

# Saturated Control of Chained Nonholonomic Systems

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*Plenty of approaches to stabilize chained nonholonomic systems have been proposed in the literature. However, the stabilization with constrained inputs is seldom addressed. This problem has practical importance since all physical nonholonomic systems have actuator limitations. In this article, a novel switching control design is proposed. The design strategy is inspired by the structural similarity between chained nonholonomic systems and multiple-integrator systems. The key idea is to make  $u_1$  to be piecewise constant, which renders the rest of the states a chain of constant weighted integrators. Moreover, since the saturation control  $u_2$  eventually works in a linear region with fixed eigenvalues, a buffer zone for  $x_1$  is introduced to ensure the convergence of the rest of the states. The effectiveness of the proposed design is verified by computer simulations.*

**Keywords:** Nonholonomic systems, chained form, feedback control, saturated control

## 1. Introduction

In past decades, plenty of effort has been devoted to the stabilization and tracking control of chained systems [1, 7, 10, 12–16, 21, 22]. It is well known that the chained form is a canonical form for many nonholonomic mechanical systems, hence control designs based on chained systems ensure their wide applicability. Since chained systems do not satisfy Brockett's necessary condition [3], discontinuous or time-varying

feedback controls have to be sought for their stabilization. In the literature, a great deal of solutions have been obtained following the lines of using discontinuous control method or time-varying control method [8]. In general, discontinuous controls can render exponential stability [2, 6, 10, 12], while time-varying controls lead to asymptotic stability [14, 17, 19]. More recent study has also seen the results of  $\rho$ -exponential stability of chained system using time-varying periodic feedback controls [13]. In [16, 21, 22], exponential convergence rates are also reported for continuous time-varying aperiodic design.

Despite these extensive studies on feedback control design, the problem of stabilization with input saturation effect is rarely addressed. In this article, we focus on designing such a control with constrained inputs. When actuator saturation is applied to the inputs, usually, there could be two types of treatments. One is to handle the saturation effect implicitly (or a posteriori), through the so-called antiwindup strategies [4, 5, 9]. The other treatment is to handle the saturation explicitly (or a priori), pursuing one of the following two techniques. The first one is the saturation avoidance method which prevents the saturation from taking place. Therefore the resulted controller always operates in the linear region of saturation nonlinearities. The second approach is the saturation allowance approach which allows the saturation to take place and take saturation effects into account from the outset of control design. The existing designs for nonholonomic systems have been following the second approach mentioned above. In [7], the saturated stabilization and tracking controls are directly synthesized from a

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unicycle-type robot model by using passivity theory and Lyapunov argument. However, the design was not generalized to nonholonomic systems in the chained form. In [10], the authors proposed a discontinuous control design, seeking to remedy the excessively large control inputs near the singular manifold resulting from the  $\sigma$ -process [1]. The state space is decomposed into two separate “good” or “bad” regions. In the “good” region, the control inputs are typically small. In the “bad” region, the controller uses the so-called linear-dominant function (L.D.F) to scale down the magnitude of the control inputs while forcing the trajectories to get into the “good” region. This article proposes a novel switching control design. The chained system is divided into two subsystems controlled by  $u_1$  and  $u_2$ , respectively. The key idea is to make  $u_1$  piecewise constant, which renders the other subsystem a chain of integrators. Then, the multiple-integrator system is transformed into a linear system with an upper triangular system matrix and control  $u_2$  is synthesized.

This article is organized as follows. In Section 2, the bounded state feedback stabilization problem for chained systems is formulated. Section 3 gives the controller design. Section 4 gives the simulation results, and Section 5 concludes the paper.

In the article,  $\|x\|$  denotes the Euclidean norm of a vector  $x$ ,  $\min\{a, b\}$  and  $\max\{a, b\}$  define the minimum and maximum of parameters  $a$  and  $b$ . The sign functions are defined as:

$$\text{sgn}(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}.$$

The saturation functions are defined as  $\text{sat}_\phi(x) = \text{sgn}(x) \min\{|x|, \phi\}$ , where  $\phi$  is the saturation bound. Moreover,  $\text{sat}_1(x)$  is written as  $\text{sat}(x)$  for short.

