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Analysis and Control of Two-Layer Frenkel–Kontorova Model *

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A one-dimensional two-layer Frenkel–Kontorova model is studied. Firstly, a feedback tracking control law is given. Then, the boundedness result for the error states of single particles of the model is derived using the Lyapunov Method. Especially, the motion of single particles can be approximated analytically for the case of sufficiently large targeted velocity. Simulations illustrate the accuracy of the derived results.

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Recently, the Frenkel–Kontorova (FK) model, which describes a chain of classical particles interacting with its nearest neighbors and subjected to a periodic one-site potential, has become a useful tool to study nanotribology.[1−6] There are several generalizations of the FK model that have been introduced with the hope of understanding friction dynamics at nanoscale. These models include a many-layer model with harmonic interactions, the Frenkel–Kontorova–Tomlinson model (FKT) and the single-layer model with harmonic interactions.[5−7] References [8,9] formulated the friction control problem and gave control laws based on a single layer FK model. The main disadvantage of the single layer FK model is that the atoms in one of the interfaces are fixed. To overcome this disadvantage, different types of two-layer FK models are studied. In Refs. [5,10], a one-dimensional (1D), two-layer FK-type model with $2N$ atoms is proposed, as shown in Fig. 1. This model displays some key features of frictional dynamics at nanoscale and therefore is used here as a controlling theory. If $x_i + i$ and $y_i + i + 1/2 (1 < i < N)$ are the atomic displacements in the first and second layer, respectively, the equations of motion of the model can be written as

$$\ddot{x}_i + \gamma\dot{x}_i = F_{xi} - b\sin 2\pi x_i + f, \ddot{y}_i + \gamma\dot{y}_i = F_{yi} + f, \quad (1)$$

where

$$F_{xi} = \begin{cases} k_c(x_2 - 3x_1/2 + y_1/2), \ i = 1, \\ k_c(x_{i+1} + x_{i-1} - 3x_i + y_i/2 + y_{i-1}/2), \\ i = 2, \ldots, N-1, \\ k_c(x_{N-1} - 2x_N + y_N/2 + y_{N-1}/2), \ i = N, \end{cases}$$

$$F_{yi} = \begin{cases} k_c(y_2 - 2y_1 + x_1/2 + x_2/2), \ i = 1, \\ k_c(y_{i+1} + y_{i-1} - 3y_i + x_i/2 + x_{i+1}/2), \\ i = 2, \ldots, N-1, \\ k_c(y_{N-1} - 3y_N/2 + x_N/2), \ i = N. \end{cases}$$

The interactions between the particles are assumed to be harmonic with spring constant k_c and the first layer is subjected to a time-averaged sinusoidal force of strength b. Symbol γ is the damping coefficient, f is the external force taken to act on all particles uniformly. The simple two-layer FK model (1) displays some key features of the large-scale simulations and can be used to understand the data from experiment analytically. Hence, in this Letter, the simple two-layer FK model (1) will be subjected to study the friction control problem by using controlling theory.

The control objective is to achieve the expected average velocity, v_0, of the model (1). Let the external force f in (1) be a feedback control, denoted by $u(t)$. Due to physical accessibility constraints, the feedback control $u(t)$ is assumed to be a function of only three measurable quantities, v_0, v_{cm} and x_{cm}, where $x_{cm} = \sum_{i=1}^{N}(x_i + y_i)/(2N)$, $v_{cm} = \sum_{i=1}^{N}(\dot{x}_i + \dot{y}_i)/(2N)$. To design feedback tracking controllers, the tracking error states are defined as $e_{a1} = x_{cm} - v_0 t$, $e_{a2} = v_{cm} - v_0$. Then asymptotic stability of the system in the error state space is equivalent to asymptotic tracking of the targeted positions and constant velocity. The dynamics of $[e_{a1}, e_{a2}]$ can be derived as

$$\dot{e}_{a1} = e_{a2}, \quad \dot{e}_{a2} = -\gamma(e_{a2} + v_0) - \sum_{i=1}^{N} b\sin(2\pi x_{i1})/(2N) + u(t). \quad (2)$$

Since system (2) is similar to the system (26) in Ref. [9]. The result derived in Ref. [9] can be used for system (2). Hence, we give the stability result for system (2) as the following theorem without proof. Readers can see the detailed proof in Ref. [9].

