S(t_k^s) : k \in \aleph\}$. Should frequency of this subsequence be known, σ can be found. This explicit result of convergence rate also applies to the extension of Theorem III.2, Theorem III.3 in the next section. ◇

C. Convergence of General Matrix Sequences

Although Theorem III.2 provides a necessary and sufficient condition on convergence, it requires that permutation matrix $\overline{T}(t)$ be time-independent. In general, $\overline{T}(t)$ would be time varying, and $\prod_{\eta=0}^{k} P(\eta) = \prod_{\eta=0}^{k} \overline{T}^T(t_\eta^s) P'_{\triangle}(\eta) \overline{T}(t_\eta^s)$. Nonetheless, a necessary and sufficient condition on convergence of this general sequence can be explicitly established and is stated as the following theorem.

Definition III.4: A nonnegative matrix sequence $\{E(k) : k \in \aleph\}$ is said to be *sequentially complete* if the sequence (or one of its regrouped versions) contains an infinite subsequence that is sequentially lower triangularly complete. ◇

Theorem III.3: Nonnegative and row-stochastic sequence $\{P(k) : k \in \aleph\}$ with $P(k)$ defined in (12) is exponentially convergent to $\mathbf{1}_{N_q} c$ with $c \in \Re^{1\times N_q}$ if and only if $\{S(t_k^s) : k \in \aleph\}$ is sequentially complete.

Proof: Necessity can be seen easily from the simple fact that, if $\{S(t_k^s) : k \in \aleph\}$ is not sequentially complete, matrices $S'_{\triangle}(k)$ become and remain block diagonal after some finite k, and so are matrices $P'_{\triangle}(k)$, and this decoupling property is invariant with respect to k. Hence, according to the proof of Theorem III.2, sequence $\{P(k) : k \in \aleph\}$ converges to a matrix of different rows.

To show sufficiency, let $\{P(k_v) : v \in \aleph\}$ denote the sequentially lower triangularly complete subsequence contained in $\{P(k) : k \in \aleph\}$. It follows that the infinite product $\prod_{k=0}^{\infty} P(k)$ can be regrouped as

$$\prod_{k=0}^{\infty} P(k) = \prod_{v=1}^{\infty} \left\{ \left[\overline{T}_c P'_{\triangle}(k_v) \overline{T}_c^T \right] F(k_v) \right\} \tag{17}$$

where $\overline{T}_c$ is a fixed and appropriate permutation matrix, $F(k_0) = [P(k_0 - 1) \cdots P(0)]$, and $F(k_v)$ with $v > 0$ is the product of $[P(k_v - 1) \cdots P(k_{v-1} + 1)]$. Since subsequence $\{P(k_v) : v \in \aleph\}$ is sequentially lower triangularly complete, it follows from Theorem III.2 that $\prod_{v=0}^{\infty} P'_{\triangle}(k_v) = \mathbf{1}_{N_q} c$ with $c = [c_1 \; 0 \cdots 0] \in \Re^{1\times N_q}$ and $c_1 \in \Re^{1\times r_1}$. On the other hand, applying permutation matrix $\overline{T}_c$ to sequence (17) yields

$$\overline{T}_c^T \left[\prod_{k=0}^{\infty} P(k) \right] \overline{T}_c = \prod_{v=1}^{\infty} \left\{ P'_{\triangle}(k_v) \left[\overline{T}_c^T F(k_v) \overline{T}_c \right] \right\}$$

in which diagonal elements of $F(k_v)$ are uniformly positive for all $v \in \aleph$, and so are all those of matrices $[\overline{T}_c^T F(k_v) \overline{T}_c]$. Thus, convergence of $\overline{T}_c^T [\prod_{k=0}^{\infty} P(k)] \overline{T}_c$, and hence, that of $\prod_{k=0}^{\infty} P(k)$ can be concluded by invoking Lemmas F.1, F.2, and D.2. □

D. Explanation, Examples, and Comparisons

Sequence $S(t_k^s)$ being sequentially complete is mathematically defined in Definition III.4. Nonetheless, it is straightforward to see that this concept is equivalent to the following definition in layman's language. Similarly, Theorem III.3 can also be restated as Theorem III.4.

Definition III.5: Systems of (1) are said to form *two (or more) sensing/communication groups* over finite time interval $[t - \Delta, t]$ for some $\Delta > 0$ [or over infinite interval $[t, \infty)$] if, over the interval, the set of vehicles can be separated into two (or more) disjoint, nonempty and complementary subsets such that there is no sensing/communication between any two vehicles chosen from two different subsets. Alternatively, systems of (1) are said to have *only one sensing/communication group* if they do not form two or more sensing/communication groups. ◇

Theorem III.4: Sequence $\{P(k) : k \in \aleph\}$ with $P(k)$ defined in (12) is exponentially convergent to $\mathbf{1}_{N_q} c$ with $c \in \Re^{1\times N_q}$ if and only if there exists a constant $\Delta > 0$ such that systems of (1) always have only one sensing/communication group over time intervals $[t_{i-1}^*, t_i^*)$ with $t_0^* = t_0$ and $t_i^* = t_{i-1}^* + \Delta$. Alternatively and mathematically, $\lim_{k\to\infty} Q(k) = \mathbf{1}_{N_q} c$ if and only if sequence $\{S^*(i) : i \in \aleph\}$ defined by

$$S^*(i) = S(t_{k_i}^s) \bigwedge \cdots \bigwedge S(t_{k_{i-1}+1}^s)$$

has the property that, in every row l of its lower triangular canonical form $S'^*_{\triangle}(i)$ in the form of (13), there is at least one $j < l$ such that $S'^*_{lj}(i) \neq 0$, where $t_{k_{i-1}+1}^s$ up to $t_{k_i}^s$ belong to interval $[t_{i-1}^*, t_i^*)$, and $\bigwedge$ denotes a binary product of binary matrices (i.e., binary multiplication and addition of the elements are used).

As discussed in Remarks 4 and 5, Theorem III.2 (except for its convergence rate analysis) can be viewed as the matrix-theoretical counterpart of known graph-theoretical results. Theorem III.3 is more general than Theorem III.2 since all time-varying permutation matrices are admissible. Although there is no need to account for any specific vehicle (or node in terms of graph theory) in applying Theorem III.3, one could still argue that Theorem III.3 nonetheless mirrors known graph-theoretical results. Theorem III.4 is novel because its concept of only one sensing/communication group is physically motivated (and more intuitive to nonexperts while being necessary and sufficient mathematically), and more importantly, it provides a simple test. Real-time calculation of binary product of the changes of $S(k)$ over any interval is trivial, its corresponding lower triangular form can be found easily and monitored periodically, and both cooperative controllability and performance measure (of exponential convergence rate) can be determined by simply checking whether the resulting product's canonical form is block-diagonal. The following examples are to illustrate the concepts of sequentially lower triangularly complete sensing/communication and sequentially complete sensing/communication.

Example III.1: Consider sensing/communication sequence $\{S(k) : k \in \aleph\}$ defined by $S(k) = S_1$ if k is even and $S(k) =$

S_2 if k is odd, where

$$S_1 = \left[\begin{array}{c|cc|c|c} 1 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \text{ and } S_2 = \left[\begin{array}{c|c|ccc} 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right]$$

lead to

$$S^* \triangleq S_2 \bigwedge S_1 = \left[\begin{array}{c|cccc} 1 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{array}\right].$$

Although matrices S_1 and S_2 are of different block sizes, $\{S^* : l \in \aleph\}$ is sequentially lower triangular complete. ◇

Example III.2: Suppose that sensing/communication sequence $\{S(k) : k \in \aleph\}$ is defined by $S(k) = S_1$ if $k = 3\eta$, $S(k) = S_2$ if $k = 3\eta + 1$, and $S(k) = S_3$ if $k = 3\eta + 2$, where $\eta \in \aleph$,

$$S_1 = \left[\begin{array}{c|c|c|c|c} 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 1 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right], \quad S_2 = \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

and

$$S_3 = \left[\begin{array}{c|c|c|c|c} 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 1 & 0 & 1 \end{array}\right].$$

It is apparent that matrices S_1, S_2, and S_3 are all reducible and have different permutation matrices with $T_s(3\eta) = T_s(3\eta + 2) = I$, and

$$T_s(3\eta + 1) \triangleq T_s^* = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Neither of three subsequences $\{S(3\eta + 1)\}$, $\{S(3\eta + 2)\}$, $\{S(3\eta)\}$ is sequentially lower triangularly complete by itself. Nonetheless, sequence $\{S(k), k \in \aleph\}$ is sequentially complete because an infinite lower triangularly complete subsequence $\{S^*(k), k \in \aleph\}$ can be constructed with $S^*(k) = S(3k + 2)S(3k + 1)S(3k)$ and the corresponding canonical lower triangular form is

$$[T_s^*]^T \left(S_3 \bigwedge S_2 \bigwedge S_1\right) T_s^* = \left[\begin{array}{cc|cc|c} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ \hline 1 & 1 & 0 & 0 & 1 \end{array}\right].$$

◇

IV. Designs of Cooperative Control

Convergence conditions revealed in Theorems III.2, III.3, and III.4 can be applied directly to cooperative control designs. As the main result of this paper, the following theorem can be concluded by invoking Theorem III.4 and by noting that the internal dynamics are input-to-state stable.