## 2. Problem Formulation

The objective of this article is to present a control design strategy which globally stabilizes the chained nonholonomic system under saturation conditions. Consider the following  $n$ th order chained system with initial condition  $x(t_0) \in \mathfrak{R}^n$ , where  $t_0 \geq 0$  is the initial time.

$$\begin{cases} \dot{x}_1 = u_1 \\ \dot{x}_2 = u_1 \cdot x_3 \\ \dot{x}_3 = u_1 \cdot x_4 \\ \vdots \\ \dot{x}_{n-1} = u_1 \cdot x_n \\ \dot{x}_n = u_2 \end{cases} \quad (1)$$

where  $x = [x_1 \ \cdots \ x_n]^T \in \mathfrak{R}^n$  is the state,  $u = [u_1, u_2]^T \in \mathfrak{R}^2$  is the control input, which is subject to the following saturation constraint:

$$-\delta_i \leq u_i \leq \delta_i, \quad i = 1, 2, \delta_i > 0. \quad (2)$$

The control design follows the second aforementioned approach, that is, the saturation effect is taken into consideration at the design phase. It follows from (1) that the chained system can be reorganized as the following two subsystems:

$$\dot{x}_1 = u_1, \quad (3)$$

and

$$\dot{z} = u_1 A z + B u_2, \quad (4)$$

where  $z \triangleq [z_1 \ z_2 \ \cdots \ z_{n-1}]^T = [x_2 \ x_3 \ \cdots \ x_n]^T$ , and

$$A \triangleq \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & & 1 \\ 0 & 0 & \cdots & & 0 \end{bmatrix}, \quad B \triangleq \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Clearly, subsystem (3) only contains  $x_1$  and is independent of the rest of the states. It can be easily stabilized with or without saturation. Subsystem (4) is a linear time-varying (LTV) system, which is very structurally similar to a multiple-integrator system, except that it is weighted by one of the control inputs. Naturally, one would think of manipulating  $u_1$  to gain advantages in controlling subsystem (4). A straightforward way is to create a piecewise constant  $u_1$  that meets the saturation condition as well as stabilize the subsystem (3). Then subsystem (4) becomes a constant-weighted multiple-integrator system whose saturation control is studied in [20, 18, 11, 23].

## 3. The Saturated Control Design

Before proceeding with the control design, we first state the following stability theorem from [23].

**Theorem 1:** *Let  $\lambda_i$ ,  $i = 1, \dots, n$  be a series of positive constants. Consider the following linear system with input constraint  $-v_{\max} \leq v \leq v_{\max}$  with  $v_{\max} > 0$ :*

$$\dot{\xi} = A_n \xi + b_n v, \quad (5)$$

where  $\xi = [\xi_1 \ \xi_2 \ \cdots \ \xi_n]$ ,  $v \in \mathfrak{R}$ ,

$$A_n = \begin{bmatrix} 0 & \lambda_2 & \cdots & \lambda_{n-1} & \lambda_n \\ 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \cdots & \lambda_{n-1} & \lambda_n \\ 0 & 0 & \cdots & 0 & \lambda_n \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \in \mathfrak{R}^{n \times n},$$

$$b_n = \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 1 \\ 1 \end{bmatrix} \in \mathfrak{R}^{n \times 1}.$$

The nonlinear control:

$$v = - \sum_{i=1}^n \epsilon_i \text{sat} \left( \frac{\lambda_i \xi_i}{\epsilon_i} \right), \quad (6)$$

where  $\epsilon_i$  satisfies:

$$\begin{cases} \epsilon_1 > 0 \\ \epsilon_j > \sum_{i=1}^{j-1} \epsilon_i, \quad j = 2, 3, \dots, n. \\ \sum_{i=1}^n \epsilon_i \leq v_{\max} \end{cases} \quad (7)$$

is a globally stabilizing control that satisfies the input constraint. Furthermore, the closed loop system will operate in a linear region in finite time with eigenvalues  $-\lambda_i$ ,  $i = 1, \dots, n$ .

*Proof:* Refer to Lemma 2 and Theorem 3 of [23].  $\square$

**Corollary 1:** The linear region of (6) is:

$$\Omega_1 = \left\{ \xi : |\xi_1| \leq \frac{\epsilon_1}{\lambda_1}, |\xi_2| \leq \frac{\epsilon_2}{\lambda_2}, \dots, |\xi_n| \leq \frac{\epsilon_n}{\lambda_n} \right\}.$$

And once control (6) gets into  $\Omega_1$ , its saturation elements will not be saturated again, that is, the control becomes a linear control law afterward.

