Methods for estimating stability regions with applications to power systems

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SUMMARY

A specialized method for constructing a hyper-ellipse that resides inside the stability regions of a class of nonlinear autonomous systems such as electric power systems is provided. This method is further generalized to estimate the stability region of a fairly general class of high dimension nonlinear autonomous systems. Applications of the introduced results to power system transient stability analysis are described, together with numerical examples. Copyright © 2006 John Wiley & Sons, Ltd.

KEY WORDS: autonomous system; stability region; lyapunov function; power system

1. INTRODUCTION

The stability region (or attractor) of a dynamic system is of interest in many fields. For example, power engineers have been looking for a solution of stability region estimation for well over 40 years [1–5]. The recent 2003 North America blackout has stimulated an ever-stronger interest in power system stability analysis [6]. In power system control design, controllers are subject to very tight control saturations [7–10]. This also raises the question of stability region estimation.

The best known result so far is a geometric characterization of stability region boundary [1,2]. This celebrated result asserts that the stability region boundary of a dynamic system is composed of the union of stable manifolds of the unstable equilibrium on the boundary. Aside from this seminal work, energy function heuristics for power system stability analysis have been extensively studied [3]. This method may produce either optimistic or pessimistic stability analysis results, which somewhat limited the acceptance by the power industry.

Many techniques for estimating a guaranteed set of a stability region have been reported. For an autonomous system with an isolated equilibrium, a subset of the stability region can be estimated by a quadratic Lyapunov Function [11]. This method requires finding a bounding function. However, there does not seem to exist a general and numerically efficient method for constructing bounding functions

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for general autonomous systems. In Reference [12], a method using quadratic Lyapunov function and grid search was reported. This method can deal with fairly general autonomous systems but it works for systems with very low dimensions only. Other results can be found in, for example, References [13–17].

Based on an exact Taylor's expansion, a specialized method for computing a guaranteed set of stability region of power system is presented in the paper. This method is further generalized to solve the problem for a fairly general class of autonomous systems: the systems that are sufficiently smooth. Applications of this general method to a power system represented by a higher order model are described. Both methods are free of the problem of 'curse of dimensionality.' The results of using first order as well as higher order Taylor's expansion terms are presented.

2. THE SPECIALIZED METHOD FOR MULTI-MACHINE POWER SYSTEMS

The right-hand-side of the classical model of a multi-machine power system contains trigonometric functions only thus can be conveniently bounded; this allows the application of Lyapunov's method [11,18]. The multi-machine system classical model with uniform damping coefficients is as follows [3]:

$$\dot{\delta}_{in} = \omega_{in} \tag{1}$$

$$\dot{\omega}_{in} = -\gamma \omega_{in} + \frac{P_{mi} - P_{ei}(\delta_1, \delta_2 \dots \delta_n)}{M_i} - \frac{P_{mn} - P_{en}(\delta_1, \delta_2 \dots \delta_n)}{M_n} \tag{2}$$

$$i = 1, 2, \dots, n-1 \tag{3}$$

where, n is the number of the generators in the system and

$$\delta_{in} = \delta_i - \delta_n \tag{4}$$

$$\omega_{in} = \omega_i - \omega_n \tag{5}$$

$$\gamma = \frac{D_1}{M_1} = \cdots = \frac{D_n}{M_n} \tag{6}$$

$$P_{ei} = E_i \sum_{l=1}^{n} E_l(G_{il} \cos \delta_{il} + B_{il} \sin \delta_{il}) \tag{7}$$

Let $x_i = \delta_{in} - \delta_{in}^0$, $y_i = \omega_{in} - \omega_{in}^0$, $G_{il}'' = E_i E_l G_{il}$, $B_{il}'' = E_i E_l B_{il}$, $x_{ij} = x_i - x_j$ for $i = 1, 2, \dots, n-1$, where $(\delta_{in}^0, \omega_{in}^0)$ denotes the post-fault system stable equilibrium, and consider *n-th* machine as

reference, the state space model of a multi-machine system with post-fault system stable equilibrium being transferred to the origin becomes,

$$\dot{x}_i = y_i \tag{8}$$

$$\dot{y}_i = g'_i(\boldsymbol{x}, \boldsymbol{y}) = -\gamma y_i + g_i(\boldsymbol{x}) \tag{9}$$

where $i = 1, 2, \ldots, n - 1$ and

$$\boldsymbol{x} = [x_1, x_2, \ldots x_{n-1}]^T, \qquad \boldsymbol{y} = [y_1, y_2, \ldots y_{n-1}]^T \tag{10}$$

$$g_i(\boldsymbol{x}) = -\frac{P_{gi}(\boldsymbol{x})}{M_i} + \frac{P_{gn}(\boldsymbol{x})}{M_n} + \frac{P_{gi}(\boldsymbol{0})}{M_i} - \frac{P_{gn}(\boldsymbol{0})}{M_n} \tag{11}$$

$$P_{gi}(\boldsymbol{x}) = E_i \sum_{l=1}^{n} E_j \left[G_{il} \cos\left(x_{il} + \delta_{il}^0\right) + B_{il} \sin\left(x_{il} + \delta_{il}^0\right) \right] \tag{12}$$

Let us expand $g'_i(\cdot)$ at the origin as follows:

$$g'_i(\boldsymbol{x}, \boldsymbol{y}) = -\gamma y_i + g_i(\boldsymbol{0}) + \sum_{j=1}^{n-1} \frac{\partial g_i(\sigma_i \boldsymbol{x})}{\partial x_j} x_j \tag{13}$$

Since $g_i(\boldsymbol{0}) = 0$ and σ_i is a constant which satisfies $0 \leq \sigma_i \leq 1$, rewrite the equation as:

$$
\begin{aligned}
g'_i(\boldsymbol{x}, \boldsymbol{y}) &= -\gamma y_i + \sum_{j=1}^{n-1} \frac{\partial g_i(\boldsymbol{0})}{\partial x_j} x_j + \sum_{j=1}^{n-1} \left(\frac{\partial g_i(\sigma_i \boldsymbol{x})}{\partial x_j} - \frac{\partial g_i(\boldsymbol{0})}{\partial x_j} \right) x_j \\
&= -\gamma y_i + \sum_{j=1}^{n-1} a_{ij} x_j + \sum_{j=1}^{n-1} h_{ij}(\sigma_i \boldsymbol{x}) x_j \\
&= -\gamma y_i + \sum_{j=1}^{n-1} a_{ij} x_j + h_i(x, \sigma_i)
\end{aligned} \tag{14}
$$

where,

$$h_{ij}(\sigma_i \boldsymbol{x}) = \frac{\partial g_i(\sigma_i \boldsymbol{x})}{\partial x_j} - \frac{\partial g_i(\boldsymbol{0})}{\partial x_j} \tag{15}$$

$$a_{ij} = \frac{\partial g_i(\boldsymbol{0})}{\partial x_j} \tag{16}$$

$$h_i(\boldsymbol{x}, \sigma_i) = \sum_{j=1}^{n-1} h_{ij}(\sigma_i \boldsymbol{x}) x_j \tag{17}$$