Theorem 1. The states of system (2) are uniformly bounded over time $[0, \infty)$ with the feedback control law

$$u(t) = -\gamma v_0 - k_1 e_{a1} - k_2 e_{a2} + b\sin(2\pi v_0 t)/2, \quad (3)$$

where constants k_1 and k_2 are positive.

Then, the question is whether the dynamics of single particles will also perform well under Eq. (3), i.e., whether the single particle state, $(x_i, \dot{x}_i)$, will convergent to $(v_0 t, v_0)$. Since the control design must account for the whole dynamics (and therefore cannot

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tolerate the instability of single particle's dynamics), the single particle's behavior has to be addressed carefully. Here the focus is to study the dynamics of single particles for the two-layer FK model with the designed controller. For studying the stability of single particles in Eq. (1), the following equations define the error states as $e_{xi1} = x_{i1} - v_0 t$, $e_{xi2} = x_{i2} - v_0$, $e_{yi1} = y_{i1} - v_0 t$ and $e_{yi2} = y_{i2} - v_0$. Let $e = [e_{x11}, e_{x12}, e_{y11}, e_{y12}, \ldots, e_{xN1}, e_{xN2}, e_{yN1}, e_{yN2}]^T$. Then under the control law (3), the corresponding error dynamics of e can be derived as

$$\dot{e} = \Big(I_{2N} \otimes \begin{bmatrix} 0 & 1 \\ 0 & -\gamma \end{bmatrix} + Q \otimes \begin{bmatrix} 0 & 0 \\ k_c & 0 \end{bmatrix} - \frac{1}{2N} J_{2N} \otimes \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix} \Big) e + [g_1^T \ldots g_N^T]^T, \quad (4)$$

where I_{2N} denotes $2N$-by-$2N$ identity matrix, J_{2N} denotes a $2N$-by-$2N$ special matrix whose elements are all 1, $g_i^T = [0 b\sin(2\pi v_0 t)/2 - b\sin(2\pi x_{i1}) 0 b\sin(2\pi v_0 t)/2]$ for $i = 1, \ldots, N$, and

$$Q = \begin{bmatrix} -3/2 & 1/2 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 1/2 & -2 & 1/2 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1/2 & -3 & 1 & 1/2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 1 & 1/2 & -3 & 1 & 1/2 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 1 & 1/2 & -2 & 1/2 \\ 0 & \cdots & 0 & 0 & 0 & 0 & 1 & 1/2 & -3/2 \end{bmatrix} \in \Re^{2N\times 2N}.$$

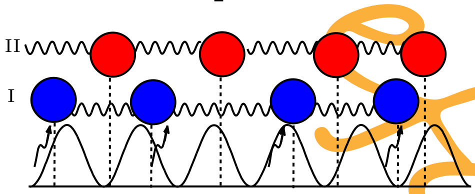

Fig. 1. The geometry of the simplified two-layer FK model.

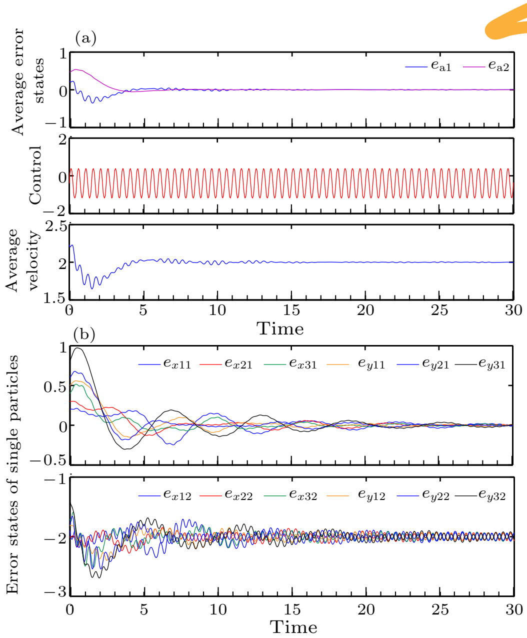

Fig. 2. Tracking performance of the average system for $v_0 = 2$ under control law (3) and its corresponding error states of single particles.

