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Brief paper

Novel iterative learning controls for linear discrete-time systems based on a performance index over iterations☆

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Abstract

An optimal iterative learning control (ILC) is proposed to optimize an accumulative quadratic performance index in the iteration domain for the nominal dynamics of linear discrete-time systems. Properties of stability, convergence, robustness, and optimality are investigated and demonstrated. In the case that the system under consideration contains uncertain dynamics, the proposed ILC design can be applied to yield a guaranteed-cost ILC whose solution can be found using the linear matrix inequality (LMI) technique. Simulation examples are included to demonstrate feasibility and effectiveness of the proposed learning controls.
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Keywords: Iterative learning control; Quadratic performance index; Iteration domain; Optimality

1. Introduction

In applications such as automation, manufacturing, and chemical processes, control systems are commanded to repetitively perform the same task. In a typical scenario, the output of the system is to track a given reference signal within a specified time interval, which is repeated trial-by-trial. In these applications, tracking accuracy can be improved by designing an iterative learning control (ILC) in which the control input in the current trial is calculated according to the control input and tracking error in the previous trial(s). In other words, different from standard feedback controls (such as PID control, adaptive control, robust control), an ILC is a functional feedback control. By taking advantage of periodicity in the reference signal, a properly designed ILC can achieve better tracking accuracy in a fixed-length time interval (than those under standard feedback controls) and can learn an unknown periodic function (Qu, 2002). Due to these advantages, ILC design has been and continues to be a useful methodology.

In 1984, Arimoto and his coworkers proposed the first D-type ILC for robot systems (Arimoto, Kawamura, & Miyazaki, 1984). Since then, different ILCs have been proposed (Arimoto, Naniwa, & Suzuki, 1990, 1991; Bien & Huh, 1989; Chen, Gong, & Wen, 1998; Heinzinger, Fenwick, Paden, & Miyazaki, 1989; Jayati & Bard, 2002; Shao, Gao, & Yang, 2003), and they have the common control objective that either $\lim_{k\to\infty} e_k(t) = \lim_{k\to\infty}[y_k(t) - y^*(t)] = 0$ or $\lim_{k\to\infty} u_k(t) = u^*(t)$ for any t in a given discrete-time interval $[0, N]$, where subscript k represents the index of trial numbers, and $y^*(t)$ and $u^*(t)$ are the desired output and control input, respectively. More recently, optimal ILCs are designed using two types of quadratic performance indices. The first type includes those indices defined in the time domain. For instance, the time domain performance index used by Frueh and Phan (2000), Gunnarsson and Norrlof (2001), Norrlof (2002) and Phan and Frueh (1996) is quadratic in both the tracking error and control input error, and it also is the same as that in the design of standard feedback optimal control. The second type of performance indices are those defined in the iteration domain, which is more suitable for ILCs by their nature. The performance index used by Amann, Owens, and Rogers (1996a,b), Kim, Chin, and Lee (2000), Lee, Lee, and Kim (2000) and Sugie and Ono (1991) is quadratic in both tracking

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error and control error between two successive trials. However, their ILC design is to locally minimize the performance index using the gradient method, and optimality of the learning process in the iteration domain is not guaranteed. In this paper, ILC is designed to ensure the optimal performance of the learning process in the iteration domain. To this end, an accumulative performance index over iterations is defined, and optimal ILCs are then designed for linear discrete-time systems.

The rest of the paper is organized as follows. In Section 2, the problem of performance-based ILC design is described, and a new performance index in the iteration domain is introduced. In Section 3, an optimal ILC with respect to the new performance index is derived for the nominal dynamics, and its stability, convergence, and optimality are analyzed. In Section 4, a guaranteed-cost ILC is proposed to achieve robustness for uncertain systems, and its solution is expressed in terms of a linear matrix inequality (LMI). In Section 5, simulation examples are presented. In Section 6, conclusions are drawn.