Theorem 1. *Assume that the linearized part of the power system* (8), *Equation* (9) *is exponentially stable, then the stability region of Equation* (8), *Equation* (9) *is non-empty and it can be estimated using a quadratic Lyapunov function.*

Proof. Let $\boldsymbol{z} = (\boldsymbol{x}^T, \boldsymbol{y}^T)^T$, Equations (8) and (9) can be expressed as:

$$\dot{z} = \boldsymbol{A}z + \boldsymbol{h} \tag{18}$$

$$\text{Or}: \quad \begin{pmatrix} \dot{\boldsymbol{x}} \\ \dot{\boldsymbol{y}} \end{pmatrix} = \boldsymbol{A} \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} + \boldsymbol{h} \tag{19}$$

where:

$$\mathbf{A} = \begin{pmatrix} 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \\ a_{11} & \cdots & a_{1,n-1} & -\gamma & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1,1} & \cdots & a_{n-1,n-1} & 0 & \cdots & -\gamma \end{pmatrix} \tag{20}$$

$$\boldsymbol{h} = \begin{pmatrix} 0 & \cdots & 0 & h_1 & \cdots & h_{n-1} \end{pmatrix}^T \tag{21}$$

By hypothesis the matrix **A** is stable and the above application of Taylor's expansion does not rely on any approximation. Since the nonlinear term $\boldsymbol{h}$ in Equation (19) contains trignometric terms only, therefore it is bounded. This makes it possible to find an estimate of stability region of Equation (19).

Choose Lyapunov function as:

$$V = \boldsymbol{z}^T \boldsymbol{P} \boldsymbol{z} \tag{22}$$

It follows that:

$$\begin{aligned} \dot{V} &= \boldsymbol{z}^T (\boldsymbol{A}^T \boldsymbol{P} + \boldsymbol{P} \boldsymbol{A}) \boldsymbol{z} + 2\boldsymbol{z}^T \boldsymbol{P} \boldsymbol{h} = -\boldsymbol{z}^T \boldsymbol{Q} \boldsymbol{z} + 2\boldsymbol{z}^T \boldsymbol{P} \boldsymbol{h} \\ &\leq -\lambda_{\min}(\boldsymbol{Q}) \|z\|_2^2 + 2\lambda_{\max}(\boldsymbol{P}) \|z\|_2 \|\boldsymbol{h}\|_2 \end{aligned} \tag{23}$$

Now the key is to find a bound for $\|\boldsymbol{h}\|_2$. Let us derive the analytical expression of the $\|\boldsymbol{h}\|_2$ upper bound. We at first derive the bounding function of $|h_i(\boldsymbol{x})|$. The expression of $h_i(\boldsymbol{x})$ and matrix **A** are given in the appendix.

$$h_i(\boldsymbol{x}) = \sum_{\substack{j=1 \\ j \neq i}}^{n-1} 2x_{ij}\left(G'_{ij}\cos\frac{\sigma_i x_{ij} + 2\delta^0_{ij}}{2} + B'_{ij}\sin\frac{\sigma_i x_{ij} + 2\delta^0_{ij}}{2}\right)\sin\frac{\sigma_i x_{ij}}{2}$$

$$+ 2x_i\left((G'_{in} - G'_{ni})\cos\frac{\sigma_i x_i + 2\delta^0_{in}}{2} + (B'_{in} + B'_{ni})\sin\frac{\sigma_i x_i + 2\delta^0_{in}}{2}\right)\sin\frac{\sigma_i x_i}{2} \quad (24)$$

$$+ \sum_{\substack{j=1 \\ j \neq i}}^{n-1} 2x_j\left(-G'_{nj}\cos\frac{\sigma_i x_j + 2\delta^0_{jn}}{2} + B'_{nj}\sin\frac{\sigma_i x_j + 2\delta^0_{jn}}{2}\right)\sin\frac{\sigma_i x_j}{2}$$

Using the following inequalities: $\sin(t) \leq |t|$, $\cos(t) \leq 1.0$, we can obtain a bounding function for $|h_i(\boldsymbol{x})|$. This is the approached suggested in Equation [19]. Here a different approach is used. Note that for any real number a, b, u we have the following *Hölder* inequality:

$$|a\cos u + b\sin u| = \left|(a \quad b)\begin{pmatrix}\cos u \\ \sin u\end{pmatrix}\right|$$
$$\leq \|(a \quad b)\|\|(\cos u \quad \sin u)^T\| \leq \sqrt{a^2 + b^2} \quad (25)$$

Let

$$r_{in} = \sqrt{(G'_{in} - G'_{ni})^2 + (B'_{in} + B'_{ni})^2}, (i = 1, \dots n) \quad (26)$$

$$r_{ij} = \sqrt{G'_{ij}2 + B'_{ij}2}, (i = 1, \dots n; \; j = 1, \dots, n-1) \quad (27)$$

Since $0 \leq \sigma_i \leq 1$ and $|\sin(\sigma_i t)| \leq |t|$,

$$|h_i(\boldsymbol{x})| \leq r_{in}x_i^2 + \sum_{\substack{j=1 \\ j \neq i}}^{n-1} r_{ij}x_{ij}^2 + \sum_{\substack{j=1 \\ j \neq i}}^{n-1} r_{nj}x_j^2$$
$$= x_i^2\left(r_{in} + \sum_{\substack{j=1 \\ j \neq i}}^{n-1} r_{ij}\right) - 2\sum_{\substack{j=1 \\ j \neq i}}^{n-1} r_{ij}x_ix_j + \sum_{\substack{j=1 \\ j \neq i}}^{n-1}(r_{nj} + r_{ij})x_j^2 \quad (28)$$

and

$$\|h(\boldsymbol{x})\|_1 = \sum_{i=1}^{n-1} |h_i(\boldsymbol{x})|$$
$$= \sum_{i=1}^{n-1}\left(x_i^2\left(r_{in} + \sum_{\substack{j=1 \\ j \neq i}}^{n-1} r_{ij}\right) - 2\sum_{\substack{j=1 \\ j \neq i}}^{n-1} r_{ij}x_ix_j + \sum_{\substack{j=1 \\ j \neq i}}^{n-1}(r_{nj} + r_{ij})x_j^2\right)$$

$$\begin{aligned}
&= \sum_{i=1}^{n-1} x_i^2 r_{in} + \sum_{i=1}^{n-1} x_i^2 \sum_{\substack{j=1\\j\neq i}}^{n-1} r_{ij} - 2\sum_{i=1}^{n-1}\sum_{\substack{j=1\\j\neq i}}^{n-1} r_{ij}x_ix_j + \sum_{i=1}^{n-1}\sum_{\substack{j=1\\j\neq i}}^{n-1} r_{nj}x_j^2 + \sum_{i=1}^{n-1}\sum_{\substack{j=1\\j\neq i}}^{n-1} r_{ij}x_j^2 \\
&= \sum_{i=1}^{n-1} x_i^2 r_{in} + \sum_{i=1}^{n-1} x_i^2 \sum_{\substack{j=1\\j\neq i}}^{n-1} r_{ij} - 2\sum_{i=1}^{n-1}\sum_{\substack{j=1\\j\neq i}}^{n-1} r_{ij}x_ix_j + (n-2)\sum_{j=1}^{n-1} r_{nj}x_j^2 + \sum_{j=1}^{n-1} x_j^2 \sum_{\substack{i=1\\i\neq j}}^{n-1} r_{ij} \\
&= \boldsymbol{x}^{\mathrm{T}} \mathbf{W}\boldsymbol{x} \leq \|\mathbf{W}\|_2 \|\boldsymbol{x}\|_2^2
\end{aligned} \tag{29}$$

where the matrix $\mathbf{W}$ is made of w_{ii} and w_{lk} as follows:

$$w_{ij} = \begin{cases} r_{in} + \sum_{\substack{j=1\\j\neq i}}^{n-1} r_{ij} + (n-2)r_{nj} + \sum_{\substack{i=1\\i\neq j}}^{n-1} r_{ij}, & (i=j) \\ -r_{ij}, & (i\neq j) \end{cases} \tag{30}$$