A main difference between the system (4) and Eq. (42) in Ref. [9] is that the sinusoidal term vanishes in the dynamics of upper particles, which makes origin of Eq. (4) no longer an equilibrium point and sufficient conditions for the stability of origin can not be obtained similarly to Ref. [9]. However, the boundedness result for Eq. (4) can be concluded as follows:

Theorem 2. The states of system (4) are globally uniformly ultimately bounded by

$$\|e\| \le 2N\alpha_2/\alpha_1, \quad (5)$$

where $\alpha_1 = \min\{\frac{\gamma}{2}\mu_{2N-1}k_c, \frac{\gamma}{2}k_1, \frac{\gamma}{2} + k_2\}$, $\alpha_2 = Nb\sqrt{\sum_{i=1}^{2N-1}(\gamma+\gamma^2+2\mu_i k_c)^2 + (\gamma+\gamma^2+\gamma k_2+2k_1)^2 + 8N}$, and $\mu_i > 0$ for $i = 1, \ldots, 2N-1$ are eigenvalues of matrix $-Q$ with $\mu_1 \ge \cdots \ge \mu_{2N-1}$, over time $[0, \infty)$.

Proof: Through,[9] we know that there exits a transformation matrix $T = (t_{ij}) \in \Re^{4N\times 4N}$ such that

$$T^T T = I_{4N}, \ T^T G T = \mathrm{diag}\{S_1, \ldots, S_{2N}\}, \quad (6)$$

where $S_i = \begin{bmatrix} 0 & 1 \\ -\mu_i k_c & -\gamma \end{bmatrix}$ for $i = 1, \ldots, 2N-1$, $S_{2N} = \begin{bmatrix} 0 & 1 \\ -k_1 & -\gamma - k_2 \end{bmatrix}$ and I_{4N} denotes $4N$-by-$4N$ identity matrix. Let $\eta = [\eta_{11}\eta_{12}\cdots\eta_{2N,1}\eta_{2N,2}]^T = T^T e$, then the dynamics of η can be presented as:

$$\dot{\eta} = \mathrm{diag}\{S_1, \ldots, S_{2N}\}\eta + [\phi_{11}\ \phi_{12}\ \cdots\ \phi_{2N,1}\ \phi_{2N,2}]^T, \quad (7)$$

where $[\phi_{11}\phi_{12}\cdots\phi_{2N,1}\phi_{2N,2}]^T = T^T[g_1^T \cdots g_N^T]^T$. Utilizing Eq. (6) yields $|t_{ij}| \le 1, \forall i, j$. Hence,

$$|\phi_{ij}| \le \sum_{i=1}^{N} (|(b\sin(2\pi v_0 t))/2 - b\sin(2\pi x_{i1})| + |(b\sin(2\pi v_0 t))/2|) \le 2Nb, \quad (8)$$

where $i = 1, \cdots, 2N$ and $j = 1, 2$. Define the following Lyapunov function $V(\eta) = \sum_{i=1}^{2N}[\rho_i\eta_{i1}^2 + (\frac{\gamma}{2}\eta_{i1} + \eta_{i2})^2]/2$, where $\rho_i = \gamma^2/4 + \mu_i k_c$ for $i = 1, \ldots, 2N-1$ and $\rho_{2N} = \gamma^2/4 + \gamma k_2/2 + k_1$. Using Eq. (8) and taking the time derivative of $V(\eta)$ along system (7) has $\dot{V}(\eta) \le -\alpha_1\|\eta\|^2 + \alpha_2\|\eta\|$. Hence, $\dot{V}(\eta) < 0$, $\forall\|\eta\| > \alpha_2/\alpha_1$, which shows that system (7) is globally uniformly ultimately bounded by $\|\eta\| \le \alpha_2/\alpha_1$.

Due to the similarity transformation $e = T\eta$, we have $\|e\| \leq \|T\|_{m_\infty}\|\eta\| \leq 2N\alpha_2/\alpha_1$, by utilizing compatible property (see Ref. [11], page 327), where $\|\cdot\|_{m_\infty}$ denotes infinity matrix norm. The proof is thus completed.