Theorem IV.1: Consider dynamical systems in (1), under Assumption 1, and under cooperative control (4) and (5). Then, systems of (1) exhibit a single cooperative behavior as

$$x^{ss} = c_0 \mathbf{1}_{N_q} \qquad \text{and} \qquad y^{ss} = c_0 \mathbf{1}_m, \qquad c_0 \in \Re \qquad (18)$$

provided that: 1) gain matrix K_c is chosen to be irreducible and row-stochastic and 2) systems in (1) have only one sensing/communication group (or a sequentially complete sensing/communication). Furthermore, convergence to the cooperative behavior is exponential with respect to certain length Δ if systems in (1) have only one sensing/communication group over consecutive time intervals of period Δ.

In the previous theorem, the first condition can easily and always be met in the control design, and the second condition defines cooperative controllability (i.e., the minimum requirement on the operational environment). Several results can be derived from the general design result (Theorem IV.1), and they are presented in the next three sections.

A. Two Special Designs of Cooperative Control

The first result, implied by Theorem IV.1 and given later as Corollary IV.2, generalizes the existing result of cooperative control design in [21] and [23] to dynamical systems of high-relative degree. Its proof is obvious as $S(t)$ being irreducible is equivalent to triangulation with $p = 1$, and as shown in [39], is also equivalent to the corresponding directed sensor graph being strongly connected.

Corollary IV.2: Under Assumption 1 and under cooperative control (4) and (5) with irreducible and row-stochastic matrix K_c, systems of (1) exhibit a single cooperative behavior as described in (18) if their sensor/communication sequence $\{S(k)\}$ contains an infinite subsequence of irreducible matrices.

The second result, implied by Theorem IV.1 and given later as Corollary IV.3, deals with the case that the communication pattern in the system has a lower triangular structure. In this case, the structure of $S'(k)$ with $T_s(k)$ being constant means that, in terms of sensing and communication, the system observes a leader–follower structure. Thus, Corollary IV.3 (or simply Theorem III.2) can be viewed as the result of leader–follower cooperative control, while Theorem IV.1 deals with the leaderless cooperative control. As pointed out in Remark 4, the corresponding graph has a spanning tree or a globally reachable node. Thus, Theorem IV.1 and the following corollary extend the result in [22] to dynamical systems of high-relative degree as well as the results of [23] and [24] to formation control of nonholonomic systems in a high-order chained form (see Example V.3 in Section V-B) and within a dynamically changing environment.

Corollary IV.3: Under Assumption 1 and under cooperative control (4) and (5) with irreducible and row-stochastic matrix K_c, systems of (1) exhibit a single cooperative behavior as described in (18) if their sensor/communication sequence $\{S(k)\}$ is sequentially lower triangularly complete.

B. Multiple-Objective Cooperative Control

Utilizing the design flexibility embedded in gain matrix K_c, Theorem IV.1 can be extended to the case that multiple cooperative behaviors are desired. Corollary IV.4, given later, specifies the cooperative control design for the case that each channel of vehicle output has a distinct behavior. Its proof is straightforward as K_c being diagonal yields completely decoupled dynamics among the m channels of all the vehicles and their associated state variables, and Theorem IV.1 can be applied to each of the channels. By analogy, one can work out a design to generate any given combination of behaviors (greater than one but less than m) among vehicles' output.

Corollary IV.4: Under Assumption 1 and under cooperative control (4) and (5) with $K_c = I_{m\times m}$, systems of (1) exhibit m distinct cooperative behaviors described by

$$x^{ss} = \mathbf{1}_{L_q} \otimes c_0 \qquad \text{and} \qquad y_i^{ss} = c_0, \qquad c_0 \in \Re^{m\times 1}$$

if their sensor/communication sequence $\{S(k)\}$ is sequentially complete.

C. Adaptive Cooperative Control

The single cooperative behavior described in (18) does not necessarily mean that, if $y^d = c_0^d \mathbf{1}_m$, the desired behavior represented by constant c_0^d is achieved. In order to ensure $c_0 = c_0^d$ in (18), we must modify cooperative control (4) by introducing an integral term. To this end, a virtual vehicle representing a hand-off operator is introduced as

$$\dot{x}_0 = \left(1 - \sum_{i=1}^{q} \epsilon_i s_{0i}(t)\right)(u_0 - x_0) + \sum_{i=1}^{q} \epsilon_i s_{0i}(t) x_i$$
$$y_0(t) = x_0(t), \quad x_0(t_0) = c_0^d \mathbf{1}_m, \quad u_0 = K_c x_0(t_0) \qquad (19)$$

where $x_0(t) \in \Re^m$, $\epsilon_i \geq 0$ are constants with $\sum_{i=1}^{q} \epsilon_i < 1$, and $s_{0i}(t) = 1$ if x_i is known to the operator and $s_{0i}(t) = 0$ if otherwise. In the simplest setting of $\epsilon_i = 0$ for all i, dynamics of the virtual vehicle reduce to $\dot{x}_0 = -x_0 + u_0$.

Communication from the virtual vehicle to the physical vehicles is also intermittent and local; thus, we can introduce the following augmented sensor/communication matrix and its associated time sequence $\{\bar{t}_k^s : k \in \aleph\}$ as $\overline{S}(t) = \overline{S}(k) \stackrel{\triangle}{=} \overline{S}(\bar{t}_k^s), \forall t \in [\bar{t}_k^s, \bar{t}_{k+1}^s)$,

$$\overline{S}(t) = \begin{bmatrix} 1 & s_{01} & \cdots & s_{0q} \\ s_{10} & & & \\ \vdots & & S(t) & \\ s_{q0} & & & \end{bmatrix} \in \Re^{(q+1)\times(q+1)}. \qquad (20)$$

Accordingly, cooperative control is modified from (4) and (5) to the following version:

$$u_i(t) = \sum_{j=0}^{q} \frac{s_{ij}(t)}{\sum_{\eta=0}^{q} s_{i\eta}(t)} K_c[s_{ij}(t) y_j], \qquad i = 1, \ldots, q \qquad (21)$$

where $s_{ij}(t)$ are piecewise constant entries (20). Since adaptive control is essentially a control that contains certain integral action, the proposed cooperative control (21) [which contains x_0, an integral term solved from (19)] can be viewed as an adaptive cooperative control. Applying Theorem IV.1 to the resulting augmented closed-loop system renders the following corollary.

Corollary IV.5: Under Assumption 1 and under cooperative control (21) with irreducible and row-stochastic matrix K_c, systems of (1) exhibit the desired cooperative behavior y^d, (i.e., $x^{ss} = \mathbf{1}_{L_q+1} \otimes y^d$ and $y_i^{ss} = y^d$), if $y^d = c_0^d \mathbf{1}_m$ for $c_0^d \in \Re$ and if their augmented sensor/communication sequence $\{\overline{S}(k)\}$ defined by (20) is sequentially complete.

If $y^d \neq c_0^d \mathbf{1}_m$, a multiobjective adaptive cooperative control can be designed by combining Corollaries IV.4 and IV.5. The results corresponding to Corollaries IV.2 and IV.3 can also be stated here.

V. Illustrative Examples

In Section V-A, a few cooperative control objectives are explicitly formulated. Then, in Section V-B, several examples are used to illustrate the decentralized vehicle-level control design [which, together with its associated state transformation, renders the canonical form of (3)]. Finally, the proposed cooperative controls are simulated in Section V-C to illustrate their performance.

A. Cooperative Control Objectives

For the vehicles described by (1), there are two choices in cooperative control design: group behavior described by vector y^d and individual behavior described by vector $\psi_i^d(t)$. As has been shown in Corollaries IV.4 and IV.5, y^d can be chosen to be anything between the two extremes of $y^d = c_0^d \mathbf{1}_m$ for $c_0^d \in \Re$ and $y^d \in \Re^m$. In other words, through the choice of K_c, one or a given number of cooperative behaviors can be achieved for the group of vehicles.

On the other hand, some or all of the vehicles can also exhibit their individual behavior(s) by choosing among the following list (and more can be added).

1) *Consensus problem:* $\psi_i^d(t) = 0$ for all $i = 1, \ldots, q$.
2) *Formation control problem:* $\psi_i^d(t) = \int_{t_0}^{t} w^d(\tau) d\tau + \psi_i^d(t_0)$, where $w^d(t)$ is the desired velocity of the whole formation and $\psi_i^d(t_0)$ is the relative position of the vehicle in the formation.

B. Vehicle Platforms

In general, the vehicles described by (1) are heterogeneous, i.e., the vehicles can be any combination of the following vehicle platforms or their equivalence/extensions. To simplify the notation, the subscript i denoting the ith vehicle is omitted later if the discussion is limited to just one vehicle.