Then the expression $\|h(\boldsymbol{x})\|_1$ will satisfies

$$\|h(\boldsymbol{x})\|_1 \leq \|\mathbf{W}\|_2 \|\boldsymbol{x}\|_2^2 \tag{31}$$

Since the matrix $\mathbf{W}$ is positive definite, it can be easily verified that:

$$\|\boldsymbol{h}\|_2 \leq \|\boldsymbol{h}\|_1 \leq \|\mathbf{W}\|_2 \|\boldsymbol{x}\|_2^2 = \lambda_{\max}(\mathbf{W})\|\boldsymbol{x}\|_2^2 \leq \lambda_{\max}(\mathbf{W})\|\boldsymbol{z}\|_2^2 \tag{32}$$

Equations (23) and (31) show that $\dot{V} < 0$ if the following inequality is satisfied.

$$\|\boldsymbol{z}\|_2 < \eta = \frac{\lambda_{\min}(\mathbf{Q})}{2\lambda_{\min}(\mathbf{P})\lambda_{\min}(\mathbf{W})} \tag{33}$$

It is now clear that inside the compact set D, where D is defined below:

$$D = \left\{ \boldsymbol{z} \in R^n \,\middle|\, \|\boldsymbol{z}\|_2 \leq \eta \right\}$$

the derivative $\dot{V}$ is negative definite and the Lyapunov function is positive definite. Once we find such a ball D, choose:

$$c = \min_{\|\boldsymbol{z}\|_2 = \eta} V(\boldsymbol{z}) = \lambda_{\min}(\boldsymbol{P})\eta^2$$

and define set Ω_c as:

$$\Omega_c = \left\{ \boldsymbol{z} \in R^n \,\middle|\, V(\boldsymbol{z}) < c \right\} \subset D$$

Then the set Ω_c is a subset of the stability region [11]. The proof is complete. $\qquad\square$
To check the transient stability of a power system, one can follow the following steps:

(1) Calculate the post-fault equilibrium;
(2) Form matrices $\mathbf{A}$, $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{W}$;
(3) Compute η and c;
(4) Run simulation to obtain the state variable z_0 at fault clearing time;
(5) Check if $V(z_0) < c$.

The above method yields a conservative estimate of stability region. Since trignometric functions and their derivatives are bounded above, higher order terms of Taylor's expansion can be used to improve the stability estimation results. This will be further illustrated in subsequent sections.

Three examples are provided in this section to further illustrate the approach described above.

Example a: A Single-Machine-Infinite-Bus System (SMIB)

Consider the SMIB system in Figure 1 where a synchronous machine is connected to an infinite bus through a transmission line and two transformers. In order to use the result of the paper we consider the infinite bus as a specialized generator with a very large inertia constant M and very small transient reactance X'_d, so this system is effectively a two-machine system. Two cases are studied. In the first case, a three-phase-to-ground short-circuit is applied at node 2 and later removed. In the second case, the fault is applied at node 3. The critical clearing angles based on the suggested method and a step-by-step simulation (SBS) method are shown in Figure 2.

Example b: A Nine-Bus system

The data of the system is from Reference [3] and the simulation results are shown in Figure 3.

Example c: New England 39 bus test system

The data is again obtained from Reference [3] and the results are shown in Figure 4.

3. A GENERAL METHOD FOR NONLINEAR AUTONOMOUS DYNAMIC SYSTEM

In this section, we generalize the specialized bounding function method in Section 2 to estimate the stability region of a class of general autonomous system.

Consider the following nonlinear autonomous system,

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}) = [\, f_1(\boldsymbol{x}), f_2(\boldsymbol{x}) \ldots f_n(\boldsymbol{x})]^T, \tag{34}$$

with $f \in C^2(R^n)$. Without loss of generality, assume that the equilibrium point is the origin, and the eigenvalues of $\boldsymbol{A} = (\partial f_i(0)/\partial x_j)_{n \times n}$ have strictly negative real parts. Note that the above system does
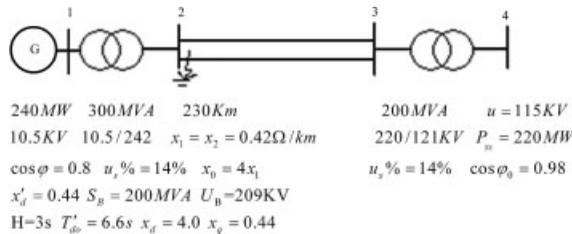

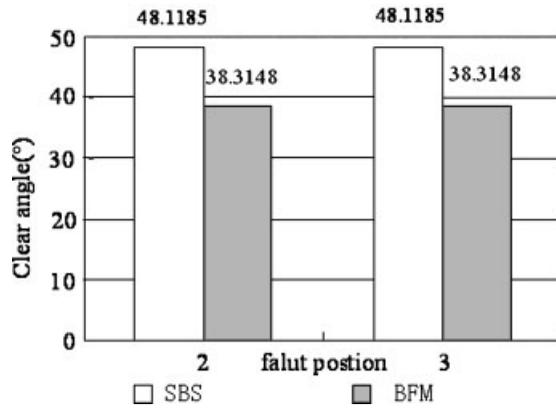Figure 1. The single-line diagram of the SMIB test system.

Figure 2. Comparison of critical clearing angles obtained using SBS and the suggested
bounding function method (BFM).

not have the special structure system (19) possess thus the specialized method presented in Section 2
does not apply.