*Proof:* From (6), the saturation functions become linear if  $|\frac{\lambda_i \xi_i}{\epsilon_i}| \leq 1$ ,  $i = 1, 2, \dots, n$ , from which,  $\Omega_1$  can be deduced.

Define:

$$v_j = - \sum_{i=1}^j \epsilon_i \text{sat} \left( \frac{\lambda_i \xi_i}{\epsilon_i} \right), \quad j = 1, 2, \dots, n.$$

Suppose at a certain moment, control (6) is in the linear region  $\Omega_1$ , then it can be rewritten as:

$$v = v_n = -\lambda_n \xi_n + v_{n-1}.$$

Consider the last state equation:

$$\dot{\xi}_n = v_n = -\lambda_n \xi_n + v_{n-1}.$$

Take the Lyapunov function candidate  $V_n = \frac{1}{2} \xi_n^2$ . It follows that:

$$\begin{aligned} \dot{V}_n &= -\lambda_n \xi_n^2 + \xi_n u_{n-1} \\ &\leq -\lambda_n |\xi_n|^2 + |\xi_n| |u_{n-1}| \\ &\leq -\lambda_n |\xi_n|^2 + |\xi_n| \epsilon_n. \end{aligned}$$

It shows if  $|\xi_n| > \frac{\epsilon_n}{\lambda_n}$ , then  $\dot{V}_n < 0$ . Therefore  $\xi_n$  will remain in  $|\xi_n| \leq \frac{\epsilon_n}{\lambda_n}$ . Consider the second to last state equation:

$$\dot{\xi}_{n-1} = \lambda_n \xi_n + v_n = -\lambda_{n-1} \xi_{n-1} + v_{n-2}.$$

Take the Lyapunov function candidate  $V_{n-1} = \frac{1}{2} \xi_{n-1}^2$ . A similar process would show that  $\xi_{n-1}$  is in  $|\xi_{n-1}| \leq \frac{\epsilon_{n-1}}{\lambda_{n-1}}$ . Repeating the same process for the state  $\xi_1, \xi_2, \dots, \xi_{n-2}$ , one would have:

$$|\xi_i| \leq \frac{\epsilon_i}{\lambda_i}, \quad i = 1, 2, \dots, n-2.$$

Therefore, the state  $\xi$  is always confined in the same set of  $\Omega_1$ , which indicates that once the state gets into  $\Omega_1$ , it cannot escape, where control (6) is linear.  $\square$

### 3.1. The Saturated Control Design

To begin the control design process, it is needed to define some new state variables in a transformed space. Let  $w = [w_1 \ w_2 \ \cdots \ w_{n-1}]^T$  and  $w' = [w'_1 \ w'_2 \ \cdots \ w'_{n-1}]^T$ , where

$$w_{n-1-i} = \sum_{j=0}^i \binom{i}{j} (-1)^j z_{n-1-j},$$

$$w'_{n-1-i} = \sum_{j=0}^i \binom{i}{j} z_{n-1-j}, \quad i = 0, 1, \dots, n-2. \quad (8)$$

with  $z$  introduced in (4), and

$$\binom{i}{j} = \frac{i!}{j!(i-j)!}.$$

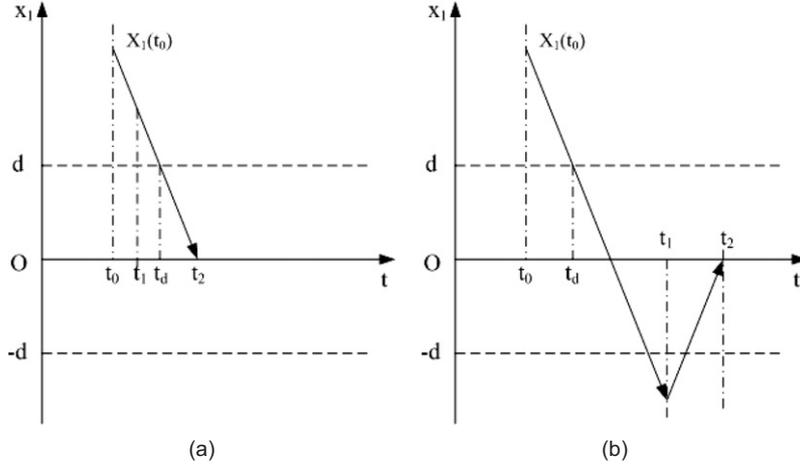


Fig. 1. Two case of control: (a)  $t_1 \leq t_d$ ; (b)  $t_1 > t_d$ .