2. Problem formulation

Consider a linear time-invariant discrete system:

$$\begin{aligned} x(i+1) &= [A+\Delta A]x(i) + [B+\Delta B]u(i) \\ y(i) &= [C+\Delta C]x(i) + [D+\Delta D]u(i), \end{aligned} \tag{1}$$

where $\Omega_N \triangleq \{0, \ldots, N\}$, and $i \in \Omega_N$ is the time index. Vectors $x(i) \in R^n$, $y(i) \in R^m$ and $u(i) \in R^r$ are state, output, and control input, respectively. Matrices A, B, C, and D are known and of appropriate dimension, and they represent the nominal dynamics of the system. Matrices ΔA, ΔB, ΔC and ΔD are uncertain dynamics (if any) of the system, respectively.

For the ease of technical development, systems of linear time-invariant model (1) are considered in the paper. Nonetheless, the learning control designs proposed here are not restricted to this class of systems. For instance, the proposed designs can directly be applied to time-variant systems, and the process is parallel to that presented in Amann et al. (1996a) and Lee et al. (2000). Nonlinear systems could also be handled in a way analogous to the Lyapunov argument presented in Ham, Qu, and Kaloust (2001) except that the optimal solution would require solving a two-point boundary value problem.

Let us first consider the nominal dynamics of system (1). Suppose that its relative degree and input–output controllability index are d and γ, respectively, that is,

$$d \triangleq \begin{cases} 0 & \text{if } D \neq 0 \\ \min\limits_{l} \left\{ l \in \aleph^+ : CA^{l-1}B \neq 0 \right\} & \text{if } D = 0, \end{cases}$$

and

$$\gamma \triangleq \min_i \left\{ \max_k \left\{ i \in \aleph^+ : \operatorname{rank}(H_i) = k \right\} \right\},$$

where $\aleph^+$ is the set of positive integers,

$$\varphi(d) \triangleq \begin{cases} D, & \text{if } d = 0 \\ CA^{d-1}B, & \text{if } d > 0 \end{cases}$$

is a nonzero matrix, and H_i with $i \geq d$ is defined by

$$H_i \triangleq \begin{bmatrix} CA^{i-1}B & CA^{i-2}B & \cdots & CA^dB & \varphi(d) \end{bmatrix}.$$

If $\operatorname{rank}(H_\gamma) = m$, the nominal system is input–output controllable. The proposed learning control design is based on the input–output controllability.

Let $\{y^*(i), i \in \Omega_N\}$ be the known signal representing the desired output sequence for the system to track. Given the knowledge of relative degree d, controllability index γ, and initial condition $x(0)$, output tracking error $e(i) \triangleq y(i) - y^*(i)$ for the nominal dynamics can be expressed as, under any control sequence $\{u(i), i \in \Omega_N\}$,

$$e(i) = \begin{cases} CA^ix(0) - y^*(i) & \text{for } i < d \\ H_i \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(i-d) \end{bmatrix} + CA^ix(0) - y^*(i) & \text{for } i \geq d. \end{cases} \tag{2}$$

It is obvious that, given input–output controllability of the nominal system, error system (2) is also input–output controllable.

In system (2), tracking error $e(i)$ of $i < \gamma$ may not be made zero under any control. In particular, tracking error $e(i)$ with $i < d$ depends solely upon $x(0)$ and $y^*(i)$, and it is not influenced by control $u(j)$ for any $j \leq i$. Thus, in the subsequent design and analysis of learning control, convergence of $e(i)$ is studied for $i \geq \gamma$. To this end, error equation (2) is rewritten into the following batch form:

$$E = GU + \Lambda x(0) - Y^*, \tag{3}$$

where $p \geq 1$ is a design integer chosen by the designer, integer l should be chosen such that $\gamma + lp \leq N < \gamma + (l+1)p$,

$$E \triangleq \begin{bmatrix} e(\gamma) \\ e(\gamma+p) \\ \vdots \\ e(\gamma+lp) \end{bmatrix}, \qquad G \triangleq \begin{bmatrix} G_\gamma \\ G_{\gamma+p} \\ \vdots \\ G_{\gamma+lp} \end{bmatrix},$$