To find a stability region of autonomous system (34), first consider the following Taylor's series
expansion,

$$f_i(\boldsymbol{x}) = f_i(\boldsymbol{0}) + \frac{\partial f_i(\boldsymbol{0})}{\partial \boldsymbol{x}} \boldsymbol{x} + \frac{1}{2} \boldsymbol{x}^T \nabla^2 f_i(\boldsymbol{\xi}^i) \boldsymbol{x} \tag{35}$$

where $\boldsymbol{\xi}^i = \sigma_i \boldsymbol{x}$, $0 < \sigma_i < 1$, for $i = 1 \ldots n$. Since $f(\boldsymbol{0}) = \boldsymbol{0}$, let us rewrite the above equation as,
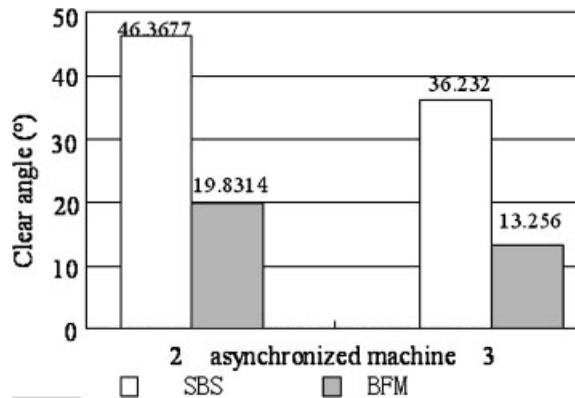

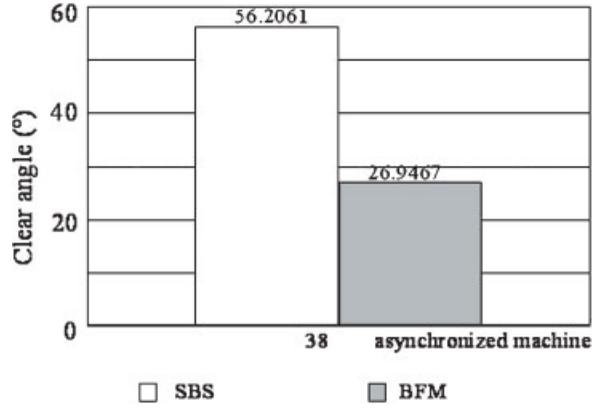Figure 3. Comparison of critical clearing angles of a nine-bus system.

Figure 4. The comparison of critical clearing angles in the New England system.

$$f(\boldsymbol{x}) = \mathbf{A}\boldsymbol{x} + \frac{1}{2}\begin{bmatrix} \boldsymbol{x}^T\nabla^2 f_1(\boldsymbol{\xi}^1)\boldsymbol{x} \\ \boldsymbol{x}^T\nabla^2 f_2(\boldsymbol{\xi}^2)\boldsymbol{x} \\ \vdots \\ \boldsymbol{x}^T\nabla^2 f_n(\boldsymbol{\xi}^n)\boldsymbol{x} \end{bmatrix} = \mathbf{A}\boldsymbol{x} + \mathbf{H}_2 \tag{36}$$

where $\mathbf{H}_2 = [\boldsymbol{h}_1, \boldsymbol{h}_2 \ldots \boldsymbol{h}_n]^T$, $h_i = \frac{1}{2}\boldsymbol{x}^T\nabla^2 f_i(\boldsymbol{\xi}^i)\boldsymbol{x}$, $\boldsymbol{\xi}^i = \sigma_i\boldsymbol{x}$, $0 < \sigma_i < 1$, for $i = 1 \cdots n$.

To proceed the analysis, pick a $\hat{r} > 0$ and define $B_1 = \bar{B}(0, \hat{r})$. Since the norm function $\left\|\nabla^2 f_i(\boldsymbol{x})\right\|$ is continuous in the compact set B_1, it has a maximum in set B_1. Let

$$l_i = \max_{\boldsymbol{x}\in B_1}\left\{\frac{1}{2}\left\|\nabla^2 f_i(\boldsymbol{x})\right\|\right\}, \quad l = \sqrt{\sum_{j=1}^{n} l_j^2} \tag{37}$$

Since $\forall\, \boldsymbol{x} \in B_1$ implies that $\boldsymbol{\xi}^i \in B_1$, the norm of $\mathbf{H}_2$ satisfies the inequality

$$\|\mathbf{H}_2\| \leq \sqrt{\sum_{i=1}^{n}\left(\left\|\frac{1}{2}\nabla^2 f_i(\boldsymbol{\xi}^i)\right\|\|\boldsymbol{x}\|^2\right)^2} \leq \sqrt{\sum_{j=1}^{n} l_j^2\|\boldsymbol{x}\|^2} = l\|\boldsymbol{x}\|^2. \tag{38}$$

Now choose $\mathbf{Q}$ to be a positive definite matrix, since all of the eigenvalues of the Jacobian matrix $\mathbf{A}$ have negative real parts, solve the following Lyapunov function to obtain a positive definite matrix $\mathbf{P}$:

$$\mathbf{P}\mathbf{A} + \mathbf{A}^T\mathbf{P} = -\mathbf{Q}. \tag{39}$$

Let the minimum eigenvalue of $\mathbf{Q}$ be $c_1 = \lambda_{\min}(\boldsymbol{Q})$, the maximum eigenvalue of $\mathbf{P}$ be $c_2 = \lambda_{\max}(\mathbf{P})$, and the minimum eigenvalue of $\mathbf{P}$ be $c_3 = \lambda_{\min}(\mathbf{P})$. Define $V(\boldsymbol{x}) = \boldsymbol{x}^T\mathbf{P}\boldsymbol{x}$, and $\dot{V}$ be its derivative along the trajectory of system (34), it follows that:

$$\dot{V} = \boldsymbol{x}^T\left(\mathbf{PA} + \mathbf{A}^T\mathbf{P}\right)\boldsymbol{x} + 2\boldsymbol{x}^T\mathbf{PH_2} = -\boldsymbol{x}^T\mathbf{Q}\boldsymbol{x} + 2\boldsymbol{x}^T\mathbf{PH_2}$$

$$\leq -c_1\|\boldsymbol{x}\|^2 + 2c_2\|\boldsymbol{x}\|\|\mathbf{H_2}\| \leq \|\boldsymbol{x}\|^2(-c_1 + 2c_2 l\|\boldsymbol{x}\|)$$

(40)

Finally, let $\breve{r} = c_1/2lc_2$, $B_2 = \bar{B}(0, \breve{r})$, $r = \min(\hat{r}, \breve{r})$, $B = \mathrm{int}(B_1 \cap B_2) = B(0, r)$, $V_{cr} = \min\limits_{\|\boldsymbol{x}\|=r} V(\boldsymbol{x}) = c_3 r^2$ and define

$$\Omega_c = \{\boldsymbol{x} \in R^n |\ V(\boldsymbol{x}) < V_{cr}\}$$

(41)

Apparently $\Omega_c \subset B$, if not so, there exists a point $\boldsymbol{x}_1$ in Ω_c satisfying $\|\boldsymbol{x}_1\| \geq r$, which implies $V(\boldsymbol{x}_1) \geq c_3 r^2$, a contradiction with the definition of Ω_c.

Now let us prove that the set Ω_c is an invariant set and also is a subset of stability region of system (34).

Theorem 2. *The set Ω_c as defined in Equation* (41) *is an invariant set and it is in the interior of the stability region of Equation* (34).