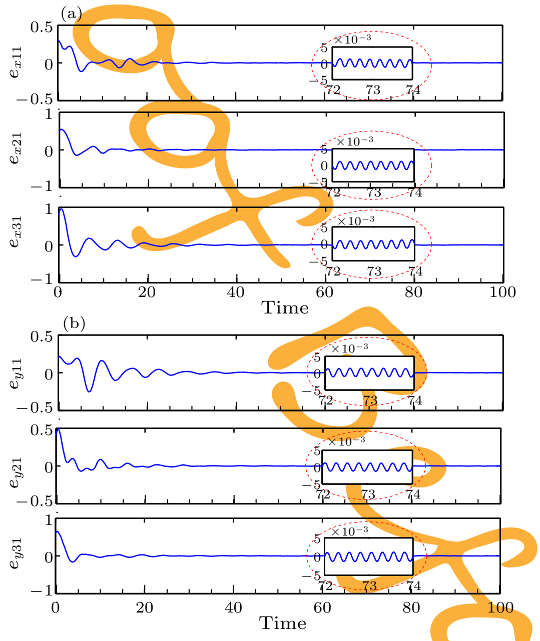

Fig. 3. The performance of error states of single particles for $v_0 = 4$ using control law (3).

Theorem 2 considers a general case for the motion of single particles. For some special cases such as extremely large v_0, the motion of single particles can be described more specifically. The following theorem gives properties of the motion of single particles for the case of extremely large v_0.

Theorem 3 If the targeted velocity v_0 is sufficiently large, the solution of Eq. (4) can be approximated by

$$\begin{aligned} e_{xi1} =& b[2\sin(2\pi(c_{xi} + v_0 t)) - \sin(2\pi v_0 t)]/(8\pi^2 v_0^2) \\ &+ c_{xi}, e_{yi1} = -b\sin(2\pi v_0 t)/(8\pi^2 v_0^2) + c_{yi}, \end{aligned} \tag{9}$$

when time goes to infinity, where c_{xi} (c_{yi}) is decided by the initial condition and satisfies

$$\Big\{\sum_{i=1}^{N}\Big[\Big(\frac{3b}{8\pi^2 v_0^2} + c_{xi}\Big)^2 + \Big(\frac{b}{8\pi^2 v_0^2} + c_{yi}\Big)^2\Big]\Big\}^{1/2} < \frac{2N\alpha_2}{\alpha_1}. \tag{10}$$

Proof: In order to study the motion of single particles for the special case that v_0 is extremely large, Eq. (4) is rewritten as

$$\begin{aligned} \ddot{e}_{xi1} =& -\gamma\dot{e}_{xi1} + F_{exi} - F_u \\ & - b\sin(2\pi(e_{xi1} + v_0 t)) + b\sin(2\pi v_0 t)/2, \\ \ddot{e}_{yi1} =& -\gamma\dot{e}_{yi1} + F_{eyi} - F_u \\ & + b\sin(2\pi v_0 t)/2, \end{aligned} \tag{11}$$

where function F_{exi} (F_{eyi}) for $i = 1, \ldots, N$ has the same form as F_{xi} (F_{yi}) except that variables x_j and y_m for $j, m = 1, \ldots, N$ in F_{xi} (F_{yi}) are replaced by e_{xj1} and e_{ym1}, $F_u = \sum_{i=1}^{N}[k_2(\dot{e}_{xi1} + \dot{e}_{yi1}) + k_1(e_{xi1} + e_{yi1})]/(2N)$. Since e_{xi1} and e_{yi1} are shown at the stay in a finite region, according to theorem 2, e_{xi1} and e_{yi1} converge either to equilibrium or to a stable periodic solution or is chaotic. Clearly the system does not have an equilibrium point. Assume chaos do not appear for this system, then e_{xi1} and e_{yi1} can be presented as

$$\begin{aligned} e_{xi1} =& \sum_{j=1}^{\infty}\Big[a_{xi}\Big(\frac{j}{q_{xi}}\Big)\sin\frac{2\pi j v_0 t}{q_{xi}} + b_{xi}\Big(\frac{j}{q_{xi}}\Big)\cos\frac{2\pi j v_0 t}{q_{xi}}\Big], \\ e_{yi1} =& \sum_{j=1}^{\infty}\Big[a_{yi}\Big(\frac{j}{q_{yi}}\Big)\sin\frac{2\pi j v_0 t}{q_{yi}} + b_{yi}\Big(\frac{j}{q_{yi}}\Big)\cos\frac{2\pi j v_0 t}{q_{yi}}\Big], \end{aligned} \tag{12}$$