It can be verified that the transformations are invertible. Define the following set for any  $n - 1$  dimensional vector  $\mathcal{X} = [\mathcal{X}_1 \ \mathcal{X}_2 \ \cdots \ \mathcal{X}_{n-1}]$ :

$$\Omega_2 = \left\{ \mathcal{X} : |\mathcal{X}_1| \leq \frac{\epsilon_1}{k\delta_1}, |\mathcal{X}_2| \leq \frac{\epsilon_2}{k\delta_1}, \dots, |\mathcal{X}_{n-1}| \leq \frac{\epsilon_{n-1}}{k\delta_1} \right\},$$

where  $0 < k \leq 1$  is a constant,  $\epsilon_i$  satisfies condition (7) with  $i = 1, 2, \dots, n - 1$  and  $v_{\max} = \delta_2$ . In fact, it would be straightforward to verify that  $\Omega_2$  is the linear region for control  $u_2$  that is proposed in (12).

The control design for the case of  $x_1(t_0) \geq 0$  is to be discussed. For the case of  $x_1(t_0) < 0$ , one can always make it positive by redefining the following coordinate system  $x'_i(t) = (-1)^i x_i(t)$ ,  $i = 1, 2, \dots, n$ , which results in a new chained system with  $x'_i(t) > 0$ .

A scheme for controlling  $x_1$  is shown in Fig. 1, where a buffer zone  $\Omega_3 = -d \leq x_1 \leq d$  is created, with  $d > 0$  as a design parameter. The objective of creating  $\Omega_3$  is by forcing  $x_1$  out of  $\Omega_3$  after subsystem (4) gets into the linear region  $\Omega_2$ , sufficient time can be ensured for subsystem (4) to converge as controls are terminated at  $x_1 = 0$ .

To better illustrate the control strategy, let's first consider a simpler control for  $t \geq t_0$ :

$$\begin{cases} u_1(t) = -\text{sgn}(x_1(t_0))k\delta_1 \\ u_2(t) = -\sum_{i=1}^{n-1} \epsilon_i \text{sat}\left(\frac{k\delta_1 w_i}{\epsilon_i}\right) \end{cases} \quad (9)$$

With  $u_1$  in (9), define  $t_d$  to be the time when  $x_1$  gets into  $\Omega_3$ , which can be calculated as:

$$t_d = \begin{cases} t_0 + \frac{x_1(t_0) - d}{k\delta_1}, & x_1(t_0) > d \\ t_0, & 0 \leq x_1(t_0) \leq d \end{cases}.$$

Accordingly, subsystem (4) becomes:

$$\begin{cases} \dot{z}_1 = -k\delta_1 z_2 \\ \dot{z}_2 = -k\delta_1 z_3 \\ \vdots \\ \dot{z}_{n-2} = -k\delta_1 z_{n-1} \\ \dot{z}_{n-1} = u_2 \end{cases}.$$

By the transformation (8), the transformed system is:

$$\dot{w} = A_w w + B_w u_2, \quad (10)$$

where

$$A_w = \begin{bmatrix} 0 & k\delta_1 & \cdots & k\delta_1 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & k\delta_1 \\ 0 & \cdots & \cdots & 0 \end{bmatrix}, \quad B_w = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}.$$

By Theorem 1,  $u_2$  in (9) is a stabilizing control for system (10). This indicates that  $\lim_{t \rightarrow \infty} \|w\| = 0$ . Hence, there is a minimal time  $t_w$  such that  $w(t) \in \Omega_2$  for  $t \geq t_w$ . Here, two cases are possible:

1.  $t_0 \leq t_w \leq t_d$ , this means system (10) gets into  $\Omega_2$  while  $x_1$  hasn't reach  $\Omega_3$ . This case is illustrated by Fig. 1(a) where we define  $t_1 = t_w$ .
2.  $t_w > t_d$ , in this case,  $x_1$  entered  $\Omega_3$  before system (10) gets into  $\Omega_2$ . It is desired to force  $x_1$  leave  $\Omega_3$ . The condition for  $x_1(t) \leq -d$  is:

$$t \geq t'_d \triangleq t_0 + \frac{x_1(t_0) + d}{k\delta_1}.$$

Since  $\lim_{t \rightarrow \infty} \|w(t)\| = 0$ , there is  $\lim_{t \rightarrow \infty} \|z(t)\| = 0$ . By transformation (8),  $\lim_{t \rightarrow \infty} \|w'(t)\| = 0$ , hence there exists a finite time  $t'_w$  such that  $w'(t) \in \Omega_2$  for