Proof: let $\tilde{l}_i = \max\limits_{\boldsymbol{x} \in B}\{\frac{1}{2}\|\nabla^2 f_i(\boldsymbol{x})\|\}$, $\tilde{l} = \sqrt{\sum_{j=1}^n \tilde{l}_i^2}$. Since $r \leq \hat{r}$ implies $B \subset B_1$, therefore $\tilde{l}_i \leq l_i$ and $\tilde{l} \leq l$. For every $\boldsymbol{x}, \boldsymbol{x} \subset B$ and $\boldsymbol{x} \neq 0$, following Equation (40), we have:

$$\dot{V} \leq \|\boldsymbol{x}\|^2\left(-c_1 + 2c_2\tilde{l}\|\boldsymbol{x}\|\right) < \|\boldsymbol{x}\|^2\left(-c_1 + 2c_2\tilde{l}r\right)$$

$$\leq \|\boldsymbol{x}\|^2\left(-c_1 + 2c_2 l\breve{r}\right) < 0$$

By LaSalle's Invariance Principle [11], the closed set Ω_c is an invariant set and it is a subset of the stability region of Equation (34). The proof is complete. $\qquad\square$

Remark 1. The constant l_i as defined in Equation (37) can be estimated by solving a nonlinear optimization problem. This gives the best estimate of l_i. It can be estimated using other forms of norms, for example, 1-norm and infinity-norm, since for a matrix $\mathbf{X}$, $\|\mathbf{X}\|_2 \leq \|\mathbf{X}\|_1$ and $\|\mathbf{X}\|_2 \leq \|\mathbf{X}\|_\infty$. The later approach is simpler but gives less favorable solution. This is the idea of the specialized method described in previous section.

Remark 2. The choice of the matrix $\mathbf{Q}$ will affect the value of r. For certain problems, it has been shown in Reference [11, pp. 206] that the best choice of matrix $\mathbf{Q}$ is the identity matrix. It remains an open question as to what the best choice of matrix $\mathbf{Q}$ is. In the numerical examples supplied in the paper, the identity matrix is used.

Apparently, once the matrix $\mathbf{Q}$ is given, the parameter r has significant impact on the solution of stability region estimation. The best choice of r can be obtained via certain iteration algorithm. We provide one such an algorithm.

Algorithm 1.
Step 1. Initialize $\hat{r}_0$ with an arbitrary number (here the subscript denotes the index of the iteration numbers), choose relaxation parameter α such that $0 < \alpha < 1$;
Step 2. Given $\hat{r}_k$ of the k-th iteration, compute $\breve{r}_k$ and $r_k = \min\{\hat{r}_k, \breve{r}_k\}$;
Step 3. Check if $|\Delta r_k| = |r_k - r_{k-1}|$ is smaller than a given tolerance level or the number of iterations reaches a given threshold, if yes, r_k is the desired solution, stop computation; if not, go to the next step;
Step 4. set $\hat{r}_{k+1} = \hat{r}_k + \alpha(\breve{r}_k - \hat{r}_k)$, and return to step 2.

The following result demonstrates that the algorithm is convergent and the execution of the algorithm improves the initial solution.

Proposition. The sequence r_k is non-decreasing.

Proof: From Equations (37) and (38), it is clear that l is a function of $\hat{r}$, denote it as $l(\hat{r})$, this function is non-decreasing as $\hat{r}$ increases [20]. It follows that the function $\breve{r}(\hat{r})$, where $\breve{r} = c_1/2lc_2$, is non-increasing. Recall the definition of the sequence r_k:

$$\Delta r_{k+1} = r_{k+1} - r_k = \min\{\hat{r}_{k+1}, \ \breve{r}_{k+1}\} - \min\{\ \hat{r}_k, \ \breve{r}_k\}$$

Now consider the case that $\hat{r}_k > \breve{r}_k$, then based on Algorithm 1, $\breve{r}_k < \hat{r}_{k+1} = \hat{r}_k - \alpha(\hat{r}_k - \breve{r}_k) < \hat{r}_k$ and $\breve{r}_{k+1}(\hat{r}_{k+1}) \geq \breve{r}_k(\hat{r}_k)$. This implies that $\min\{\hat{r}_{k+1}, \ \breve{r}_{k+1}\} \geq \breve{r}_k$, therefore:

$$\Delta r_{k+1} = \min\{\hat{r}_{k+1}, \ \breve{r}_{k+1}\} - \breve{r}_k \geq 0$$

For the cases where $\hat{r}_k < \breve{r}_k$, then based on Algorithm 1, $\breve{r} > \hat{r}_{k+1} = \hat{r}_k + \alpha(\breve{r}_k - \hat{r}_k) > \hat{r}_k$ and $\breve{r}_{k+1}(\hat{r}_{k+1}) \leq \breve{r}_k$. This implies that $\min\{\ \hat{r}_{k+1}, \ \breve{r}_{k+1}\} \geq \hat{r}_k$, therefore:

$$\Delta r_{k+1} = \min\{\hat{r}_{k+1}, \ \breve{r}_{k+1}\} - \ \hat{r}_k \geq 0$$

For all the cases we have $\Delta r_{k+1} \geq 0$ as the iteration advances, this completes the proof. $\qquad\square$

Remark 3. If $\hat{r}_k - \breve{r}_k = 0$, then the sequence r_k reaches its maximum point.

Remark 4. The third order term of Taylor's expansion can be used in Equation (36) to improve the quality of the stability region estimation. The computational results of the third order method will be presented below for a comparison.

Two examples are provided below.

Example d. Consider the stability region of the second-order system:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = 1 - \sqrt{2}\sin(x_1 + \pi/4) - x_2$$

Let us try the second-order method. First linearize the system and compute the Taylor expansion terms as:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, h_1 = 0, h_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} \sin(\xi_1 + \pi/4) & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Then choose $\hat{r}_0 = \pi/5$, $B_1 = \{\boldsymbol{x} \in R^n | \|\boldsymbol{x}\| \leq \pi/5\}$, and compute $l = \max_{\boldsymbol{x} \in B_1} \|\mathbf{H}_2(\boldsymbol{x})\| = \sin(\hat{r}_0 + \pi/4)/\sqrt{2} = 0.6984$.

Now let $\mathbf{Q} = \mathbf{I}$, so we have:

$$\mathbf{P} = \begin{bmatrix} 1.5000 & -0.5000 \\ -0.5000 & 1.0000 \end{bmatrix},$$

$c_1 = 1$, $c_2 = \lambda_{\max}(\mathbf{P}) = 1.8090$, $c_3 = \lambda_{\min}(\boldsymbol{P}) = 0.6910$.

It follows that $\breve{r}_0 = 0.3958 < \hat{r}_0$. After executing a search based on Algorithm 1, we have $r = 0.4187$. Finally let $V_{cr} = c_3 r^2 = 0.1211$, so the inequality

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 1.5000 & -0.5000 \\ -0.5000 & 1.0000 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} < 0.1211$$

characterizes a subset of the stability region (see Figure 5).

The third order method yields a better solution as follows (see Figure 5):

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 1.5000 & -0.5000 \\ -0.5000 & 1.0000 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} < 0.2224$$

Example e [12]. Consider the stability region of the second order system

$$\dot{x}_1 = -x_2$$
$$\dot{x}_2 = -x_1 - (1 - x_1^2)x_2$$

Let us start with the second-order method. First calculate $\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$, $h_1 = 0$,

$$h_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} \xi_2^2 & \xi_1^2 \\ \xi_1^2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Now choose $\hat{r}_0 = 1$, $B_1 = \{\mathbf{x} \in R^n | \; \|\mathbf{x}\| \leq 1\}$, so $l = \max_{\mathbf{x} \in B_1} \|\mathbf{H}_2(\mathbf{x})\| = \hat{r}_0 = 1$.