$$U \triangleq \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(N-d) \end{bmatrix},$$

$$\Lambda \triangleq \begin{bmatrix} CA^\gamma \\ CA^{\gamma+p} \\ \vdots \\ CA^{\gamma+lp} \end{bmatrix}, \qquad Y^* \triangleq \begin{bmatrix} y^*(\gamma) \\ y^*(\gamma+p) \\ \vdots \\ y^*(\gamma+lp) \end{bmatrix},$$

and matrix block $G_i \in R^{m \times [(N-d+1)r]}$ is defined by

$$G_i \triangleq \begin{cases} \begin{bmatrix} CA^{i-1}B & CA^{i-2}B & \cdots & CAB & CB & D & 0 \end{bmatrix} \\ \quad \text{if } \gamma \leq i < N \\ \begin{bmatrix} CA^{N-1}B & CA^{N-2}B & \cdots & CAB & CB & D \end{bmatrix} \\ \quad \text{if } i = N. \end{cases} \tag{4}$$

Note that all of row blocks G_i in (4) contain H_γ and have the same rank as H_γ. From one block row to the next in matrix

G, locations of sub-matrix H_γ contained therein are shifted horizontally by pr columns. Thus, it becomes apparent that, upon properly choosing p (e.g., as a sufficient condition, any choice satisfying $p \geq \gamma$), matrix G becomes to be of full rank if matrix H_γ is so (i.e., the nominal system is input–output controllable).

In a manner similar to the derivation of error equation (3), one can show that error dynamics in the batch form and of uncertain system (1) are:

$$E = G'U + \Lambda' x(0) - Y^*, \tag{5}$$

where matrices G' and Λ' are defined in the same way as their counterparts G and Λ except that matrices A, B, C, and D are replaced by $(A + \Delta A)$, $(B + \Delta B)$, $(C + \Delta C)$, and $(D + \Delta D)$, respectively. It is easy to see from the relationship between Eqs. (3) and (2) that their stabilization problems are equivalent. Hence, the subsequent discussions will be in terms of batch model (5).

In a typical application of iterative learning control methodology, dynamical model of the system needs to be re-stated in terms of trial index k, which is usually done under the two assumptions[1] that their dynamics (including uncertainties) are repetitive and that initial conditions at the beginning of the trials are re-set as $x_k(0) \triangleq x_0$. Following this convention, we let E_k and U_k denote the values of E and U during the kth trial, respectively, and then rewrite Eq. (5) into the following iterative error model between any two successive trials:

$$E_{k+1} = E_k + [G + \Delta G]\, V_k, \quad k \in \aleph \tag{6}$$

where $\aleph$ is the set of non-negative integers, $\Delta G \triangleq G' - G$ is the lumped uncertainty, and $V_k \triangleq U_{k+1} - U_k$ is the incremental control input for which an ILC control law is to be designed.[2]

System (6) remains to be in the standard linear form. Accordingly, an optimal ILC law can be designed for the case that $\Delta G = 0$. If $\Delta G \neq 0$, a guaranteed-cost ILC law can be found. Both designs are presented in the subsequent sections based on the performance index

$$J = \lim_{k\to\infty} J^k, \tag{7}$$

where

$$J^k = \sum_{l=0}^{k} J_l, \tag{8}$$

$$J_l = E_l^T \Theta_1 E_l + V_l^T \Theta_2 V_l, \tag{9}$$

and Θ_1 and Θ_2 are the positive definite matrices of proper dimension. It will be shown that, under input–output controllability, the resulting iterative learning controls are static and output feedback. Should system (1) be input-state controllable but not input–output controllable, the proposed designs can still be pursued with the aid of a standard linear state observer to yield observer-based learning controls.

[1] The assumptions can be relaxed by incorporating such methods as robust control methodology, etc. (Qu, 2002).

[2] U_0 can be chosen to be either zero or any other standard feedback law.

3. Design of the optimal ILC

The proposed optimal ILC and its properties are summarized in the following theorem and its corollary.