Example V.1: A point-mass agent whose equation of motion is

$$\dot{\phi}_j = \phi_{j+1}, \qquad j = 1, \ldots, l-1, \qquad \dot{\phi}_l = v, \qquad \psi = \phi_1 \tag{22}$$

where $\phi_j \in \Re^m$ are the state subvectors, ψ is the output, and $v \in \Re^m$ is the control. Then, under the state and input transformations of

$$x_j = \sum_{k=0}^{j-1} \frac{(j-1)!}{(j-1-k)!k!} \left(\phi_{k+1} - \frac{d^k}{dt^k} \psi_i^d(t) \right)$$

$$v = -\sum_{k=0}^{l-1} \frac{l!}{(l-k)!k!} \left(\phi_{k+1} - \frac{d^k}{dt^k} \psi_i^d(t) \right) + \frac{d^l}{dt^l} \psi_i^d(t) + u$$

a dynamical model of (22) is transformed into (3), and both are mathematically equivalent. ◇

Example V.2: A simple model of *unmanned aerial vehicle* is [40]

$$\begin{aligned}
\dot{P}_x &= P_V \cos(P_\gamma)\cos(P_\phi), \quad \dot{P}_y = P_V \cos(P_\gamma)\sin(P_\phi) \\
\dot{P}_h &= P_V \sin(P_\gamma), \quad \dot{P}_V = \frac{T-D}{M} - g\sin(P_\gamma) \\
\dot{P}_\gamma &= \frac{g}{P_V}\left(n\cos\delta - \cos(P_\gamma)\right), \quad \dot{P}_\phi = \frac{L\sin(P_\delta)}{mP_V\cos(P_\gamma)}
\end{aligned} \tag{23}$$

where P_x is the down-range displacement, P_y is the cross-range displacement, P_h is the altitude, P_V is the ground speed and is assumed to be equal to the airspeed, P_γ is the flight path angle, P_ϕ is the heading angle, T is the aircraft engine thrust, D is the drag, M is the aircraft mass, g is the gravity acceleration, L is the aerodynamic lift, and δ is the banking angle. Control variables are δ, engine thrust T, and load factor $n = L/gm$. Then, by defining the output $\psi = [P_x \ P_y \ P_h]^T \in \Re^3$ and under the input transformation of $v = [v_1, v_2, v_3]^T \in \Re^3$ where

$$\delta = \tan^{-1}\left[\frac{v_2 \cos(P_\phi) - v_1 \sin(P_\phi)}{\cos(P_\gamma)(v_3+g) - \sin(P_\gamma)(v_1 \cos(P_\phi) + v_2 \sin(P_\phi))}\right]$$

$$n = \frac{\cos(P_\gamma)(v_3+g) - \sin(P_\gamma)(v_1 \cos(P_\phi) + v_2 \sin(P_\phi))}{g \cos \delta}$$

$$\begin{aligned}
T = {} & [\sin(P_\gamma)(v_3+g) + \cos(P_\gamma)(v_1 \cos(P_\phi) \\
& + v_2 \sin(P_\phi))]m + D
\end{aligned}$$

system (23) is transformed into (22) with $m = 3$, $l = 2$, and no internal dynamics. ◇

Example V.3: It is well known [41] that many nonholonomic systems (such as differential-driven wheeled mobile robots, car-like mobile robots, etc.) can be transformed into the chained form by state and control transformations. Although the chained form is not feedback linearizable, the following discussion shows that the proposed framework applies to cooperative formation control of a groups of vehicles, some or all of which involve those nonholonomic dynamics.

Without loss of any generality, let us assume that the two-dimensional formation control objective be given by $\psi_i^d(t) = \int_{t_0}^t w^d(\tau)d\tau + \psi_i^d(t_0)$ where $w^d(t) \triangleq [w_1^d(t) \ w_2^d(t)]^T \in \Re^2$ and $\inf_{t \geq t_0} |w_j^d(t)| \geq \underline{w}^d > 0$ for $j = 1, 2$ and that the ith vehicle be described by the following $(4,2)$ chained form [while other vehicles are described by either chained forms of same or different orders or by system (3)]:

$$\begin{cases} \dot{\phi}_{i,1} = v_{i,1}, \quad \dot{\phi}_{i,2} = \phi_{i,3} v_{i,1}, \\ \dot{\phi}_{i,3} = \phi_{i,4} v_{i,1}, \quad \dot{\phi}_{i,4} = v_{i,2}, \end{cases} \qquad \psi_i = \begin{bmatrix} \phi_{i,1} \\ \phi_{i,2} \end{bmatrix}. \tag{24}$$

Then, for any given $w^d(t)$ and $\psi_i^d(t_0)$ and for a given $\Delta t > 0$, there exist individual open-loop steering controls [41], [42] and smooth closed-loop exponentially stabilizing controls [43] such that, for the ith vehicle in chained form (24) as well as for other vehicles,

$$\|\psi_{j,1}(t_0+\Delta t) - \psi_{j,1}^d(t_0+\Delta t)\| < \underline{w}^d/(2q), \quad j \in \{1, \ldots, q\}. \tag{25}$$

Now, for $t \geq t_0 + \Delta t$, let us define the decentralized state transformation

$$x_i(t) = \begin{bmatrix} \phi_{i,1} - \psi_{i,1}^d, \\ \phi_{i,2} - \psi_{i,2}^d, \\ \phi_{i,1} - \psi_{i,1}^d + v_{i,1} - w_1^d, \\ \phi_{i,3} v_{i,1} - w_2^d + \phi_{i,2} - \psi_{i,2}^d, \\ \phi_{i,1} + 2v_{i,1} + \dot{v}_{i,1} \\ -\psi_{i,1}^d - 2w_1^d - \dot{w}_1^d, \\ \phi_{i,2} + 2\phi_{i,3} v_{i,1} + \phi_{i,4} v_{i,1}^2 \\ +\phi_{i,3}\dot{v}_{i,1} - \psi_{i,2}^d - 2w_2^d - \dot{w}_2^d, \end{bmatrix} \triangleq \begin{bmatrix} x_{i,1} \\ x_{i,2} \\ x_{i,3} \\ x_{i,4} \\ x_{i,5} \\ x_{i,6} \end{bmatrix} \tag{26}$$

and decentralized control mapping

$$\begin{aligned}
\ddot{v}_{i,1}(t) = {} & \psi_{i,1}^d + 3w_1^d + 3\dot{w}_1^d + \ddot{w}_1^d - \phi_{i,1} \\
& - 3v_{i,1} - 3\dot{v}_{i,1} + u_{i,1} \\
v_{i,2}(t) = {} & \frac{1}{v_{i,1}^2} \{ -[\phi_{i,2} + 2\phi_{i,3} v_{i,1} + \phi_{i,4} v_{i,1}^2 + \phi_{i,3}\dot{v}_{i,1} \\
& -\psi_{i,2}^d - 2w_2^d - \dot{w}_2^d] - [2\phi_{i,4} v_{i,1} + 2\phi_{i,3}\dot{v}_{i,1} - 2\dot{w}_2^d \\
& +\phi_{i,3} v_{i,1} - w_2^d + 3\phi_{i,4} v_{i,1}\dot{v}_{i,1} + \phi_{i,3}\ddot{v}_{i,1} - \ddot{w}_2^d] + u_{i,2} \}
\end{aligned} \tag{27}$$

where initial conditions of $v_{i,1}(t_0+\Delta t)$ and $\dot{v}_{i,1}(t_0+\Delta t)$ are set to be

$$\begin{cases} x_{i,3}(t_0+\Delta t) = \phi_{i,1}(t_0+\Delta t) - \psi_{i,1}^d(t_0+\Delta t) \\ \qquad\qquad + v_{i,1}(t_0+\Delta t) - w_1^d(t_0+\Delta t) = 0 \\ x_{i,5}(t_0+\Delta t) = \phi_{i,1}(t_0+\Delta t) + 2v_{i,1}(t_0+\Delta t) \\ \qquad\qquad + \dot{v}_{i,1}(t_0+\Delta t) - \psi_{i,1}^d(t_0+\Delta t) \\ \qquad\qquad - 2w_1^d(t_0+\Delta t) - \dot{w}_1^d(t_0+\Delta t) = 0. \end{cases} \tag{28}$$

It is straightforward to verify that, under transformations (26) and (27), chained form (24) is mapped into canonical form (3) with $m = 2$ and $l_i = 3$. Applying the multiple-objective cooperative control design in Section IV-B and applying the initial conditions in (25) and (28), we know that transformations

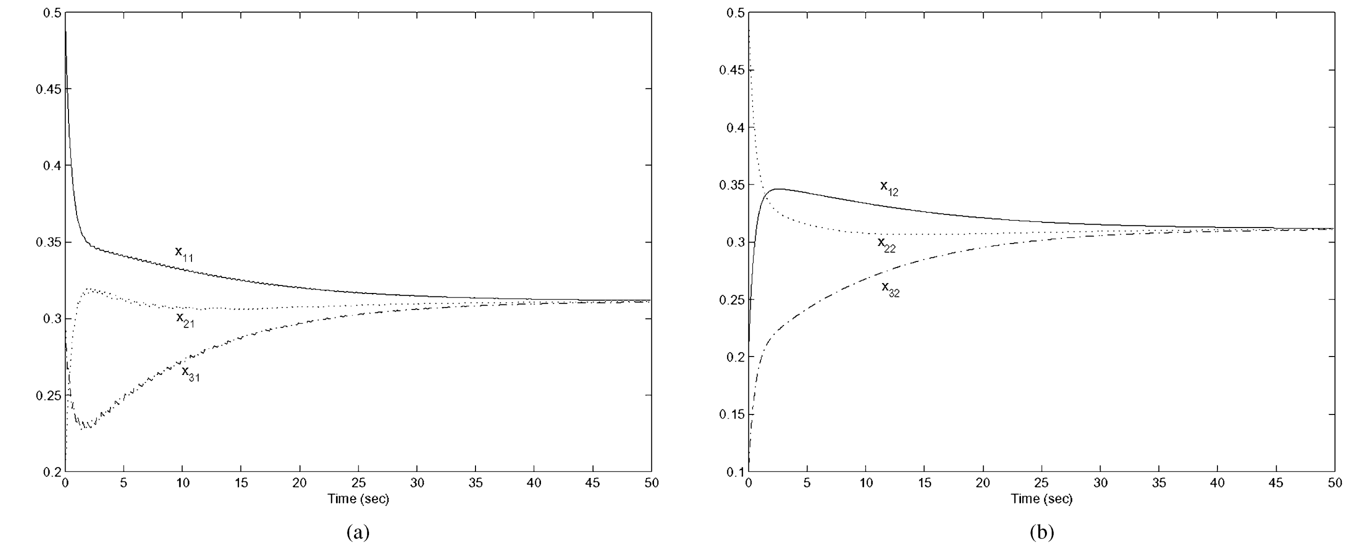

Fig. 1. Single cooperative behavior: Velocity convergence.