where q_{xi} and q_{yi} are positive integers, $a_{xi}(j/q_{xi})$ and $b_{xi}(j/q_{xi})$ ($a_{yi}(j/q_{xi})$ and $b_{yi}(j/q_{xi})$) are coefficients of the terms containing $\sin(2\pi j v_0 t/q_{xi})$ ($\sin(2\pi j v_0 t/q_{yi})$) and $\cos(2\pi j v_0 t/q_{xi})$ ($\cos(2\pi j v_0 t/q_{yi})$) separately; $a_{xi}(j/q_{xi})$, $b_{xi}(j/q_{xi})$, $a_{yi}(j/q_{xi})$ and $b_{yi}(j/q_{xi})$ are limited to be constants due to the boundedness of e_{xi1} and e_{yi1}, according to theorem 2. The term $-b\sin(2\pi(e_{xi1} + v_0 t)) + b\sin(2\pi v_0 t)/2$ is also periodic, hence it can be presented as

$$\begin{aligned} & - b\sin(2\pi(e_{xi1} + v_0 t)) + b\sin(2\pi v_0 t)/2 \\ =& \sum_{j=1}^{\infty}\Big[c_i\Big(\frac{j}{q_i}\Big)\sin(2\pi j v_0 t/q_i) + d_i\Big(\frac{j}{q_i}\Big)\cos(2\pi j v_0 t/q_i)\Big], \end{aligned} \tag{13}$$

where q_i is a positive integer, $c_i(j/q_i)$ and $d_i(j/q_i)$ are coefficients of the terms containing $\sin(2\pi j v_0 t/q_i)$ and $d_i(j/q_i)$, respectively. Substituting Eqs. (12) and (13) into Eq. (11) and utilizing the fact that the terms contain $\sin(2\pi j v_0 t/q_i)$ ($\cos(2\pi j v_0 t/q_i)$) should be separately equal has

$$\begin{aligned} & a_{xi}(j/q_{xi})4\pi^2 j^2 v_0^2/q_{xi}^2 + \gamma b_{xi}(j/q_{xi})2\pi j v_0/q_{xi} \\ =& F_{xsi} - F_{u2} + F_{u1} - c_i(j/q_{xi}), \\ & b_{xi}(j/q_{xi})4\pi^2 j^2 v_0^2/q_{xi}^2 - \gamma a_{xi}(j/q_{xi})2\pi j v_0/q_{xi} \\ =& F_{xci} + F_{u2} + F_{u1} - d_i(j/q_{xi}), \\ & a_{yi}(j/q_{yi})4\pi^2 j^2 v_0^2/q_{yi}^2 + \gamma b_{yi}2\pi j v_0/q_{yi} \\ =& F_{ysi} - F_{u2} + F_{u1} - g_j, \\ & b_{yi}(j/q_{yi})4\pi^2 j^2 v_0^2/q_{yi}^2 - \gamma a_{yi}(j/q_{yi})2\pi j v_0/q_{yi} \\ =& F_{yci} + F_{u1} + F_{u2}, \end{aligned} \tag{14}$$

where functions F_{xsi} and F_{xci} (F_{ysi} and F_{yci}) for $i = 1, \cdots, N$ have the same form as F_{xi} (F_{yi}) except that variables x_n and y_m for $n, m = 1, \cdots, N$ in F_{xsi} and F_{xci} (F_{ysi} and F_{yci}) are replaced by $a_{xn}(j/q_{xi})$ and $b_{xm}(j/q_{xi})$ ($a_{yn}(j/q_{xi})$ and $b_{ym}(j/q_{xi})$), $F_{u1} = \sum_{i=1}^{N}k_1[b_{xi}(j/q_{yi}) + b_{yi}(j/q_{yi})]/(2N)$, $F_{u2} = \sum_{i=1}^{N}k_2[a_{xi}(j/q_{yi})2\pi j v_0/q_{yi} + b_{yi}(j/q_{yi})2\pi j v_0/q_{yi}]/(2N)$, $g_j = b/2$ when $j = q_{yi}$, otherwise, $g_j = 0$. Since $a_{xi}(j/q_{xi})$, $b_{xi}(j/q_{xi})$, $a_{yi}(j/q_{yi})$ and $b_{yi}(j/q_{yi})$ for all i and j are limited, the values of the left side of Eq. (14) are limited. Hence, $a_{xi}(j/q_{xi})$, $b_{xi}(j/q_{xi})$, $a_{yi}(j/q_{yi})$ and $b_{yi}(j/q_{yi})$ for all i and j are small due to sufficiently large v_0. Therefore, e_{xi1} and e_{yi1} for all