Theorem 1. *Consider the ILC control*

$$V_k = -(\Theta_2 + G^T P G)^{-1} G^T P E_k \triangleq -K E_k, \tag{10}$$

where P is the solution to the Riccati equation

$$PG(\Theta_2 + G^T P G)^{-1} G^T P - \Theta_1 = 0. \tag{11}$$

Suppose that matrix G is full rank and that $\Delta G = 0$. Then, under control (10), system (6) is both exponentially stabilizing and optimal with respect to performance index (7).

Proof. With $\Delta G = 0$, system (6) reduces to

$$E_{k+1} = E_k + G V_k, \quad k \in \aleph.$$

It follows from matrix G being full rank that the above system with pair $\{I, G\}$ is controllable. Therefore, there is a unique symmetric positive definite matrix solution P to Riccati equation (11) for arbitrary choices of matrices $\Theta_1, \Theta_2 > 0$. Substituting control (10) into the above yields the closed-loop system

$$E_{k+1} = \left[I - G(\Theta_2 + G^T P G)^{-1} G^T P \right] E_k, \tag{12}$$

where I is the identity matrix of appropriate dimension.

Let Lyapunov function be

$$\mathscr{L}_k = E_k^T P E_k, \quad k \in \aleph. \tag{13}$$

It follows from (11) and (12) that

$$\begin{aligned} \mathscr{L}_{k+1} - \mathscr{L}_k &= E_{k+1}^T P E_{k+1} - E_k^T P E_k \\ &= -E_k^T \left[\Theta_1 + K^T \Theta_2 K \right] E_k \\ &< 0. \end{aligned}$$

Hence, the closed-loop system (12) is exponentially stable, and the corresponding value of performance index (7) is

$$\begin{aligned} J^* &= \sum_{k=0}^{\infty} \left[E_k^T \Theta_1 E_k + V_k \Theta_2 V_k \right] \\ &= -\sum_{k=0}^{\infty} \left[\mathscr{L}_{k+1} - \mathscr{L}_k \right] = E_0^T P E_0. \end{aligned}$$

To demonstrate optimality of control (10), consider any other ILC law $V_k' \neq V_k$ under which the closed-loop system is

$$E_{k+1} = E_k + G V_k',$$

along which the corresponding change in Lyapunov function (13) is

$$\begin{aligned} \mathscr{L}_{k+1} - \mathscr{L}_k &= (E_k + G V_k')^{\mathrm{T}} P (E_k + G V_k') - E_k^T P E_k \\ &= -\left[E_k^T \Theta_1 E_k + V_k'^T \Theta_2 V_k' \right] \\ &\quad + (V_k' - V_k)^{\mathrm{T}} G^T P G (V_k' - V_k) + V_k'^T \Theta_2 V_k' - V_k^T \Theta_2 V_k \\ &\quad + 2(V_k' - V_k)^{\mathrm{T}} G^T P (I + GK) E_k. \end{aligned}$$

It follows from (11) that

$$(V_k' - V_k)^{\mathrm{T}} G^T P(I + GK)E_k = -V_k'^T \Theta_2 V_k + V_k^T \Theta_2 V_k.$$

Hence, the performance index value under control V_k' is

$$J' = \sum_{k=0}^{\infty} \left[E_k^T \Theta_1 E_k + V_k' \Theta_2 V_k' \right]$$
$$= J^* + \sum_{k=0}^{\infty} (V_k' - V_k)^{\mathrm{T}} (G^T PG + \Theta_2)(V_k' - V_k) > J^*,$$

from which the optimality is concluded. □

It follows from closed-loop system (12) that error state E_k converges exponentially as

$$\|E_{k+1}\| \leq \lambda \|E_k\|,$$

where $0 \leq \lambda < 1$ is defined in Euclidean norm and by

$$\lambda \triangleq \|I - G(\Theta_2 + G^T PG)^{-1} G^T P\| = \|I - P^{-1}\Theta_1\|.$$

Through the choices of Θ_1 and Θ_2, the convergence rate λ can be optimized according to its definition above and Eq. (11). This is important because λ provides a measure of robustness as shown in the following corollary.