Let $\mathbf{Q} = \mathbf{I}$, we obtain $\mathbf{P} = \begin{bmatrix} 1.5000 & 0.5000 \\ 0.5000 & 1.0000 \end{bmatrix}$, $c_1 = 1$, $c_2 = 1.8090$, $c_3 = 0.6910$, furthermore, we have $\breve{r}_0 = 0.2764 < \hat{r}_0$, and $r = 0.5257 > \min(\hat{r}_0, \breve{r}_0) = 0.2764$.
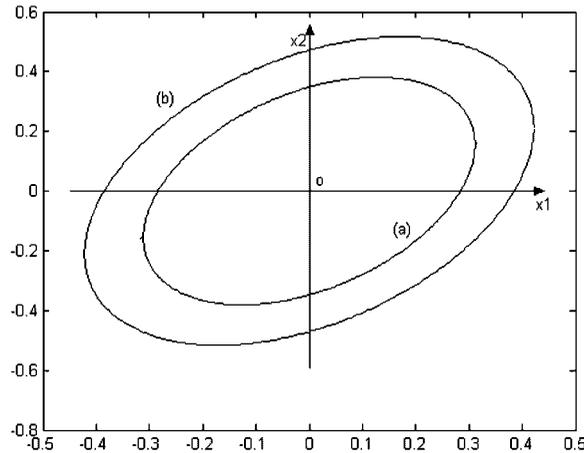

Figure 5. The result of example d. (a) Stability region boundary estimated using the second Method. (b) Stability region boundary estimated using the third Method.
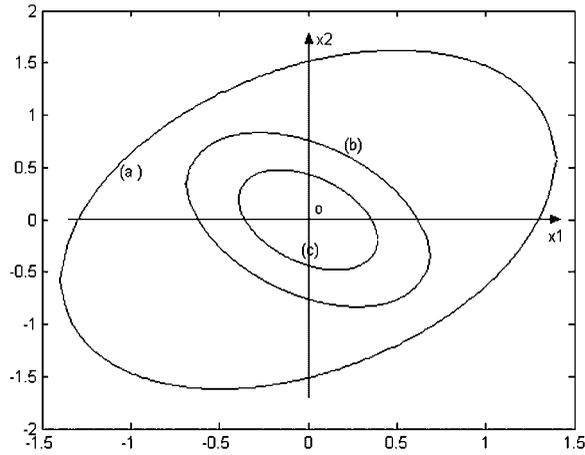
Figure 6. The result of the example e. (a) Result of Reference [4]. (b) Result of the second-order method. (c) Result of the third-order method.

Finally let $V_{cr} = 0.1910$, so the inequality

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 1.5000 & 0.5000 \\ 0.5000 & 1.0000 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} < 0.1910$$

characterizes a subset of the stability region (see Figure 6).

Again, the third-order method gives a better estimate of the stability region as follows (see Figure 6).

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 1.5000 & 0.5000 \\ 0.5000 & 1.0000 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} < 0.5778$$

Example f. Consider the stability region of a single machine system with flux decay model:

$$M\ddot{\delta} = P_m - \frac{E_q' E}{x_{12} + x_d'} \sin\delta - D\dot{\delta} \tag{42}$$

$$\dot{E}_q' = -\frac{1}{T_{do}'} E_q' - \frac{(x_d - x_d')}{T_{do}'} I_d + \frac{1}{T_{do}'} E_{fd} \tag{43}$$

where,

$$I_d = \frac{E'_q}{(x_{12} + x'_d)} - \frac{E}{x_{12}} \cos\delta$$

The definitions of the variables and equations can be found in Reference [3]. The stable equilibrium point is given by $\delta = \delta^s$ and $E'_q = e$. It is obtained by setting the derivatives of the right hand side equal to zero and solving for δ and E'_q. The equilibrium point is then shifted to the origin by defining new state variables as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \dot{\delta} \\ \dot{x}_3 &= E'_q - e \end{aligned} \tag{44}$$

The above system can be re-written as:

$$\dot{x} = \mathbf{A}x + \mathbf{H}_2 \tag{45}$$

where, for $i = 1, 2, 3$:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{K_1 e \cos\delta^s}{M} & -\lambda & -\frac{K_1 \sin\delta^s}{M} \\ -\eta_2 \sin\delta^s & 0 & -\eta_1 \end{bmatrix}, \quad \mathbf{H}_2 = \begin{bmatrix} 0 \\ h_2 \\ h_3 \end{bmatrix},$$

$$K_1 = \frac{E}{(x_{12} + x'_d)}, \quad \eta_1 = \frac{x_{12} + x_d}{(x_{12} + x'_d)T'_{do}}, \quad \eta_2 = \frac{(x_d - x'_d)E}{(x_{12} + x'_d)T'_{do}}$$

$$h_2 = \frac{1}{2}x^T \begin{bmatrix} b_{11} & 0 & b_{31} \\ 0 & 0 & 0 \\ b_{13} & 0 & 0 \end{bmatrix} x, \quad h_3 = \frac{1}{2}x^T \begin{bmatrix} b'_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x,$$

$$b_{11} = \frac{K_1}{M}(e + \xi_3)\sin(\xi_2 + \delta^s), \quad b_{13} = b_{31} = -\frac{K_1}{M}\cos(\xi_2 + \delta^s),$$

$$b'_{11} = -\eta_2 \cos(\xi'_1 + \delta^0), \quad \xi_i = \sigma_i x_i, \quad \xi'_i = \sigma'_i x_i, \quad 0 \le \sigma_i, \sigma'_i \le 1.$$

In the set $B_1 = \{x \in R^3 | \ \|x\| \le r\}$, we have

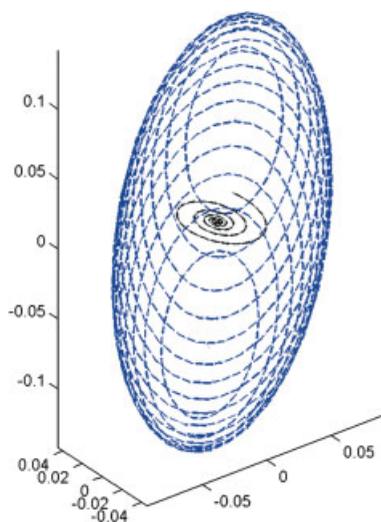$$|h_2| \le (|b_{11}| + |b_{13}|)x^2 \le K_1(e + r + 1)x^2/M \tag{46}$$

Figure 7. Stability region of a single machine system with flux decay model.

$$|h_3| = |b'_{11}||x^2 \leq \eta_2 x^2 \qquad (47)$$

The final system is as follows:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = 0.8318 - 2(x_3 + 1.02)\sin(x_1 + 0.42) - 0.2x_2$$
$$\dot{x}_3 = -0.2295 - 0.405x_3 + 0.255\cos(x_1 + 0.42)$$

The stability region estimated using the general method is illustrated in Figure 7.