(26) and (27) are globally well defined because

$$\begin{aligned}|v_{i,1}(t)| &= |x_{i,3}(t) - x_{i,1}(t) + w_1^d| \\ &\geq \underline{w}^d - |x_{i,3}(t) - x_{i,1}(t)| \\ &= \underline{w}^d - \left| [Q_{i,3}(t) - Q_{i,1}(t)] \begin{bmatrix} x_{i,1}(t_0+\Delta t) \\ x_{i,3}(t_0+\Delta t) \\ x_{i,5}(t_0+\Delta t)\end{bmatrix} \right| \\ &\geq \underline{w}^d - 2\sum_{j=1}^{q} |x_{j,1}(t_0+\Delta t)| > 0 \end{aligned} \tag{29}$$

where $Q(t) = \prod_{t \geq t_k^s} P(k)$ is the row-stochastic matrix solution, and $Q_{i,3}(t)$ and $Q_{i,1}(t)$ are the rows corresponding to $x_{i,3}(t)$ and $x_{i,1}(t)$, respectively. Further research is needed to remove the condition of $\inf_{t \geq t_0} |w_j^d(t)| \geq \underline{w}^d > 0$. One possibility is to use the smooth time-varying state and control transformations recently introduced in [43] for global and smooth regulation of chained systems. ◇

C. Simulation Results

In this section, the proposed cooperative control is simulated for a group of three vehicles described by (3). For the ease of presentation, two-dimensional output (i.e., vehicles moving in a plane) is considered.

In the simulations, the sensing/communication matrix $S(t)$ is randomly switched among the following four topologies:

$$S_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad S_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$S_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \qquad S_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}. \tag{30}$$

Since the union of infinite subsequences of S_1 and S_3 is sequentially lower triangularly complete, thus the sensing/communication consisting of randomly switching S_1, S_2, S_3, and S_4 is sequentially complete.

To achieve a single cooperative behavior, K_c is set to be

$$K_c = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

which is irreducible and row-stochastic. To have multiple cooperative behaviors for the two-dimensional outputs of vehicles, $K_c = I_{2\times 2}$ is chosen according to Corollary IV.4.

1) Consensus of Velocity/Motion: $m = 2$ and $l_i = 1$ in (3) Fig. 1 shows the single cooperative behavior for (velocity) states of the vehicles and Fig. 2 shows two different cooperative behaviors by output channel. In both simulations, the initial velocities are set to be $[0.5 \ \ 0.2]^T$, $[0.2 \ \ 0.5]^T$, and $[0.3 \ \ 0.1]^T$, respectively.

2) Rendezvous (Consensus of Positions): $m = 2$ and $l_i = 2$ *in (3):* Suppose that initial positions are at $[6 \ \ 3]^T$, $[2 \ \ 5]^T$, and $[4 \ \ 1]^T$, respectively. Fig. 3(a) shows a convergent consensus of both channels, while Fig. 3(b) shows convergence of separate consensus for the two channels.

If the target position is specified to be $[3.5 \ \ 4]^T$, the adaptive cooperative control design in Section IV-C can be used. In the simulation, it is assumed that vehicle 1 receives information from the virtual vehicle about the target position. Then, the augmented sensor/communiction matrix $\overline{S}(t)$ is given by (20) together with (30), where s_{10} assumes a binary value randomly assigned, and $s_{20} = s_{30} = 0$. The control gains $G_{ij}(t)$ can be chosen according to Remark 1 and given by

$$G_{ij} = \frac{w_{ij} s_{ij}}{\sum_{\eta=0}^{3} w_{i\eta} s_{i\eta}} K_c, \qquad j = 0, 1, 2, 3$$

with $w_{ij} = 0.9$ for $i \neq j$ and $w_{ii} = 0.1$. Convergence to the given target position is shown by Fig. 3(c).

3) Formation Control: $m = 2$ and $l_i = 2$ *in (3)*: The desired formation trajectory is chosen to be one with velocity

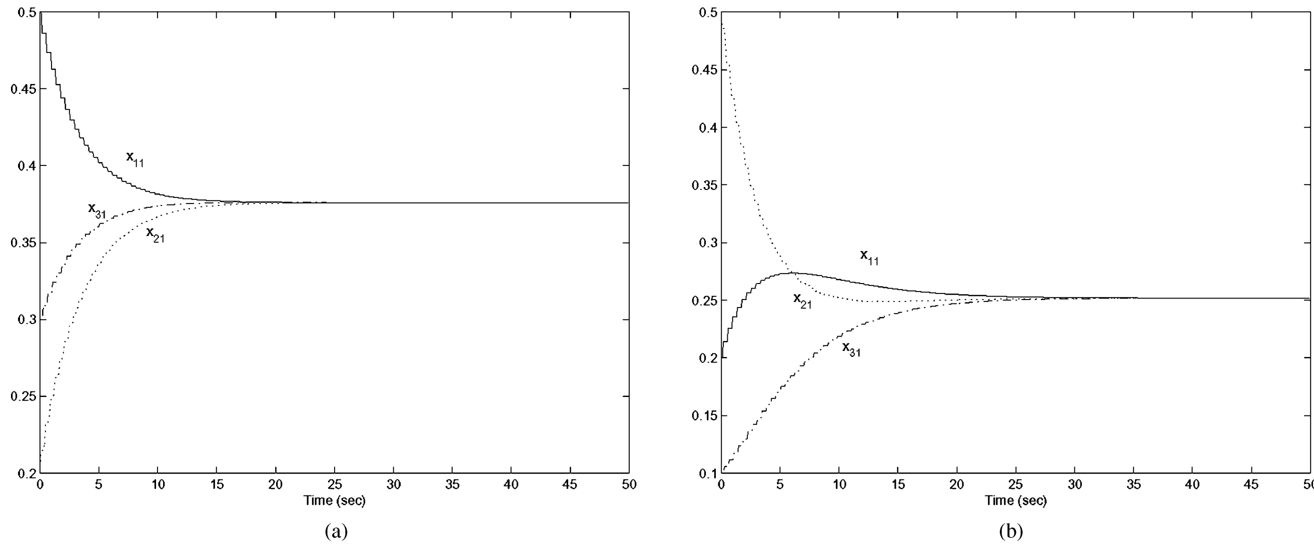

Fig. 2. Cooperative behaviors by output channel: Velocity convergence.

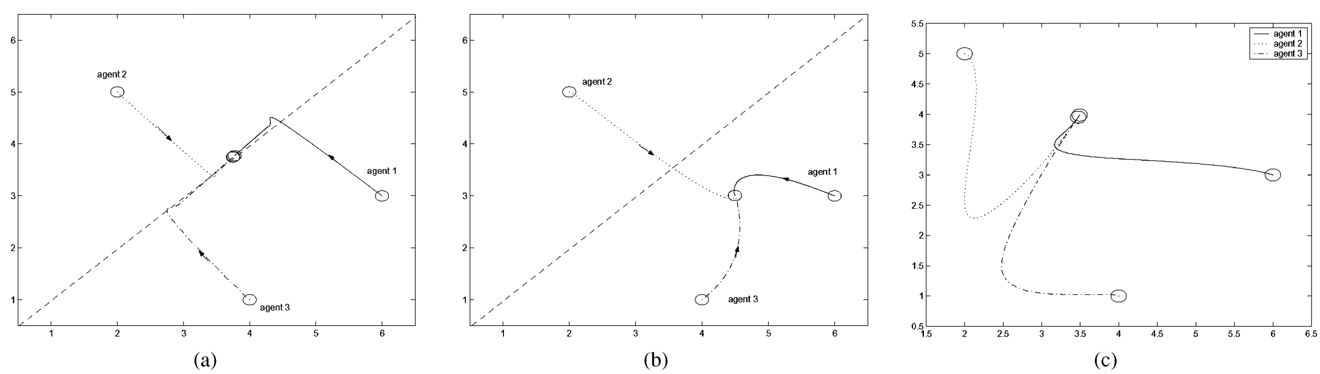

Fig. 3. Rendezvous under cooperative controls.

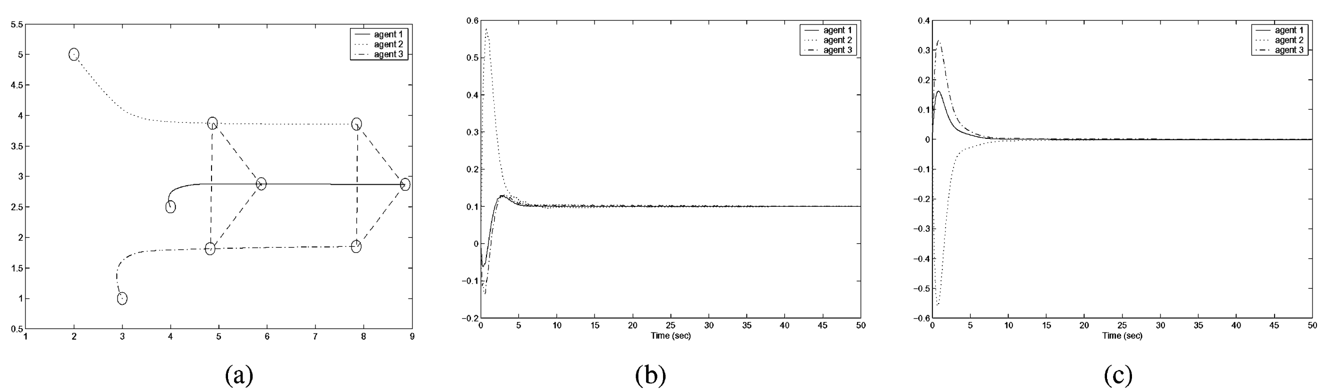

Fig. 4. Responses under formation cooperative control. (a) Phase portrait. (b) Horizontal velocities. (c) Vertical velocities.