Corollary 1. *Suppose that pair $\{I, G\}$ is controllable. Then, under iterative learning control (10), uncertain system (6) is exponentially stable provided that the lumped uncertainty ΔG is bounded from above as*

$$\|\Delta G\| < \frac{1 - \lambda}{\|(\Theta_2 + G^T PG)^{-1} G^T P\|}. \tag{14}$$

Proof. In the presence of uncertainty ΔG, the closed-loop system is

$$E_{k+1} = [I - (G + \Delta G)(\Theta_2 + G^T PG)^{-1} G^T P]E_k.$$

Thus, it follows from (14) that

$$\begin{aligned}\|E_{k+1}\| &\leq \|[I - G(\Theta_2 + G^T PG)^{-1} G^T P]E_k\| \\ &\quad + \|\Delta G(\Theta_2 + G^T PG)^{-1} G^T PE_k\| \\ &\leq \lambda \|E_k\| + \|\Delta G\| \cdot \|(\Theta_2 + G^T PG)^{-1} G^T P\| \cdot \|E_k\| \\ &< \|E_k\|,\end{aligned}$$

from which stability can be concluded. □

Compared with several of existing ILC designs, the proposed optimal design renders better performance. For example, consider the ILC design in Amann et al. (1996a) and Lee et al. (2000) which is based on the gradient method. That design locally minimizes performance index J_k in (9) and, when applied to system (6) with $\Delta G = 0$, yields the following learning control:

$$V_k'' = -(\Theta_2 + G^T \Theta_1 G)^{-1} G^T \Theta_1 E_k. \tag{15}$$

Clearly, performance index J in (7) is common to this gradient-based design and our proposed design. However, control (15) is not optimal with respect to the performance index (7) unless Riccati Eq. (11) has the specific solution of $P = \Theta_1$, which is unlikely for most systems. Hence, by Theorem 1, we know that the optimal control (10) has better performance than that of control (15).

4. Design of a guaranteed-cost ILC

In the presence of uncertainty $\Delta G \neq 0$, ILC control (10) is shown in Corollary 1 to be robust. However, control (10) has two shortcomings. First, for system (6), it no longer ensures certain level of performance as measured by performance index (7). Second, condition (14) is quite conservative for most systems in general. Consequently, it is both interesting and useful to design a guaranteed-cost ILC which is defined by the following definition.

Definition 1. An iterative learning control V_k is to be guaranteed-cost for system (6) if, for a class of admissible class of uncertainties ΔG and along all possible trajectories of the system, the value of performance index (7) is upper bounded as $J \leq E_0^T PE_0$, where P is a positive definite symmetric matrix.

As stipulated in the above definition, guaranteed-cost ILC is designed for some fixed-matrix P and a class of admissible uncertainties. Parallel to many robust control results (Zhou, Doyle, & Glover, 1996), the following assumption on uncertainties is introduced.

Assumption 1. *Uncertainty ΔG in system (6) is norm bounded in the time domain, that is,*

$$\Delta G = L\Xi M, \tag{16}$$

where L and M are known constant matrices of appropriate dimensions, and Ξ is a matrix of independent and unstructured uncertainties satisfying the inequality of $\Xi^T \Xi \leq I$.

In the following theorem, the proposed guaranteed-cost ILC design is shown to be a problem of solving a linear matrix inequality (LMI). Proof of the theorem requires a technical lemma adopted from Xie (1996) and Schmitendorf and Stalford (1997).