$t \geq t'_w$ .  $t_1$  is defined to be  $t_1 = \max\{t'_d, t'_w\}$ . This case is illustrated by Fig. 1(b).

Note that in both cases,  $|x_1(t_1)| \geq d$ .

Depending on which situation occurs, different controls are applied for  $t \geq t_1$ . The full controllers are proposed in (11) and (12).

$$u_1(t) = \begin{cases} -\text{sgn}(x_1(t_0))k\delta_1, & t_0 \leq t \leq t_1 \\ -\text{sgn}(x_1(t_1))k\delta_1, & t_1 < t \leq t_2 \\ 0, & t > t_2 \end{cases}, \quad (11)$$

$$u_2(t) = \begin{cases} -\sum_{i=1}^{n-1} \epsilon_i \text{sat}\left(\frac{k\delta_1 w_i}{\epsilon_i}\right), & t_0 \leq t \leq t_1 \\ -\sum_{i=1}^{n-1} \epsilon_i \text{sat}\left(\frac{k\delta_1 w_i}{\epsilon_i}\right), & t_1 < t \leq t_2, t_1 \leq t_d \\ -\sum_{i=1}^{n-1} \epsilon_i \text{sat}\left(\frac{k\delta_1 w'_i}{\epsilon_i}\right), & t_1 < t \leq t_2, t_1 > t_d \\ 0, & t > t_2 \end{cases}, \quad (12)$$

where  $t_1$  is redefined for both cases:

$$t_1 = \inf\{\{t_0 \leq t \leq t_d : w(t) \in \Omega_2, x_1(t) \geq d\} \cup \{t \geq t_d : w'(t) \in \Omega_2, x_1(t) \leq -d\}\}.$$

$t_2$  is the moment when the control goal is considered to be accomplished and it can be quantified as:

$$t_2 = t_1 + \frac{|x_1(t_1)|}{k\delta_1}.$$

Due to the fact  $|x_1(t_1)| \geq d$ , the following relation holds

$$t_2 \geq t_1 + \frac{d}{k\delta_1}. \quad (13)$$

The following theorem proves that the proposed controls have practical stability.

**Theorem 2:** *Control (11) and (12) practically stabilize the chained system (1) while satisfying the bound condition (2). Moreover, for any constant  $\rho > 0$ , there exists a constant  $d_0 > 0$ , such that when  $d > d_0$ ,  $\|x(t)\| < \rho$  for  $t \geq t_2$ .*

*Proof:* Consider subsystem (3), since  $0 < k \leq 1$ , obviously  $u_1$  satisfies  $|u_1| \leq \delta_1$ . Moreover, no matter where  $x_1(t_1)$  is,

$$\begin{aligned} x_1(t_2) &= x_1(t_1) + u_1 \times (t_2 - t_1) \\ &= x_1(t_1) - \text{sgn}(x_1(t_1))k\delta_1 \times \frac{|x_1(t_1)|}{k\delta_1} \\ &= 0. \end{aligned}$$

For  $t \in [t_0, t_1]$ , subsystem (4) becomes (10). Moreover, if  $t_0 \leq t_1 \leq t_d$ ,  $u_1$  and  $u_2$  are kept unchanged for  $t_1 < t \leq t_2$ . Therefore, for  $t_1 < t \leq t_2$ , subsystem (4) still has the transformed system (10).