Remark 5. The specialized bounding function presented in Section 2 fails to deal with examples e and f. Both the general method presented in this section and the method in Reference [12] can solve the case. The advantage of the suggested method is that it has the potential of dealing with higher dimension systems while the method in Reference [12] cannot.

4. CONCLUSION

In this paper, a specialized bounding function method for estimating stability region of a special class of systems such as power systems in classical model is described. This specialized method is further generalized to estimate the stability region of a fairly general class of nonlinear autonomous systems. The general method is applicable for systems that are sufficiently smooth, and it is particularly useful for higher dimension systems. Extension of the method to transient stability analysis subject to parameter uncertainties is presented in Reference [21].

5. LIST OF SYMBOLS AND ABBREVIATIONS

7.1. Symbols:

E_i	Constant voltage behind direct axis transient reactance of the *i*-th generator
γ	Uniform damping coefficient
δ_i, ω_i	Rotor angle and speed of the *i*-th generator
P_{mi}, P_{ei}	Mechanical input power and electrical output power of the *i*-th generator
M_i, D_i	Moment of inertia and the damping constant of the *i*-th generator
G_{ij}, B_{ij}	Transfer conductance and susceptance of the *i-j* element in the reduced admittance matrix of the system
$\boldsymbol{z}, \boldsymbol{x}, \boldsymbol{y}$	state variable
$\dot{\boldsymbol{z}}, \dot{\boldsymbol{x}}, \dot{\boldsymbol{y}}$	Derivatives of the corresponding state variable
$\mathbf{x}^T$	Transpose of vector $\mathbf{x}$
$\mathbf{I}$	Identity matrix
$\mathbf{P}, \mathbf{Q}$	Positive definite matrixes
$\lambda_{\min}(\mathbf{P})$	Minimum eigenvalue of matrix $\mathbf{P}$
$\lambda_{\max}(\mathbf{P})$	Maximum eigenvalue of matrix $\mathbf{P}$
a, b, u, σ_i	Algebraic variables
∂	Partial derivation
R^n	*n*-dimension Euler domain
$f(\cdot)$	A map from R^n to R^n
$V(\cdot)$	Lyapunov function
$\mathbf{A}$	Jacobian matrix of a full system
$C^2(\cdot)$	Functions with two continuous derivatives
$\text{int}(B)$	Interior of set B
∇^2	Hessian Operator
$\|\cdot\|$	Norm of an element in a vector space, a 2-norm if not stated specifically
$\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_\infty$	1-norm, 2-norm and ∞-norm of an element in a vector space

7.2 Abbreviations:

$$\bar{B}(\boldsymbol{x}_0, r) = \{\boldsymbol{x} \in R^n | \|\boldsymbol{x} - \boldsymbol{x}_0 \le r\}$$

$$B(\boldsymbol{x}_0, r) = \{\boldsymbol{x} \in R^n | \|\boldsymbol{x} - \boldsymbol{x}_0\| < r\}$$

$$\|\boldsymbol{x}\|_2 = \sqrt{\boldsymbol{x}^T \boldsymbol{x}}, \quad \|\boldsymbol{x}\|_\infty = \max_i(|x_i|), \quad \|\boldsymbol{x}\|_1 = \sum_i |x_i|$$

$$\|\mathbf{A}\|_1 = \max_j \sum_{i=1}^{m} |a_{ij}|, \quad \|\mathbf{A}\|_2 = [\lambda_{\max}(\mathbf{A}^T \mathbf{A})]^{1/2}, \quad \|\mathbf{A}\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|$$

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APPENDIX

In this appendix, we give the expression of $\partial g_i / \partial x_j$, a_{ij}, h_{ij} and the expressions of the bounding function of $h_i(\boldsymbol{x})$.

First we derive the expressions of $\partial g_i / \partial x_j$:

$$\frac{\partial P_{gi}}{\partial x_j} = \begin{cases} G'_{ij} \sin\left(x_{ij} + \delta^0_{ij}\right) - B'_{ij} \cos\left(x_{ij} + \delta^0_{ij}\right), & (j \neq i) \\ \sum\limits_{\substack{l=1 \\ l \neq i}}^{n} \left(-G'_{il} \sin\left(x_{il} + \delta^0_{il}\right) + B'_{il} \cos\left(x_{il} + \delta^0_{il}\right)\right), & (j = i) \end{cases} \tag{A1}$$

Let

$$G'_{ij} = E_i E_j G_{ij} / M_i, \quad B'_{ij} = E_i E_j B_{ij} / M_i \tag{A2}$$

So the expression of $\partial g_i(\cdot) / \partial x_j$ is given by:

$$\begin{aligned} \frac{\partial g_i(\cdot)}{\partial x_j} &= -\frac{1}{M_i} \frac{\partial P_{gi}}{\partial x_j} + \frac{1}{M_n} \frac{\partial P_{gn}}{\partial x_j} \\ &= \sum_{\substack{l=1 \\ l \neq i}}^{n} \left(G'_{il} \sin\left(x_{il} + \delta^0_{il}\right) - B'_{il} \cos\left(x_{il} + \delta^0_{il}\right)\right) \\ &\quad + \left(G'_{nj} \sin\left(x_{nj} + \delta^0_{nj}\right) - B'_{nj} \cos\left(x_{nj} + \delta^0_{nj}\right)\right), (j = i) \end{aligned} \tag{A3}$$

$$\begin{aligned} \frac{\partial g_i(\cdot)}{\partial x_j} &= -\frac{1}{M_i} \frac{\partial P_{gi}}{\partial x_j} + \frac{1}{M_n} \frac{\partial P_{gn}}{\partial x_j} \\ &= -G'_{ij} \sin\left(x_{ij} + \delta^0_{ij}\right) + B'_{ij} \cos\left(x_{ij} + \delta^0_{ij}\right) \\ &\quad + G'_{nj} \sin\left(x_{nj} + \delta^0_{nj}\right) - B'_{nj} \cos\left(x_{nj} + \delta^0_{nj}\right), \quad (j \neq i) \end{aligned} \tag{A4}$$

From Equations (50) and (51), let $\boldsymbol{x} = \boldsymbol{0}$, we obtain:

$$a_{ij} = \begin{cases} \sum\limits_{\substack{l=1 \\ l \neq i}}^{n} \left(G'_{il} \sin \delta^0_{il} - B'_{il} \cos \delta^0_{il}\right) + \left(G'_{nj} \sin \delta^0_{nj} - B'_{nj} \cos \delta^0_{nj}\right), & (j = i) \\ -G'_{ij} \sin \delta^0_{ij} + B'_{ij} \cos \delta^0_{ij} + G'_{nj} \sin \delta^0_{nj} - B'_{nj} \cos \delta^0_{nj}, & (j \neq i) \end{cases} \tag{A5}$$