$\omega^d(t) = [0.1t\ \ 0]^T$ and with a triangular formation of vertices at $\psi_1^d = [4\ \ 3]^T, \psi_2^d = [3\ \ 4]^T$, and $\psi_3^d = [3\ \ 2]^T$. Suppose that initial positions are $[4\ \ 2.5]^T, [5\ \ 2]^T$, and $[3\ \ 1]^T$, respectively. Under the proposed cooperative control and the randomly switched sensing/communication sequence, cooperative performance is shown by the simulation results in Fig. 4.

VI. Conclusion

In this paper, a matrix-theory-based framework is presented to design cooperative controls for a group of dynamical systems networked by dynamically changing communication/sensing. The framework contains a set of new results on augmentation of reducible and irreducible matrices so that, beginning with a simple sensing/communication matrix, dynamics of arbitrarily finite orders can be admitted into analysis and designs. For stability and convergence analysis, the framework develops a set of new results in terms of lower triangulation of reducible matrices for the overall closed-loop system, including necessary and sufficient conditions for convergence on a time-varying multiplicative matrix sequence.

As enrichments to control theory, lower triangulation of reducible matrices, and the corresponding convergence condition are fundamental to the understanding of the properties of networked dynamical systems, and their roles are analogous to that of Jordan decomposition for one linear dynamical system itself. Using the lower triangulation method, dynamical systems of any finite dimension can be studied, convergence rate can be explicitly obtained, and nonlinear and nonholonomic systems such as those in the chained form become admissible. By introducing a canonical form for cooperative controls, different objectives of cooperation such as individual behaviors, single group behavior, multiple group behaviors, and adaptive cooperation can also be embedded as a part of the proposed design framework so that the corresponding cooperative controls can be designed systematically in the same way.

The proposed framework is also complementary to the existing results obtained using graph theory. Since it is rooted in matrix theory and conducive to further incorporation of advanced control theory such as nonlinear systems and control, we believe that the proposed framework provides the means for solving more complicated problems, especially analysis and control of networked systems whose dynamics are nonlinear and uncertain and whose sensing and/or communication are both time-varying and uncertain.

Appendix A

Nonnegative and Row-Stochastic Matrices

Consider two matrices/vectors $E, F \in \Re^{r_1 \times r_2}$. The notations of $E = F$, $E \geq F$, and $E > F$ are defined with respect to all their elements. Operation $E = |F|$ of any matrix F is defined element-by-element as $e_{ij} = |f_{ij}|$. Matrix/vector E is *positive* (*nonnegative*) if $E > 0$ ($E \geq 0$). Matrix $\mathbf{J}_{r\times r} \in \Re^{r\times r}$ and vector $\mathbf{1}_r \in \Re^r$ are the special positive matrix and vector, respectively, whose elements are all 1. Matrix E is said to be *binary* if its elements are either 0 or 1. Matrix $E \in \Re^{r\times r}$ is said to be *diagonally positive* if $e_{ii} > 0$ for all $i = 1, \ldots, r$.

A nonnegative square matrix $E \in \Re^{r\times r}$ is said to be *squarely row-stochastic* or simply *row-stochastic* if all the sums of its rows equal 1, that is, $E\mathbf{J}_{r\times r} = \mathbf{J}_{r\times r}$ or $E\mathbf{1}_r = \mathbf{1}_r$. Similarly, a nonnegative rectangular matrix $E \in \Re^{r_1\times r_2}$ is said to be *rectangularly row-stochastic* if $E\mathbf{1}_{r_2} = \mathbf{1}_{r_1}$.

Given a squarely or rectangularly row-stochastic matrix $E \in \Re^{r_1\times r_2}$, one can define the following two measures [44]:

$$\delta(E) = \max_{1\leq j\leq r_2} \max_{1\leq i_1, i_2\leq r_1} |e_{i_1 j} - e_{i_2 j}|$$

$$\lambda(E) = 1 - \min_{1\leq i_1,i_2\leq r_1} \sum_{j=1}^{r_2} \min(e_{i_1 j}, e_{i_2 j}). \tag{31}$$

It is obvious that $0 \leq \delta(E), \lambda(E) \leq 1$ and that $\lambda(E) = 0$ if and only if $\delta(E) = 0$. Both quantities measure how different the rows of E are: $\delta(E) = 0$ if all the rows of E are identical, and $\lambda(E) < 1$ implies that, for every pair of rows i_1 and i_2, there exists a column j (which may depend on i_1 and i_2) such that both $e_{i_1 j}$ and $e_{i_2 j}$ are positive.

Appendix B

Existing Results on Reducibility

A nonnegative matrix $E \in \Re^{r\times r}$ with $r \geq 2$ is said to be *reducible* if the set of its indices, $\mathcal{I} \triangleq \{1, 2, \ldots, r\}$, can be divided into two disjoint nonempty sets $\mathcal{S} \triangleq \{i_1, i_2, \ldots, i_\mu\}$ and $\mathcal{S}^c \triangleq \mathcal{I}/\mathcal{S} = \{j_1, j_2, \ldots, j_\nu\}$ (with $\mu + \nu = r$) such that $e_{i_\alpha j_\beta} = 0$, where $\alpha = 1, \ldots, \mu$ and $\beta = 1, \ldots, \nu$. Matrix E is said to be *irreducible* if it is not reducible. The following theorem provides the most basic property of a reducible matrix and that of an irreducible matrix, and its proof can be done by definition as shown in standard texts [45], [46]. Hence, the lower triangular structure of matrix $F_\triangle$ is the canonical form for reducible matrices.

Lemma B.1: Consider matrix $E \geq 0$ where $E \in \Re^{r\times r}$ and $r \geq 2$. If E is reducible, there exist an integer $p > 1$ and a permutation matrix T such that

$$T^T E T = \begin{bmatrix} F_{11} & 0 & \cdots & 0 \\ F_{21} & F_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ F_{p1} & F_{p2} & \cdots & F_{pp} \end{bmatrix} \triangleq F_\triangle$$

where $F_{ii} \in \Re^{r_i \times r_i}$ are square irreducible submatrices, and $\sum_{i=1}^p r_i = r$. If E is irreducible, vector $z' = (I_{r\times r} + E)z$ has more than η positive entries for any vector $z \geq 0$ containing exactly η positive entries, where $1 \leq \eta < r$ and $I_{r\times r} \in \Re^{r\times r}$ is the identity matrix.

The following corollaries can directly be concluded from the previous lemma.

Corollary B.2: Consider matrix $E \geq 0$ where $E \in \Re^{r\times r}$. Then, if and only if E is irreducible, inequality $\gamma z \geq Ez$ with constant $\gamma > 0$ and vector $z \geq 0$ implies either $z = 0$ or $z > 0$.

Corollary B.3: Consider matrix $E \geq 0$ where $E \in \Re^{r\times r}$. Then, E is irreducible if and only if $(cI_{r\times r} + E)^{r-1} > 0$ for any scalar $c > 0$. If all the matrices in sequence $\{E(k)\}$ are irreducible and diagonally positive, $E(k+\eta)\cdots E(k+1)E(k) > 0$ for some $1 \leq \eta \leq r-1$ and for all k.

Appendix C

New Results on Irreducibility

Lemma B.1 and Corollary B.2 are instrumental to establish the following results. These results together show that irreducibility and the canonical form of lower triangulation of closed-loop system matrix $\overline{G}(t)$ are equivalent to those of sensor/communication matrix $S(t)$.

Lemma C.1: Consider matrix $E \in \Re^{(qm)\times(qm)}$ with sub-blocks $E_{ij} \in \Re^{m\times m}$. Suppose that $E \geq 0$ and that $\overline{E} \in$

$\Re^{[(q+1)m]\times[(q+1)m]}$ is defined by

$$\overline{E} = \begin{bmatrix} 0 & W_1 & W_2 & \cdots & W_q \\ E_{11} & F_1 & E_{12} & \cdots & E_{1q} \\ \vdots & \vdots & \vdots & & \vdots \\ E_{q1} & F_q & E_{q2} & \cdots & E_{qq} \end{bmatrix}$$

where W_1 is a diagonal matrix satisfying $\underline{c}I_{m\times m} \le W_1 \le \overline{c}I_{m\times m}$, $W_i \ge 0$ for $i = 2,\dots,q$ with constants $\underline{c} > 0$ and $\overline{c} > 0$, and $F_j \ge 0$ for $j = 1,\dots,q$. Then, if E is irreducible, so is $\overline{E}$. Furthermore, if $W_2 = \cdots = W_q = F_1 = F_2 = \cdots = F_q = 0$ and if $\overline{E}$ is irreducible, E is irreducible.