Lemma 1 (*Xie, 1996*). *Consider the matrix inequality*

$$F + H\Xi S + S^T \Xi^T H^T \leq 0, \tag{17}$$

where F is a symmetric matrix, matrices H, Ξ, and S are of appropriate dimensions. Then, the above inequality holds for all choices of Ξ satisfying $\Xi^T \Xi \leq I$ if and only if there exists a scalar $\varepsilon > 0$ such that

$$F + \varepsilon HH^T + \varepsilon^{-1} S^T S \leq 0.$$

Theorem 2. *Consider system (6) under Assumption 1. Then, $V_k = -\Gamma PE_k$ is a guaranteed-cost ILC if and only if there exist scalar $\varepsilon > 0$ and gain matrix Γ such that, for a (given) positive definite choice of performance matrix P, matrix inequality*

$$\begin{bmatrix} \varepsilon LL^T - P^{-1} & P^{-1} - G\Gamma & 0 & 0 & 0 \\ P^{-T} - \Gamma^T G^T & -P^{-1} & -\Gamma^T M^T & P^{-1} & -\Gamma^T \\ 0 & -M\Gamma & -\varepsilon I & 0 & 0 \\ 0 & P^{-1} & 0 & -\Theta_1^{-1} & 0 \\ 0 & -\Gamma & 0 & 0 & -\Theta_2^{-1} \end{bmatrix} \leq 0 \tag{18}$$

holds.

Proof. To show that $V_k = -\Gamma P E_k$ is a guaranteed-cost ILC, consider Lyapunov function $\mathscr{L}_k = E_k^T P E_k$. It follows from system (6) and performance index (7) that the cost inequality

$$\begin{aligned} E_0^T P E_0 &= \sum_{k=0}^{\infty}\left[\mathscr{L}_{k+1} - \mathscr{L}_k\right] \\ &= \sum_{k=0}^{\infty} E_k^T \Big[(I - G\Gamma P - \Delta G\Gamma P)^{\mathrm{T}} \\ &\quad \times P(I - G\Gamma P - \Delta G\Gamma P) - P\Big] E_k \\ &\geq \sum_{k=0}^{\infty} E_k^T \left[\Theta_1 + P\Gamma^T \Theta_2 \Gamma P\right] E_k = J \end{aligned}$$

is ensured if and only if the following matrix inequality holds for the class of uncertainties ΔG:

$$(I - G\Gamma P - \Delta G\Gamma P)^{\mathrm{T}} P (I - G\Gamma P - \Delta G\Gamma P) + PRP \leq 0, \tag{19}$$

where $R \triangleq -P^{-1} + P^{-1}\Theta_1 P^{-1} + \Gamma^T \Theta_2 \Gamma$. It follows from Schur complement (Boyd, Ghaoui, Feron, & Balakrishnan, 1994) that inequality (19) can be expressed in a matrix form as

$$\begin{bmatrix} -P^{-1} & I - G\Gamma P - \Delta G\Gamma P \\ (I - G\Gamma P - \Delta G\Gamma P)^{\mathrm{T}} & PRP \end{bmatrix} \leq 0.$$

Upon first multiplying both sides by positive definite symmetric matrix $\mathrm{diag}\{I, \quad P^{-1}\}$ and then invoking Eq. (16) of Assumption 1, the above matrix inequality can be rewritten into (17), where $S = [0 \ -M\Gamma]$,

$$F = \begin{bmatrix} -P^{-1} & P^{-1}I - G\Gamma \\ (P^{-1} - G\Gamma)^{\mathrm{T}} & R \end{bmatrix}, \quad \text{and} \quad H = \begin{bmatrix} L \\ 0 \end{bmatrix}.$$

Hence, we know from Lemma 1 that inequality (19) is equivalent to

$$\begin{bmatrix} \varepsilon L L^T - P^{-1} & P^{-1} - G\Gamma \\ (P^{-1} - G\Gamma)^{\mathrm{T}} & R + \varepsilon^{-1}\Gamma^T M^T M \Gamma \end{bmatrix} \leq 0.$$

Schur complement is again used to convert the above inequality into inequality (18), which completes the proof. □

Usually, there are maybe more than one solutions for LMI (18). And how to get the optimal one is investigated in the following corollary.