Further more, h_{ij} can be expressed as

$$h_{ij} = \frac{\partial g_i(\sigma_i \boldsymbol{x})}{\partial x_j} - \frac{\partial g_i(\boldsymbol{0})}{\partial x_j}$$

$$
= \sum_{\substack{l=1 \\ l \neq i}}^{n} \left(G'_{il} \sin \left(\sigma_i x_{il} + \delta^0_{il} \right) - B'_{il} \cos \left(\sigma_i x_{il} + \delta^0_{il} \right) \right)
$$

$$
+ \left(G'_{ni} \sin \left(\sigma_i x_{ni} + \delta^0_{ni} \right) - B'_{ni} \cos \left(\sigma_i x_{ni} + \delta^0_{ni} \right) \right)
$$

$$
- \sum_{\substack{l=1 \\ l \neq i}}^{n} \left(G'_{il} \sin \delta^0_{il} - B'_{il} \cos \delta^0_{il} \right)
$$

$$
- \left(G'_{ni} \sin \delta^0_{ni} - B'_{ni} \cos \delta^0_{ni} \right)
$$

$$
= \sum_{\substack{l=1 \\ l \neq i}}^{n} 2 \left(G'_{il} \cos \frac{\sigma_i x_{il} + 2\delta^0_{il}}{2} + B'_{il} \sin \frac{\sigma_i x_{il} + 2\delta^0_{il}}{2} \right) \sin \frac{\sigma_i x_{il}}{2} \qquad \text{(A6)}
$$

$$
+ 2 \left(G'_{ni} \cos \frac{\sigma_i x_{ni} + 2\delta^0_{ni}}{2} + B'_{ni} \sin \frac{\sigma_i x_{ni} + 2\delta^0_{ni}}{2} \right) \sin \frac{\sigma_i x_{ni}}{2}
$$

$$
= \sum_{\substack{l=1 \\ l \neq i}}^{n-1} 2 \left(G'_{il} \cos \frac{\sigma_i x_{il} + 2\delta^0_{il}}{2} + B'_{il} \sin \frac{\sigma_i x_{il} + 2\delta^0_{il}}{2} \right) \sin \frac{\sigma_i x_{il}}{2}
$$

$$
+ 2 \left(-G'_{ni} \cos \frac{\sigma_i x_i + 2\delta^0_{in}}{2} + B'_{ni} \sin \frac{\sigma_i x_i + 2\delta^0_{in}}{2} \right) \sin \frac{\sigma_i x_i}{2}
$$

$$
+ 2 \left(G'_{in} \cos \frac{\sigma_i x_i + 2\delta^0_{in}}{2} + B'_{in} \sin \frac{\sigma_i x_i + 2\delta^0_{in}}{2} \right) \sin \frac{\sigma_i x_i}{2}, \ (j = i)
$$

$$
h_{ij}(\sigma_i \boldsymbol{x}) = \frac{\partial g_i(\sigma_i \boldsymbol{x})}{\partial x_j} - \frac{\partial g_i(\boldsymbol{0})}{\partial x_j}
$$

$$
= -G'_{ij} \sin \left(\sigma_i x_{ij} + \delta^0_{ij} \right) + B'_{ij} \cos \left(\sigma_i x_{ij} + \delta^0_{ij} \right)
$$

$$
+ G'_{nj} \sin \left(\sigma_i x_{nj} + \delta^0_{nj} \right) - B'_{nj} \cos \left(\sigma_i x_{nj} + \delta^0_{nj} \right)
$$

$$
+ G'_{ij} \sin \delta^0_{ij} - B'_{ij} \cos \delta^0_{ij} - G'_{nj} \sin \delta^0_{nj} + B'_{nj} \cos \delta^0_{nj} \qquad \text{(A7)}
$$

$$
= -2 \left(G'_{ij} \cos \frac{\sigma_i x_{ij} + 2\delta^0_{ij}}{2} + B'_{ij} \sin \frac{\sigma_i x_{ij} + 2\delta^0_{ij}}{2} \right) \sin \frac{\sigma_i x_{ij}}{2}
$$

$$
+ 2 \left(-G'_{nj} \cos \frac{\sigma_i x_j + 2\delta^0_{jn}}{2} + B'_{nj} \sin \frac{\sigma_i x_j + 2\delta^0_{jn}}{2} \right) \sin \frac{\sigma_i x_j}{2}, \ (j \neq i)
$$

The expression of $h_i(\boldsymbol{x})$ is derived as:

$$
\begin{aligned}
h_i(\boldsymbol{x}) &= \sum_{j=1}^{n-1} h_{ij}(\sigma_i \boldsymbol{x}) x_j \\
&= \sum_{\substack{l=1 \\ l \neq i}}^{n-1} 2x_i \left(G'_{il} \cos \frac{\sigma_i x_{il} + 2\delta^0_{il}}{2} + B'_{il} \sin \frac{\sigma_i x_{il} + 2\delta^0_{il}}{2} \right) \sin \frac{\sigma_i x_{il}}{2} \\
&\quad + 2x_i \left(-G'_{ni} \cos \frac{\sigma_i x_i + 2\delta^0_{in}}{2} + B'_{ni} \sin \frac{\sigma_i x_i + 2\delta^0_{in}}{2} \right) \sin \frac{\sigma_i x_i}{2} \\
&\quad + 2x_i \left(G'_{in} \cos \frac{\sigma_i x_i + 2\delta^0_{in}}{2} + B'_{in} \sin \frac{\sigma_i x_i + 2\delta^0_{in}}{2} \right) \sin \frac{\sigma_i x_i}{2} \\
&\quad - \sum_{\substack{j=1 \\ j \neq i}}^{n-1} 2x_j \left(G'_{ij} \cos \frac{\sigma_i x_{ij} + 2\delta^0_{ij}}{2} + B'_{ij} \sin \frac{\sigma_i x_{ij} + 2\delta^0_{ij}}{2} \right) \sin \frac{\sigma_i x_{ij}}{2} \\
&\quad + \sum_{\substack{j=1 \\ j \neq i}}^{n-1} 2x_j \left(-G'_{nj} \cos \frac{\sigma_i x_j + 2\delta^0_{jn}}{2} + B'_{nj} \sin \frac{\sigma_i x_j + 2\delta^0_{jn}}{2} \right) \sin \frac{\sigma_i x_j}{2} \\
&= \sum_{\substack{j=1 \\ j \neq i}}^{n-1} 2x_{ij} \left(G'_{ij} \cos \frac{\sigma_i x_{ij} + 2\delta^0_{ij}}{2} + B'_{ij} \sin \frac{\sigma_i x_{ij} + 2\delta^0_{ij}}{2} \right) \sin \frac{\sigma_i x_{ij}}{2} \\
&\quad + 2x_i \left(\left(G'_{in} - G'_{ni} \right) \cos \frac{\sigma_i x_i + 2\delta^0_{in}}{2} + \left(B'_{ni} + B'_{in} \right) \sin \frac{\sigma_i x_i + 2\delta^0_{in}}{2} \right) \sin \frac{\sigma_i x_i}{2} \\
&\quad + \sum_{\substack{j=1 \\ j \neq i}}^{n-1} 2x_j \left(-G'_{nj} \cos \frac{\sigma_i x_j + 2\delta^0_{jn}}{2} + B'_{nj} \sin \frac{\sigma_i x_j + 2\delta^0_{jn}}{2} \right) \sin \frac{\sigma_i x_j}{2}
\end{aligned}
\tag{A8}
$$