Proof: Let us first prove that $\overline{E}$ is irreducible if E is irreducible. Suppose that $z \ge 0$ and $z \ne 0$ and that, for some constant $\gamma > 0$, $\gamma z \ge \overline{E}z$ holds. Partition vector z and define vectors z' and z'' as $z = [\,z_1^T \quad z_2^T \quad z_3^T \quad \cdots \quad z_{q+1}^T\,]^T$, $z' = [\,z_1^T \quad z_3^T \quad \cdots \quad z_{q+1}^T\,]^T$, $z'' = [\,z_2^T \quad z_3^T \quad \cdots \quad z_{q+1}^T\,]^T$, where $z_i \in \Re^m$. It follows from the definition of $\overline{E}$ that

$$\gamma z_1 \ge W_1 z_2 + \sum_{i=2}^{q} W_i z_{i+1} \ge \underline{c} z_2 \tag{32}$$

and that

$$\gamma z'' \ge E z' + F z_2 \ge E z' \tag{33}$$

where $F = [\,F_1^T \quad F_2^T \quad \cdots \quad F_q^T\,]^T$. Combining (32) and (33) yields

$$\gamma \max\left\{\frac{\gamma}{\underline{c}}, 1\right\} z' \ge \gamma z'' \ge E z'.$$

Since E is irreducible, we know from Corollary B.2 in Appendix B that $z' > 0$. It follows from (33) that $z' > 0$ implies $z'' > 0$, and hence, $z_2 > 0$. Thus, we have $z > 0$, and by Corollary B.2, $\overline{E}$ is irreducible.

Next, consider the case that $W_2 = \cdots = W_q = F_1 = F_2 = \cdots = F_q = 0$. This part of the proof is done by contradiction. To this end, suppose that $\overline{E}$ is irreducible but E is reducible. Matrix E being reducible implies that $\gamma\xi \ge E\xi$ holds for some constant $\gamma > 0$ and some vector ξ, where $\xi = [\,\xi_1^T \quad \cdots \quad \xi_q^T\,]^T$ with $\xi_i \in \Re^m$, $\xi \ge 0$, $\xi \ne 0$, and $\xi \not> 0$. Define $\overline{\xi} = [\,\xi_1^T \quad \xi_1^T \quad \xi_2^T \quad \cdots \quad \xi_q^T\,]^T$. Then, it follows that $\gamma \max\{\overline{c}/\gamma, 1\}\overline{\xi} \ge \overline{E}\,\overline{\xi}$ while $\overline{\xi} \ge 0$, $\overline{\xi} \ne 0$, and $\overline{\xi} \not> 0$. According to Corollary B.2 again, this result contradicts with $\overline{E}$ being irreducible; hence, E must be irreducible. □

By applying Lemma C.1 inductively and together with appropriate permutations, one can easily conclude the following corollary. Corollary C.2 provides the property needed to study the cooperative control problem of general systems whose dynamics are of different relative degrees.

Corollary C.2: Given any nonnegative matrix $G(t) \in \Re^{(qm)\times(qm)}$ with subblocks $G_{ij}(t) \in \Re^{m\times m}$, let $\overline{G}(t) \in \Re^{(L_q m)\times(L_q m)}$ with $L_q = l_1 + \cdots + l_q$ be the augmentation of $G(t)$ as defined by (9). Then, $\overline{G}(t)$ is irreducible at time t if and only if $G(t)$ is irreducible at time t.

The following lemma shows the invariance of irreducibility under the Kronecker product, and its proof becomes a straightforward application of Corollary B.2 upon realizing the two facts that, for any $\overline{z} \in \Re^{mq}$, inequality $z' \otimes \mathbf{1}_m \le \overline{z} \le z'' \otimes \mathbf{1}_m$ always holds for some $z', z'' \in \Re^q$ and that, for any $z \in \Re^q$ with $z \ge 0$, $\gamma(z \otimes \mathbf{1}_m) \ge (S \otimes F)(z \otimes \mathbf{1}_m)$ holds if and only if $\gamma z \ge Sz$. Corollary C.4 can be concluded from Lemma C.3 since irreducibility is invariant under the operation of multiplying any row of a matrix by a positive constant.

Lemma C.3: Consider a pair of nonnegative matrices $S \in \Re^{q\times q}$ and $F \in \Re^{m\times m}$, where F is irreducible and row-stochastic. Then, matrix $E = S \otimes F$ is irreducible if and only if S is irreducible.

Corollary C.4: Given an irreducible and row-stochastic matrix K_c, matrix $G(t)$ defined in (5) is irreducible at time t if and only if matrix $S(t)$ is irreducible at time t.

Appendix D

Existing Results on Convergence

Lemma D.1 given next summarizes the relevant results in [23] and [47], and it links positiveness of the state-transient matrix of a continuous-time system to irreducibility of its corresponding design matrix. It follows from (31) that $P > 0$ implies that both $\lambda(P) < 1$ and $\delta(P) < 1$, which provides the condition necessary for applying Lemma D.2 in convergence analysis.

Lemma D.1: Consider $P = e^{(-I+E)\tau}$ where $E \in \Re^{r\times r}$ is a row-stochastic matrix. Then, for every finite $\tau > 0$, matrix P is also row-stochastic, and it is positive if and only if E is irreducible.

The following lemma provides the convergence result on a sequence of products of row-stochastic matrices. It was first reported in [44] and then restated in [23] and [48], and its proof is based upon the simple yet powerful inequality of

$$\delta\left(\prod_{\eta=0}^{k} P(\eta)\right) \le \prod_{\eta=0}^{k} \lambda(P(\eta)) \tag{34}$$

for any $k > 0$ and for any sequence of (squarely) row-stochastic matrices $\{P(k)\}$. Most of the existing results on cooperative control use this result for analysis of stability and convergence.

Lemma D.2: Given a sequence of (squarely) row-stochastic matrices $\{P(k) \in \Re^{r\times r} : k = 1,\dots\}$, consider the product $\prod_{\eta=0}^{k} P(\eta) = P(k)P(k-1)\cdots P(2)P(1)$. If inequality $0 \le \lambda(P(k)) \le c_p < 1$ holds for all k and for matrix function $\lambda(\cdot)$ defined in (31), there exists a row vector $c \in \Re^{1\times r}$ such that $\lim_{k\to\infty}\prod_{\eta=0}^{k} P(\eta) = \mathbf{1}_r c$. That is, the multiplicative sequence converges to a matrix of identical rows.

It should be noted that, if there are finite many distinct matrices $P(\eta)$ and if all their power sequences are known to be convergent, there are results available in [44] and [48] to conclude convergence of the infinite sequence $\prod_{\eta=0}^{\infty} P(\eta)$.

Appendix E

Lemmas Needed to Establish Theorem III.2

Lemma E.1 shows that, for a multiplicative sequence of matrices with positive diagonal blocks, uniformly nonvanishing lower triangular blocks in the individual matrices result in positive lower triangular blocks for the sequence.

Lemma E.1: Consider the sequence $\{P'_{\lrcorner}(k) : k \in \aleph\}$ in which $P'_{\lrcorner}(k)$ is defined by (15) and its diagonal submatrices $P'_{ii}(k)$

are square, of fixed dimension, and uniformly positive with respect to k. If $\{P'_{ij}(k)\} \succ 0$ for some $1 \leq j < i \leq p$, the product of $\prod_{k=0}^{\infty} P'_{\triangle}(k)$ can be grouped such that $\prod_{\eta=0}^{\infty} E_{\triangle}(\eta) = \prod_{k=0}^{\infty} P'_{\triangle}(k)$ and $E_{ij}(\eta) > 0$ for all $\eta \geq 2$.

Proof: By definition, $\{P'_{ij}(k)\} \succ 0$ implies that there is a subsequence $\{k_v, v \in \aleph, v > 0\}$ of $\aleph$ such that $P'_{ij}(k_v) \neq 0$ for all k_v and that $\lim_{v\to\infty} k_v = +\infty$. Now, choose a subsequence $\{k'_\eta, \eta \in \aleph, \eta > 0\}$ of $\{k_v\}$ such that $k'_1 = \min_{k_v \geq 3} k_v$ and $k'_\eta - k'_{\eta-1} \geq 3$, and the new corresponding sequence $\{E_{\triangle}(\eta) : \eta \in \aleph\}$ is $E_{\triangle}(0) = P'_{\triangle}(k'_1 - 2) \cdots P'_{\triangle}(0)$, and for $\eta \geq 0$,

$$E_{\triangle}(\eta+1) \triangleq \prod_{k=k'_\eta - 1}^{k'_{\eta+1}-2} P'_{\triangle}(k) = P'_{\triangle}(k'_{\eta+1} - 2) \cdots P'_{\triangle}(k'_\eta - 1).$$

It is obvious that $\prod_{\eta=0}^{l} E_{\triangle}(\eta) = \prod_{k=0}^{k'_l - 2} P'_{\triangle}(k)$.