Corollary 2. *If there is a solution* $(\varepsilon^*, P^*, \Gamma^*)$ *for the optimization problem:*

$$\begin{cases} \min\limits_{\varepsilon, P, \Gamma} \{\mathrm{trace}(P)\} \\ s.t.\ (18) \text{ and } \varepsilon > 0, \end{cases} \tag{20}$$

$V_k^* = -\Gamma^* P^* E_k$ *is, in statistics, the optimal guaranteed-cost ILC for system* (6) *with performance index* (7).

Proof. According to Theorem 2, the corresponding value of index (7) for system (6) under guaranteed-cost ILC $V_k = -\Gamma P E_k$ satisfies $J < E_0^T P E_0$.

For E_0 is the output error in the first trial, which varies according to the control input U_0 (selected subjectively in the first trial). In statistics, assuming that E_0 be a zero mean random vector with expectation $\mathbf{E}\{E_0 E_0^T\} = I$. Then, the expectation of index (7) satisfies

$$\mathbf{E}\{J\} < \mathbf{E}\{E_0^T P E_0\} = \mathrm{trace}(P).$$

Therefore, if solution $(\varepsilon^*, P^*, \Gamma^*)$ is obtained for problem (20), $\mathbf{E}\{E_0^T P^* E_0\} \leq \mathbf{E}\{E_0^T P E_0\}$ is found for any $P \neq P^*$. That is $V_k^* = -\Gamma^* P^* E_k$ is, in statistics, the optimal guaranteed-cost ILC for system (6). □

5. Simulations

To demonstrate effectiveness of the proposed ILCs, consider an uncertain system which is of form (1) with $N = 20$, whose nominal dynamics are characterized by matrices

$$A = \begin{bmatrix} 1 & -0.3 & 0.2 \\ 0 & 1 & 0.2 \\ 0.7 & 0 & 0.6 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \\ 0.5 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \qquad D = 0,$$

and whose uncertainties are defined by

$$\Delta A = 0, \qquad \Delta B = \delta_1 B, \qquad \Delta C = \delta_2 C, \qquad \Delta D = 0,$$

where δ_1 and δ_2 are unstructured and independent uncertain elements bounded as $|\delta_1| \leq \bar{\delta_1}$, $|\delta_2| \leq \bar{\delta_2}$.

It is straightforward to verify that the nominal dynamics of the above system are input–output controllable and that the corresponding controllability index and relative degree are $\gamma = 2$ and $d = 1$, respectively. It follows from (4) and (5) that, for any $j \leq i$,

$$\Delta G_{ij} \triangleq G'_{ij} - G_{ij} = \bar{\delta}\, \delta\, C A^{i-j} B,$$

where $\bar{\delta} = \bar{\delta_1} + \bar{\delta_2} + \bar{\delta_1}\bar{\delta_2}$, $\delta = (\delta_1 + \delta_2 + \delta_1\delta_2)/\bar{\delta}$. And $G_{ij} \triangleq 0$, for any $j > i$. Hence, the lumped uncertainty ΔG is of the form (16), where $L = \bar{\delta} I$, $\quad \Xi = \delta I$, and $M = \left[M_{ij}\right]$ with $M_{ij} = C A^{i-j} B$ if $j \leq i$ and $M_{ij} = 0$ if $j > i$.

In the simulations, the following choices are made: weighting parameters are $\Theta_1 = 3I$ and $\Theta_2 = I$; upper bounds on the uncertain elements are $\bar{\delta_1} = \bar{\delta_2} = 0.20$; the "uncertain" elements are chosen in the simulation to be $\delta_1 = 0.20$ and $\delta_2 = 0.15$; the desired output to track is

$$y^*(i) = [y_1^*(i), y_2^*(i)]^{\mathrm{T}},$$

where

$$y_1^*(i) = 5\sin(i/3) + i^2/50, \qquad y_2^*(i) = 15 + i^2/30;$$

and the initial condition of the state is

$$x_k(0) \triangleq [5 \ \ 0 \ \ -5]^{\mathrm{T}}, \quad k = 0, 1, 2, \ldots.$$

In what follows, ILCs (10) and (15) are simulated for the case that the nominal dynamics are considered, and they are compared to demonstrate the superior performance of ILC (10). For the uncertain system, the guaranteed-cost ILC is simulated to demonstrate its effectiveness.