However, if  $t_1 > t_d$ , for  $t_1 < t \leq t_2$ , control  $u_1$  and subsystem (4) becomes:

$$u_1 = k\delta_1,$$

and

$$\begin{cases} \dot{z}_1 = k\delta_1 z_2 \\ \dot{z}_2 = k\delta_1 z_3 \\ \vdots \\ \dot{z}_{n-2} = k\delta_1 z_{n-1} \\ \dot{z}_{n-1} = u_2 \end{cases}.$$

By transformation (8), subsystem (4) becomes:

$$\dot{w}' = A_w w' + B_w u_2. \quad (14)$$

No matter which case occurs, the closed loop system for  $t > t_1$  is in the following form:

$$\dot{\zeta} = A_c \zeta, \quad (15)$$

where either  $\zeta = w$  or  $\zeta = w'$ , and

$$A_c = \begin{bmatrix} -k\delta_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ -k\delta_1 & \ddots & -k\delta_1 & 0 \\ -k\delta_1 & -k\delta_1 & \cdots & -k\delta_1 \end{bmatrix} \in \mathfrak{R}^{(n-1) \times (n-1)}.$$

Therefore,

$$\begin{aligned} \|\zeta(t_2)\| &= \|e^{A_c(t_2-t_1)} \zeta(t_1)\| \\ &\leq \|e^{A_c \cdot \frac{d}{k\delta_1}}\| \|\Omega_2\|, \end{aligned}$$

where

$$\|\Omega_2\| = \left[ \left(\frac{\epsilon_1}{k\delta_1}\right)^2 + \left(\frac{\epsilon_2}{k\delta_1}\right)^2 + \cdots + \left(\frac{\epsilon_{n-1}}{k\delta_1}\right)^2 \right]^{\frac{1}{2}}.$$

Here, we used the relation (13) and  $\|\zeta(t_1)\| \leq \|\Omega_2\|$ . Since  $A_c$  is Hurwitz,  $\lim_{d \rightarrow \infty} \|\zeta(t_2)\| = 0$ . Therefore  $\lim_{d \rightarrow \infty} \|z(t_2)\| = 0$ , hence for any given  $\rho > 0$ , there exists a finite  $d_0$  such that for  $d > d_0$ ,  $\|z(t_2)\| < \rho$ . Since  $\|x(t_2)\| = \|[x_1(t_2) \ z(t_2)^T]^T\| = \|z(t_2)\|$ , that is  $\|x(t_2)\| < \rho$ . Moreover, since  $u_1 = u_2 = 0$  for  $t > t_2$ ,  $\|x(t)\| < \rho$  holds for  $t > t_2$ .  $\square$

### 3.2. Choice of $k$ and $d$

To meet the saturation condition, the design parameter  $k$  is restricted by  $0 < k \leq 1$ . Intuitively,  $k$  should be chosen large. Because with a larger  $k$ , the connections among the states of subsystem (4) are stronger and the magnitude of control  $u_2$  tends to be larger (within the saturation bound). This contributes to a faster convergence rate before the controller reaches the linear region. Moreover, in the linear operation region  $\Omega_2$ , the closed loop system of subsystem (4) is equation (15). The system matrix  $A_c$  shows that  $-k\delta_1$  is the  $(n-1)$ th order eigenvalue,  $k$  also decides the convergence rate when the controls work in the linear operation region. So, where convergence speed is concerned,  $k$  needs to be chosen as large as possible, that is,  $k = 1$ .

For the choice of  $d$ , it follows from (15) that the Laplace transformation of the state transition matrix is:

$$\mathcal{L}(e^{A_c t}) = (sI - A_c)^{-1} = \begin{bmatrix} \frac{1}{s + k\delta_1} & 0 & \cdots & 0 & 0 \\ \frac{-k\delta_1}{(s + k\delta_1)^2} & \frac{1}{s + k\delta_1} & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \frac{-k\delta_1 s^{n-4}}{(s + k\delta_1)^{n-2}} & \ddots & \ddots & \frac{1}{s + k\delta_1} & 0 \\ \frac{-k\delta_1 s^{n-3}}{(s + k\delta_1)^{n-1}} & \frac{-k\delta_1 s^{n-4}}{(s + k\delta_1)^{n-2}} & \cdots & \frac{-k\delta_1}{(s + k\delta_1)^2} & \frac{1}{s + k\delta_1} \end{bmatrix}.$$