i are small. Sufficiently large v_0 corresponds to the situation when the frequency $2\pi v_0$ of the nonlinear term $-b\sin(2\pi(e_{xi1}+v_0t)) + b\sin(2\pi v_0 t)/2$ ($b\sin(2\pi v_0 t)/2$) in Eq. (11) is much larger than the natural frequency of Eq. (11). A well known technique which is historically connected with the problem of the Kapitza pendulum is applied.[12,13] The motion is assumed to traverse a smooth path and at the same time execute small oscillations of frequency $2\pi v_0$ about that path. Accordingly, the function $e_1(t)$ is represented as a sum:

$$e_1(t) = \bar{e}_1(t) + \xi_1(t), \tag{15}$$

where $e_1(t) = [e_{x11}, e_{y11}, \ldots, e_{xN1}, e_{yN1}]^T$, $\bar{e}_1(t) = [\bar{e}_{x11}, \bar{e}_{y11}, \ldots, \bar{e}_{xN1}, \bar{e}_{yN1}]^T$ and $\xi_1(t) = [\xi_{x11}, \xi_{y11}, \ldots, \xi_{xN1}, \xi_{yN1}]^T$. The symbol $\xi_1(t)$ corresponds to these small oscillations. The mean value of the function $\xi_{xi1}(t)$ ($\xi_{yi1}(t)$) over its period $2\pi/v_0$ is zero and the function $\bar{e}_1(t)$ changes slightly in that period of time. Denoting this average by a bar, we have $\bar{\bar{e}}_1(t) = \bar{e}_1(t)$. Substituting Eq. (15) into Eq. (11) and expanding in the powers of $\xi_1(t)$ as far as the first-order terms obtain:

$$\begin{aligned}\ddot{\bar{e}}_{xi1} + \ddot{\xi}_{xi1} =& -\gamma(\dot{\bar{e}}_{xi1} + \dot{\xi}_{xi1}) + F_{exi} + b\sin(2\pi v_0 t)/2\\ &- b\sin(2\pi(\bar{e}_{x11} + v_0 t)) - F'_u\\ &- 2b\pi\cos(2\pi(\bar{e}_{x11} + v_0 t))\xi_{xi1},\\ \ddot{\bar{e}}_{yi1} + \ddot{\xi}_{yi1} =& -\gamma(\dot{\bar{e}}_{yi1} + \dot{\xi}_{yi1}) + F_{eyi} + b\sin(2\pi v_0 t)/2\\ &- F'_u,\end{aligned} \tag{16}$$

where $F'_u = \sum_{i=1}^N[k_1(\bar{e}_{xi1}+\xi_{xi1}+\bar{e}_{yi1}+\xi_{yi1}) + k_2(\dot{\bar{e}}_{xi1} + \dot{\xi}_{xi1} + \dot{\bar{e}}_{yi1} + \dot{\xi}_{yi1})]/(2N)$. Equation (16) involves both oscillatory and 'smooth' terms, which must evidently be separately equal. For the oscillating terms we can put

$$\begin{aligned}\ddot{\xi}_{xi1} &= b[\sin(2\pi v_0 t) - 2\sin(2\pi(\bar{e}_{xi1} + v_0 t))]/2,\\ \ddot{\xi}_{yi1} &= b\sin(2\pi v_0 t)/2.\end{aligned} \tag{17}$$

The other terms contain the small factor $\xi_{xi1}(t)$ ($\xi_{yi1}(t)$) and are therefore of a higher order of smallness (but the derivative $\ddot{\xi}_{xi1}$ ($\ddot{\xi}_{yi1}$) is proportional to the large quantity v_0^2 and so is not small). Integrating Eq. (17) regarding $\bar{e}_i$ as a constant has

$$\begin{aligned}\xi_{xi1} &= b[2\sin(2\pi(\bar{e}_{xi1} + v_0 t)) - \sin(2\pi v_0 t)]/(8\pi^2 v_0^2),\\ \xi_{yi1} &= -b\sin(2\pi v_0 t)/(8\pi^2 v_0^2).\end{aligned} \tag{18}$$