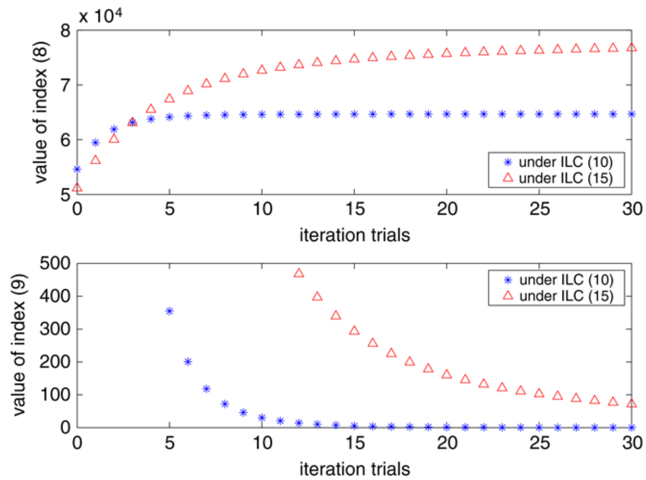

Fig. 1. Value of indices (8) and (9) for the nominal dynamics.

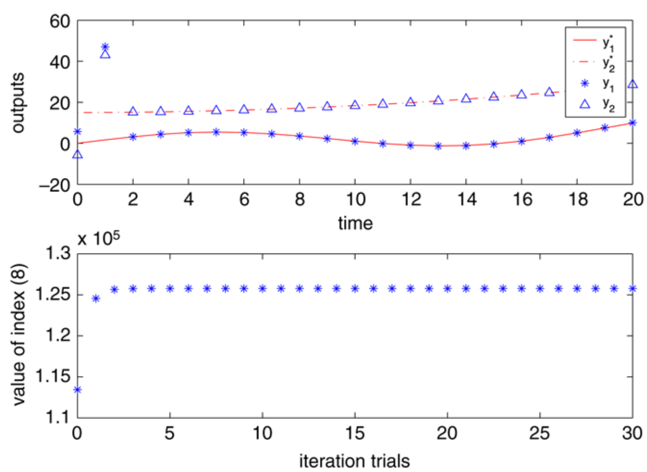

Fig. 2. Outputs at fifth trial and index (8) for uncertain system (1).

Simulation in the absence of unknown dynamics: In this case, only the nominal dynamics are considered, they are defined by matrices A, B, C, D given above. The corresponding values of indices (8) and (9) of the nominal system with respect to the iteration number are shown in Fig. 1. It is apparent that, after a few trials, ILC (10) has better performance (as its index value is smaller and converges faster than that under ILC (15)).

Simulation in the presence of unknown dynamics: In this case, we consider system (1) whose uncertainties satisfy the size bounds $\bar{\delta_1} = \bar{\delta_2} = 0.20$. It follows from Theorem 2 and Corollary 2 that the optimal guaranteed-cost ILC $V_k = -\Gamma^* P^* E_k$ can be solved using $\varepsilon = \varepsilon^* = 9.1376 \times 10^{-5}$, where the matrix solutions $\Gamma^* \in \Re^{40\times 38}$ and $P^* \in \Re^{38\times 38}$ are omitted here for briefness. The outputs during the fifth trial and the corresponding values of performance index (8) under the guaranteed-cost ILC are showed in Fig. 2, which demonstrates its effectiveness.

6. Conclusions

In this paper, a novel design of ILCs is presented to optimize an accumulative quadratic performance index over iterations. Without considering uncertain dynamics, the proposed optimal ILC is shown to have better performance than those under existing designs (Amann et al., 1996a; Lee et al., 2000). For uncertain systems, the proposed ILC design can be extended to yield a guaranteed-cost ILC. Superior performance of the proposed ILC design is demonstrated by simulation examples.

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