Therefore,

$$e^{A_c t} = \begin{bmatrix} e^{-k\delta_1 t} & 0 & \cdots & 0 & 0 \\ -k\delta_1 t e^{-k\delta_1 t} & e^{-k\delta_1 t} & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \sum_{i=2}^{n-2} (-1)^{i-1} \frac{(k\delta_1 t)^{i-1}}{(i-1)!} e^{-k\delta_1 t} & \ddots & \ddots & e^{-k\delta_1 t} & 0 \\ \sum_{i=2}^{n-1} (-1)^{i-1} \frac{(k\delta_1 t)^{i-1}}{(i-1)!} e^{-k\delta_1 t} & \sum_{i=2}^{n-2} (-1)^{i-1} \frac{(k\delta_1 t)^{i-1}}{(i-1)!} e^{-k\delta_1 t} & \cdots & -k\delta_1 t e^{-k\delta_1 t} & e^{-k\delta_1 t} \end{bmatrix}.$$

With the information of  $k$  and  $\delta_1$ , one can solve for the time  $T_m$  that is needed for maneuvering in the linear region. Then  $d$  is obtained by  $d \geq k\delta_1 T_m$ . For example, with the choice  $k = 1$  and the saturation bound  $\delta_1 = \delta_2 = 1$ , the state transition matrix for a chained system with  $n = 3$  is:

$$e^{A_c t} = \begin{bmatrix} e^{-t} & 0 \\ -te^{-t} & e^{-t} \end{bmatrix}.$$

If one chooses  $T_m = 4$  or  $T_m = 5$ , the final state is around 7% or 3% of the value it takes when it enters the linear operation region  $\Omega_2$ .

## 4. Simulations

In this section, simulation results for the proposed control are presented. The simulation is conducted on a chained system with  $n = 3$ . The saturation limit is chosen to be  $\delta_1 = \delta_2 = 1$ , the gain parameter for  $u_1$  is  $k = 1$  and  $d$  is set to be  $d = 4$ . Satisfying the condition (7),  $\epsilon_1$  and  $\epsilon_2$  are chosen to be  $\epsilon_1 = 0.499$  and  $\epsilon_2 = 0.5$ . To illustrate the two types of control actions, two sets of initial conditions are selected in the simulation. The results for both cases show that the proposed control is successful under the saturation condition.

In the first case, the initial condition is set to be  $x(t_0) = [12 \ 5 \ 3]$ . Then, it can be obtained that  $t_d = 8$ . By running the simulation, it is obtained that  $t_1 = 6.2146$  and  $t_2 = 12$ . The simulation results for this case are shown in Fig. 2. Fig. 2(a) shows the state response, since  $t_1 < t_d$ ,

subsystem (4) reaches  $\Omega_2$  before  $x_1$  gets into the region  $[-d \ d]$ . Therefore, the controller knows the time for maneuvering subsystem (4) is sufficient. Hence when  $x_1$  reaches zero, the controls are set to zero, and the states  $x_2$  and  $x_3$  are stopped at 0.0027 and 0.0039, respectively.

In the second case, the initial condition is set to be  $x(t_0) = [6 \ 5 \ 3]$ . Therefore,  $t_d = 2$ . It is obtained from the simulation that  $t_1 = 10.0012$  and  $t_2 = 14.0024$ . The simulation results for this case are shown in Fig. 3. Fig. 3(a) shows the state response, since  $t_1 > t_d$ , subsystem (4)