Next, consider the pair of i and j (with $j < i$) at which $P'_{ij}(k'_s) \neq 0$. It follows that the block in product $P'_{\triangle}(k'_s + 1)P'_{\triangle}(k'_s)$ and corresponding to $P'_{ij}(k)$ is

$$\begin{aligned}[P'_{\triangle}(k'_s)P'_{\triangle}(k'_s - 1)]_{ij} &= \sum_{w=1}^{p} P'_{iw}(k'_s)P'_{wj}(k'_s - 1) \\ &= \sum_{w=j}^{i} P'_{iw}(k'_s)P'_{wj}(k'_s - 1) \\ &\geq P'_{ij}(k'_s)P'_{jj}(k'_s - 1).\end{aligned}$$

Thus, since $P'_{jj}(k) > 0$ for all k, a single positive element in any row of $P'_{ij}(k'_s)$ makes all the elements in the corresponding row of $[P'_{\triangle}(k'_s)P'_{\triangle}(k'_s - 1)]_{ij}$ positive. Similarly, it follows that

$$\begin{aligned}&[P'_{\triangle}(k'_s + 1)P'_{\triangle}(k'_s)P'_{\triangle}(k'_s - 1)]_{ij} \\ &\quad = \sum_{w=j}^{i} P'_{iw}(k'_s + 1)[P'_{\triangle}(k'_s)P'_{\triangle}(k'_s - 1)]_{wj} \\ &\quad \geq P'_{ii}(k'_s + 1)[P'_{\triangle}(k'_s)P'_{\triangle}(k'_s - 1)]_{ij}\end{aligned}$$

which together with $P'_{ii}(k) > 0$ implies that a positive row in $[P'_{\triangle}(k'_s)P'_{\triangle}(k'_s - 1)]_{ij}$ makes the whole block of $[P_{\triangle}(k'_s + 1)P_{\triangle}(k'_s)P_{\triangle}(k'_s - 1)]_{ij}$ positive. By induction, we have that, for all $s \in \aleph^+$, $E_{ij}(s+1) \geq P'_{ii}(k'_{s+1} - 2) \cdots P'_{ii}(k'_s + 2)[P'_{\triangle}(k'_s + 1)P'_{\triangle}(k'_s)P'_{\triangle}(k'_s - 1)]_{ij} > 0$, which completes the proof. □

Lemmas E.2 and E.3 are needed to conclude an exponential convergence rate.

Lemma E.2: Given any two row-stochastic matrices $Q \in \Re^{r_1 \times r_2}$ and $W \in \Re^{r_1 \times r_1}$. Then, $|Q - WQ| \leq \delta(Q)\mathbf{J}_{r_1 \times r_2}$, where $\delta(\cdot)$ and $|\cdot|$ are defined in Appendix A.

Proof: Letting $F = WQ$ yields that, for any $1 \leq j \leq r_2$, $f_{ij} = \sum_{l=1}^{r_1} e_{il} q_{lj}$, and hence,

$$\begin{aligned}\min_{i\in\{1,\ldots,r_1\}} q_{ij} &\leq \min_{i\in\{1,\ldots,r_1\}} f_{ij} \\ &\leq \max_{i\in\{1,\ldots,r_1\}} f_{ij} \\ &\leq \max_{i\in\{1,\ldots,r_1\}} q_{ij}\end{aligned}$$

which implies $\delta(WQ) \leq \delta(Q)$. It follows from the aforementioned relationship that, for any $1 \leq j \leq r_2$,

$$\begin{aligned}\max_{i\in\{1,\ldots,r_1\}} (q_{ij} - f_{ij}) &\leq \max_{i_1\in\{1,\ldots,r_1\}} q_{i_1 j} - \min_{i_2\in\{1,\ldots,r_1\}} f_{i_2 j} \\ &\leq \max_{i_1\in\{1,\ldots,r_1\}} q_{i_1 j} - \min_{i_2\in\{1,\ldots,r_1\}} q_{i_2 j}\end{aligned}$$

and

$$\begin{aligned}\min_{i\in\{1,\ldots,r_1\}} (q_{ij} - f_{ij}) &\geq \min_{i_2\in\{1,\ldots,r_1\}} q_{i_2 j} - \max_{i_1\in\{1,\ldots,r_1\}} f_{i_1 j} \\ &\geq \min_{i_2\in\{1,\ldots,r_1\}} q_{i_2 j} - \max_{i_1\in\{1,\ldots,r_1\}} q_{i_1 j}.\end{aligned}$$

Thus, $|q_{ij} - f_{ij}| \leq \max_{i_1\in\{1,\ldots,r_1\}} q_{i_1 j} - \min_{i_2\in\{1,\ldots,r_1\}} q_{i_2 j}$ from which inequality $|Q - WQ| \leq \delta(Q)\mathbf{J}_{r_1\times r_2}$ can be concluded using the definition of $\delta(\cdot)$ in (31). □

Lemma E.3: Consider the matrix equation: $\forall k \in \aleph$,

$$Q_i(k+1) = E_{ii}(k+1)Q_i(k) + \sum_{v=1}^{l} E_{ij_v}(k+1)R_{j_v}(k) \tag{35}$$

where $Q_i(k) \in \Re^{r_i \times r}$ with $Q_i(0)$ being rectangularly row-stochastic, $E_{ij_v}(k) \in \Re^{r_i \times r_{j_v}}$ and $E_{ii}(k) \in \Re^{r_i \times r_i}$ are uniformly positive, and composite matrix $\overline{E}_i(k) \triangleq [E_{ij_1}(k) \ \cdots \ E_{ij_l}(k) \ E_{ii}(k)]$ is rectangularly row-stochastic. If $R_{j_v}(k) \in \Re^{r_{j_v} \times r}$ is row-stochastic and $|R_{j_v}(k) - \mathbf{1}_{r_{j_v}} c| \leq \sigma_{j_v}^k \mathbf{J}_{r_{j_v} \times r}$ for some vector c and some constant $0 \leq \sigma_{j_v} < 1$ and for all v, then $Q_i(k)$ converges to $\mathbf{1}_{r_i} c$ exponentially as

$$|Q_i(k) - \mathbf{1}_{r_i} c| \leq \sigma_i^k \mathbf{J}_{r_i \times r} \tag{36}$$

for some constant $0 \leq \sigma_i < 1$.

Proof: It is straightforward to recursively verify through (35) that sequence $\{Q_i(k)\}$ is also rectangularly row-stochastic. Define $\tilde{Q}_i(k) = Q_i(k) - \mathbf{1}_{r_i} c$ and $\tilde{R}_{j_v}(k) = R_{j_v}(k) - \mathbf{1}_{r_{j_v}} c$. It follows from (31) and from $\overline{E}_i(k) = \mathbf{1}_{r_i}$ that

$$\begin{aligned}&\tilde{Q}_i(k+1) \\ &\quad = E_{ii}(k+1)\tilde{Q}_i(k) + \sum_{v=1}^{l} E_{ij_v}(k+1)\tilde{R}_{j_v}(k) \\ &\quad = \Phi(k+1,2)\tilde{Q}_i(1) + \sum_{v=1}^{l}\Bigg[E_{ij_v}(k+1)\tilde{R}_{j_v}(k) \\ &\qquad + \sum_{\eta=0}^{k-1} \Phi(k+1,\eta+2)E_{ij_v}(\eta+1)\tilde{R}_{j_v}(\eta)\Bigg]\end{aligned} \tag{37}$$

where $\Phi(k_2,k_1) \triangleq \prod_{\eta=k_1}^{k_2} E_{ii}(\eta) = E_{ii}(k_2)\cdots E_{ii}(k_1)$. It follows from $|\tilde{R}_{j_v}(k)| \leq \sigma_{j_v}^k \mathbf{J}_{r_{j_v}\times r}$ that

$$\begin{aligned}&|\tilde{Q}_i(k+1)| \\ &\quad \leq \Phi(k+1,2)|\tilde{Q}_i(1)| + \sum_{v=1}^{l}\Bigg[\sigma_{j_v}^k \ E_{ij_v}(k+1)\mathbf{J}_{r_{j_v}\times r} \\ &\qquad + \sum_{\eta=0}^{k-1} \sigma_{j_v}^s \ \Phi(k+1,\eta+2)E_{ij_v}(\eta+1)\mathbf{J}_{r_{j_v}\times r}\Bigg].\end{aligned} \tag{38}$$

On the other hand, it follows from $E_{ij_v}(k), E_{ii}(k) > 0$, and $\overline{E}_i(k)\mathbf{1}_{r_i} = \mathbf{1}_{r_i}$ that, for all k, inequalities

$$\begin{aligned} E_{ij_v}(k)\mathbf{J}_{r_{j_v}\times r} &\le \sigma'_{j_v}\ \mathbf{J}_{r_i\times r} \\ E_{ii}(k)\mathbf{J}_{r_i\times r} &\le \sigma'_i\ \mathbf{J}_{r_i\times r} \\ \Phi(k_2,k_1)\mathbf{J}_{r_i\times r} &\le (\sigma'_i)^{k_2-k_1+1}\mathbf{J}_{r_i\times r} \end{aligned} \tag{39}$$

hold for some constants σ'_{j_v} with $\sigma'_i \in (0,1)$. Noting that $|\tilde{Q}_i(1)| \le \mathbf{J}_{r_i\times r}$ and substituting (39) into (38) yield

$$\begin{aligned} &|\tilde{Q}_i(k)| \\ &\le \left\{ (\sigma'_i)^k + \sum_{v=1}^{l}\left[\sigma^k_{j_v} + \sum_{\eta=0}^{k-1}\sigma^s_{j_v}\ (\sigma'_i)^{k-\eta}\right]\ \sigma'_{j_v}\right\}\mathbf{J}_{r_i\times r} \\ &\triangleq \sigma^k_i\mathbf{J}_{r_i\times r} \end{aligned} \tag{40}$$

in which $0 < \sigma_i < 1$ since power sequence $\{\sigma^k_i\}$ is the sum of scalar power sequences of $\{\sigma^k_{j_v}\}$, $\{(\sigma'_i)^k\}$, and their convolutions (all of which are convergent to zero). Obviously, inequality (40) implies the results in (36). □

APPENDIX F
LEMMAS NEEDED TO ESTABLISH THEOREM III.3

The following lemma restates the result of Theorem III.2 in a form useful for the proof of Theorem III.3.