Equation (16) is averaged with respect to time. Since $-\overline{2b\pi\xi_{xi1}\cos(2\pi(\bar{e}_{xi1} + v_0 t))} = -b^2\pi\sin(2\pi\bar{e}_{xi1})/2$ and the mean values of $b\sin(2\pi(\bar{e}_{xi1} + v_0 t)) - b\sin(2\pi v_0 t)/2$ ($-b\sin(2\pi v_0 t)/2$), $\dot{\xi}_{xi1}$ ($\dot{\xi}_{yi1}$) and ξ_{xi1} (ξ_{yi1}) are zeros, the result is

$$\begin{aligned}\ddot{\bar{e}}_{xi1}(t) =& -\gamma\dot{\bar{e}}_{xi1} + F_{\bar{e}xi} - F_{\bar{u}}\\ &- b^2\pi\sin(2\pi\bar{e}_{xi1})/2,\\ \ddot{\bar{e}}_{yi1}(t) =& -\gamma\dot{\bar{e}}_{yi1} + F_{\bar{e}xi} - F_{\bar{u}},\end{aligned} \tag{19}$$

where function $F_{\bar{e}xi}$ ($F_{\bar{e}yi}$) for $i = 1, \ldots, N$ has the same form as F_{exi} (F_{eyi}) except that variables e_{xj} and e_{ym} for $j, m = 1, \ldots, N$ in F_{exi} (F_{eyi}) are replaced by $\bar{e}_{xj1}$ and $\bar{e}_{ym1}$, $F_{\bar{u}} = \sum_{i=1}^N[k_1(\bar{e}_{xi1} + \bar{e}_{yi1}) + k_2(\dot{\bar{e}}_{xi1} + \dot{\bar{e}}_{yi1})]/(2N)$. Considering the following function $U = \sum_{i=1}^N[(\dot{\bar{e}}_{xi1}^2 + \dot{\bar{e}}_{yi1}^2)/2 + b^2(1-\cos 2\pi\bar{e}_{xi1})/4 + k_c(\bar{e}_{xi1} - \bar{e}_{yi1})^2/4] + \sum_{i=1}^{N-1} k_c\{[(\bar{e}_{xi+1,1} - \bar{e}_{xi1})^2 + (\bar{e}_{yi+1,1} - \bar{e}_{yi1})^2]/2 + (\bar{e}_{xi+1,1} - \bar{e}_{yi1})^2/4\}$. The time derivative of U along system (16) is $\dot{U} = -\gamma\sum_{i=1}^N(\dot{\bar{e}}_{xi1}^2 + \dot{\bar{e}}_{yi1}^2)$. Hence, utilizing invariance theorem in Ref. [14] has

$$\dot{\bar{e}}_{xi1} \to 0,\ (\dot{\bar{e}}_{yi1} \to 0),\ \bar{e}_{xi1} \to c_{xi},\ (\bar{e}_{yi1} \to c_{yi}), \tag{20}$$

as time goes to infinity, where constant c_{xi} (c_{yi}) is decided by initial condition. Utilizing Eqs. (15), (18) and (20) yields Eq. (9) when time goes to infinity. Using theorem 2 and Eq. (20), Eq. (10) holds. The proof is thus completed.

Simulations on a particle array of two chains is performed. The system parameters used are $\gamma = 0.1$, $k_c = 0.26$ and $b = 0.5\pi$.[7] Firstly, Fig. 2(a) demonstrates the tracking performances of an average system using control law (3) with random initial conditions and for $v_0 = 2$ and Fig. 2(b) shows the corresponding error states of single particles. From Fig. 2(b), it can be seen that the error states of dynamical system (4) are bounded which is claimed in theorem 2. Figure 3 demonstrates the performances of dynamical system (4) with $N = 3$ and for $v_0 = 4$. The initial condition for (4) is randomly chosen as $e = [0.2, 0.1, 0.3, 0, 0.4, 0.3, 0.5,\ 0.2, 0.6, 0.2, 0.8, 0.6]^T$ and control parameter are $k_1 = 1$ and $k_2 = 1$. The simulation results in Fig. 3 show that the theorem 3 gives a satisfactory approximation for the solution of Eq. (4) with sufficiently large v_0.

In summary, we have studied the 1D two-layer FK model. A control law is given to make the average system have a desired velocity. The main contribution is that the boundedness result for error states of individual particles is obtained by the Lyapunov Method. Especially, if v_0 is sufficiently large, the error states of individual particles can be approximated analytically. Simulation results are presented to illustrate the accuracy of the results obtained. The present results are applicable to various nonlinear systems that can be described by the two-layer FK model.

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