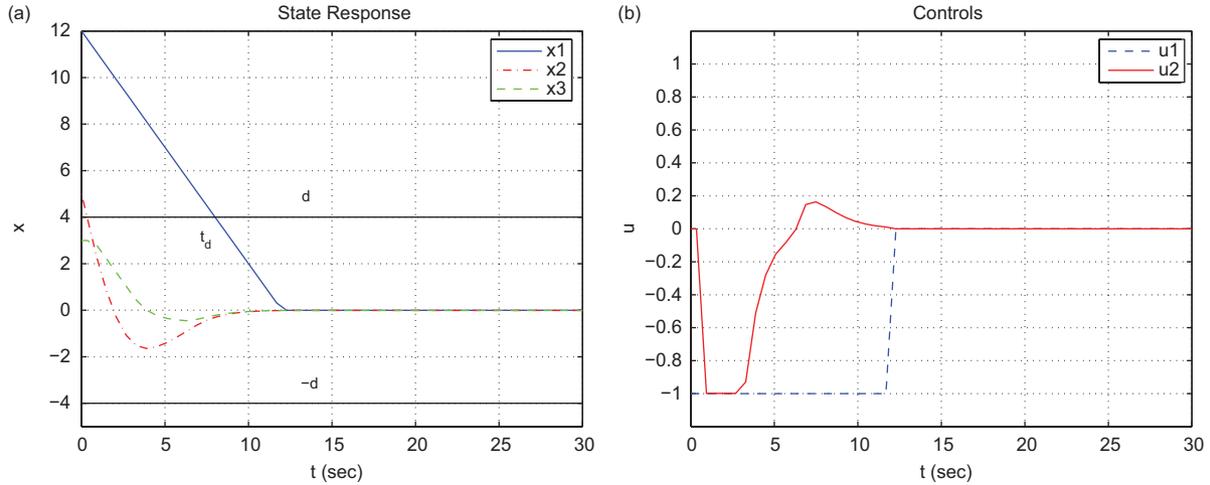


Fig. 2. State and control for the case  $t_1 \leq t_d$ . (a) state; (b) control.

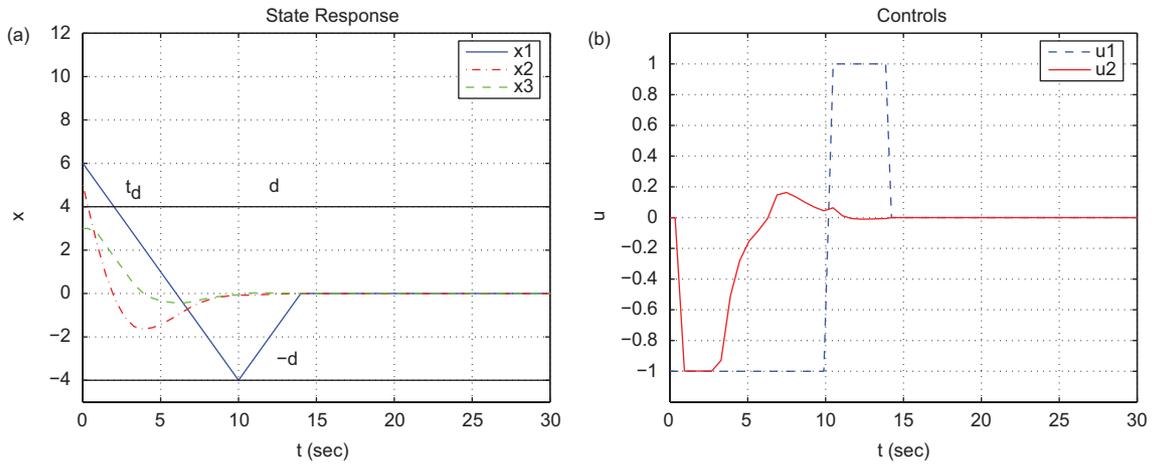


Fig. 3. State and control for the case  $t_1 > t_d$ . (a) state; (b) control.

reaches  $\Omega_2$  later than  $x_1$  gets into the region  $[-d, d]$ . Therefore, the controller thinks the time for maneuvering subsystem (4) is not sufficient. Hence, it steers  $x_1$  cross zero until  $w'$  gets into the linear region  $\Omega_2$  then steers  $x_1$  back to zero. In this case, at  $t_2$ , the states  $x_2$  and  $x_3$  are stopped at  $-0.0082$  and  $0.0077$ , respectively. It is seen that in both cases, the residual errors for  $x_2$  and  $x_3$  are very small as expected.

**Remark 1:** The controls proposed in (11) and (12) can be roughly verified by the daily experience of parking a car. A car is a 4th order nonholonomic system.  $x_1$  is the displacement from the parking position and  $u_1$  relates to its linear velocity. Subsystem (4) is its orientation and  $u_2$  is its angular control. When the car's initial position is far away from the parking position, one usually can drive directly to the parking position. The car's body angle can be aligned without difficulties and no more maneuvers

are needed. However, when the car's initial position is close to the parking position, it might not be feasible to get to the parking position while aligning the car's body angle at the same time. Therefore a straightforward solution would be to slightly get beyond the parking position for aligning the body angle and then back into the parking position.

### 5. Conclusion

In this article, we studied the feedback stabilization problem of chained nonholonomic systems with input constraints, and a switching control design scheme is proposed. The essential idea is that by making  $u_1$  to be piecewise constant, subsystem (4) becomes multiple integrators that have a constant weight  $u_1$ . Then, a state transformation is applied to convert the multiple integrator

into a linear system with an upper triangular system matrix, based on which the saturated control is obtained. Simulation study shows the effectiveness of the proposed control.

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