Lemma F.1: Consider the sequence of squarely row-stochastic and lower triangular matrices $\{P'_{\triangle}(k) : k \in \aleph\}$ defined by (15) and in Lemma III.1. Then, $\prod_{k=0}^{\infty} P'_{\triangle}(k) = \mathbf{1}_r c$ for some $c \in \Re^{1\times r}$ if and only if the product of $\prod_{k=0}^{\infty} P'_{\triangle}(k)$ can be grouped into another sequence of form $\prod_{\eta=0}^{\infty} E'_{\triangle}(\eta)$ such that $\prod_{k=0}^{\infty} P'_{\triangle}(k) = \prod_{\eta=0}^{\infty} E'_{\triangle}(\eta)$ and

$$0 \le \lambda(E'_{\triangle}(\eta)) \le c_e < 1 \qquad \forall \eta \in \aleph \tag{41}$$

where c_e is a constant.

Proof: Sufficiency of (41) to ensure $\prod_{\eta=0}^{\infty} E'_{\triangle}(\eta) = \mathbf{1}_r c$ follows directly from Lemma D.2. Hence, $\prod_{k=0}^{\infty} P'_{\triangle}(k) = \prod_{\eta=0}^{\infty} E'_{\triangle}(\eta) = \mathbf{1}_r c$.

The proof of necessity is done by grouping $\prod_{k=0}^{\infty} P'_{\triangle}(k)$ first into $\prod_{\mu=0}^{\infty} F'_{\triangle}(\mu)$ and then into $\prod_{\eta=0}^{\infty} E'_{\triangle}(\eta)$. The goal is to show that, for all $i \ge 2$, $\{F'_{i1}(\mu)\} \succ 0$ (i.e., nonvanishing property of the subsequence) and $E'_{i1}(\eta) > 0$ for any η (i.e., positive property of submatrix blocks) can be established. Recall that $P'_{ii}(k) > 0$ implies $F'_{ii}(\mu), E'_{ii}(\eta) > 0$ for all i and for all μ and η. Once such a sequence $\{E'_{\triangle}(\eta)\}$ is found, inequality (41) can readily be concluded by the definition of $\lambda(\cdot)$ in (31).

Both productive sequences $\prod_{\mu=0}^{\infty} F'_{\triangle}(\mu)$ and $\prod_{\eta=0}^{\infty} E'_{\triangle}(\eta)$ are constructed inductively with respect to their row block index i that varies from 1 to p. That is, in all the steps except for the first step, element $F'_{\triangle}(\mu_i)$ is constructed by grouping a finite length of $E'_{\triangle}(\eta_{i-1})$, and element $E'_{\triangle}(\eta_i)$ is generated by grouping a finite length of $F'_{\triangle}(\mu_i)$. For $i = 1$, no grouping is needed by setting $F'_{\triangle}(\mu_1)$ and $E'_{\triangle}(\eta_1)$ to be the same as $P'_{\triangle}(k)$. Hence, $F'_{11}(\mu_1) > 0$ and $E'_{11}(\eta_1) > 0$. For $i = 2$, note from Theorem III.2 that $\prod_{\eta_1=0}^{\infty} E'_{\triangle}(\eta_1) = \mathbf{1}_r c$ implies $\{E'_{21}(\eta_1)\} \succ 0$. Thus, setting $F'_{\triangle}(\mu_2)$ to be the same as $E'_{\triangle}(\eta_1)$ implies $\{F'_{21}(\mu_2)\} \succ 0$. It then follows from Lemma E.1 that $\prod_{\mu_2=0}^{\infty} F'_{\triangle}(\mu_2)$ can be grouped into $\prod_{\eta_2=0}^{\infty} E'_{\triangle}(\eta_2)$ such that $E'_{21}(\eta_2) > 0$ for all η_2. Now, as the ith step, assume that $\{E'_{\triangle}(\eta_i)\}$ is found such that $E'_{j1}(\eta_i) > 0$ for all η_i and for all $j \le i$. Again, by Theorem III.2, $\{E'_{(i+1)j}(\eta_i)\} \succ 0$ holds for some $j < i+1$ since $\prod_{\eta_i=0}^{\infty} E'_{\triangle}(\eta_i) = \mathbf{1}_r c$. Therefore, upon choosing $F'_{\triangle}(\mu_{i+1}) \triangleq E'_{\triangle}(\eta_i)E'_{\triangle}(\eta_i - 1)$, inequality

$$F'_{(i+1)1}(\mu_{i+1}) \ge E'_{(i+1)j}(\eta_i)E'_{j1}(\eta_i - 1)$$

holds, and we conclude from $\{E'_{(i+1)j}(\eta_i)\} \succ 0$ and $E'_{j1}(\eta_i - 1) > 0$ that $\{F'_{(i+1)1}(\mu_{i+1})\} \succ 0$. Invoking Lemma E.1 again, we know that $\prod_{\mu_{i+1}=0}^{\infty} F'_{\triangle}(\mu_{i+1})$ can be grouped into $\prod_{\eta_{i+1}=0}^{\infty} E'_{\triangle}(\eta_{i+1})$ such that $E'_{(i+1)1}(\eta_{i+1}) > 0$ for all η_{i+1}, which shows that the inductive proof is completed. □

The following lemma shows the invariance of property $\lambda(\cdot) < 1$ no matter how a diagonally positive row-stochastic matrix is introduced into a product of row-stochastic matrices.

Lemma F.2: Consider two squarely row-stochastic matrices $E, F \in \Re^{r\times r}$ satisfying $\lambda(EF) < 1$. Then, if row-stochastic matrix $W \in \Re^{r\times r}$ has positive diagonal elements (i.e., $w_{ii} > 0$ for all $i = 1, \ldots, r$), $\lambda(EWF) < 1$, $\lambda(WEF) < 1$, and $\lambda(EFW) < 1$.

Proof: Let us first show that $\lambda(EF) < 1$ implies $\lambda(EWF) < 1$. By definition, $\lambda(EF) < 1$ says that, for any i_1 and i_2, there exists j (depending on i_1 and i_2) such that $h_{i_1 j} > 0$ and $h_{i_2 j} > 0$ where $H = EF$,

$$h_{i_1 j} = \sum_{k=1}^{r} e_{i_1 k} f_{kj} \qquad \text{and} \qquad h_{i_2 j} = \sum_{k=1}^{r} e_{i_2 k} f_{kj}.$$

In other words, there exist k_1, k_2, and j (all depending on i_1 and i_2) such that $e_{i_1 k_1}, f_{k_1 j}, e_{i_2 k_2}, f_{k_2 j} > 0$. On the other hand, we have

$$\begin{aligned} [EWF]_{i_1 j} &= \sum_{k=1}^{r} [EW]_{i_1 k} f_{kj} \\ &\ge [EW]_{i_1 k_1} f_{k_1 j} \\ &= \left[\sum_{\mu=1}^{r} e_{i_1 \mu} w_{\mu k_1}\right] f_{k_1 j} \\ &\ge e_{i_1 k_1} w_{k_1 k_1} f_{k_1 j} \\ &> 0 \end{aligned}$$

and similarly $[EWF]_{i_2 j} \ge e_{i_2 k_2} w_{k_2 k_2} f_{k_2 j} > 0$. Since i_1 and i_2 are arbitrary, $\lambda(EWF) < 1$ is readily concluded.

Since $\lambda(EF) < 1$ implies $\lambda(EWF) < 1$, we know that, by setting $E' = EF$ and $F' = I$, $\lambda(EF) = \lambda(E'F') < 1$ implies $\lambda(EFW) = \lambda(E'WF') < 1$. Similarly, $\lambda(WEF) < 1$ can be shown. □

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Zhihua Qu (M'90–SM'93) received the Ph.D. degree in electrical engineering from Georgia Institute of Technology, Atlanta, in 1990.

Since 1990, he has been with the University of Central Florida, Orlando. His current research interests include nonlinear systems, robust and adaptive control designs, and robotics. He is the author of two books: *Robust Control of Nonlinear Uncertain Systems* (Wiley Interscience) and *Robust Tracking Control of Robotic Manipulators* (IEEE Press).

Jing Wang (M'01) received the Ph.D. degree in control theory from the Central South University, Changsha, China, in 1997.

Since March 2002, he has been with the School of Electrical Engineering and Computer Science, University of Central Florida, Orlando, where he is currently a Research Assistant Professor. His current research interests include cooperative control, nonlinear controls, trajectory optimization, and control applications.

Dr. Wang is a member of the American Institute of Aeronautics and Astronautics (AIAA).

Richard A. Hull (M'88) received the B.S. degree in engineering science and mechanics from the University of Florida, Gainesville, in 1972, and the M.S. and Ph.D. degrees in electrical engineering from the University of Central Florida, Orlando, in 1993 and 1996, respectively.

He is currently a Principal Engineer at the Science Applications International Corporation (SAIC), Orlando. His current research interests include synthesis, simulation, and analysis of guidance and control systems for interceptor missiles, space vehicles, supersonic turbo-jets, rockets, and fighter aircrafts.

Dr. Hull is a member of the American Institute of Aeronautics and Astronautics (AIAA).