Cooperative control strategy for multiple photovoltaic generators in distribution networks

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Abstract: A cooperative control strategy was provided to regulate the active and reactive power outputs of multiple photovoltaic (PV) generators installed on a distribution network. The proposed control strategy not only makes a group of PVs converge and operate at the same ratio of available power, but also regulates the total active and reactive power outputs of the PVs such that the active power across a concerned line and the voltage of a critical bus are kept to a referenced value. The stability of the closed-loop dynamical system was analysed by considering some special properties of classical distribution networks, and the minimal requirement of the communication topology among the PVs was provided. Simulations on a radial distribution power system network were provided to verify the validness of the proposed control strategy.

1 Introduction

The world is approaching peak oil and the ability to produce high-quality, inexpensive and economically extractable oil on demand is diminishing. Therefore in recent years, there have been an increasing amount of photovoltaics (PVs) integrated into the modern distribution networks due to their clean and renewable features, and the connection of a large amount of PVs will have far reaching consequences in the distribution network [1, 2].

As we know, if a distribution network with many PVs that constitute a high level of penetration is considered, the intermittence of PVs' energy will result in some problems, such as voltage fluctuation, frequency fluctuation and even the voltage collapse [3]. Thus, it is necessary to control and regulate the outputs of those PV units [3, 4] in accordance with the requirement of distribution network. For example, we should let those PVs provide some ancillary services such as voltage and frequency regulation, which is very useful for the distribution network to provide the reliable power of high quality. However, it is very difficult to control the outputs of the distributed PVs that may eventually be thousands in number within a single distribution network [5, 6].

Usually, there are three types of control modes to manage the PVs' outputs: the centralised mode, the decentralised mode and the distributed mode. In the centralised mode, a central controller sends its command, which is deliberately calculated online or offline, to the PVs through the designed communication channels. The methods based on the optimal power flow (OPF) strategy [7–9] are of the centralised mode, which were successfully applied in distribution networks that have several distributed generators (DGs) [10, 11]. This mode was also successfully used in some micro grids [12] in which the number of DGs in those micro grids is not large. However, this mode needs to collect system-wide information and sends command globally. For a power system whose distribution networks have numerous and geographically dispersed PVs, such centralised control mode is too expensive to be implemented and, even if it could be accomplished, the resulting system is not robust or efficient due to the long distance communication.

The decentralised mode is another choice. The control strategies based on this mode include the constant PQ of operation, the maximum photovoltaic power tracking, the constant voltage and frequency ($V–F$) with droop control [13–15] and so on. The decentralised mode is robust since all used information is from local measurement. However, for many PVs with this control mode, it is difficult to guarantee their appropriate operating point under varying factors (such as the load changes) since those PVs cannot change their outputs in accordance with the requirement of the distribution network. Thus, the PVs which use the decentralised control cannot easily provide appropriate ancillary service (e.g. to maintain the voltage profile). Consequently, when the number of PVs becomes too large and their power outputs are intermittent, some necessary tasks such as the power balance and voltage maintenance will have to be done by traditional generators, which lead to very expensive cost but ineffective results.

The third mode to dispatch a large amount of PVs is the distributed control mode. It can use local communication networks and combine the positive features of both centralised and decentralised controls while limiting their disadvantages [16–19]. That is to say, a PV will incorporate the information from neighbouring units into its control strategy. This type of control is actually a network

control [17, 20, 21], which has been systematically studied in the field of cooperative robotic control for many years [17, 18]. The distinctive feature of an appropriately designed network control is that it allows local and changing communication networks, is robust with respect to intermittency and latency of its feedbacks and also tolerates connection and disconnection of network components.

In fact, as the level of PV penetration increases, intermittent changes of these PV outputs become too many to consider, and hence it becomes intractable for either the decentralised or centralised mode to adequately manage a distribution network. Thus, given the advances in modern communication, the distributed control mode is a practical way to implement and also necessary to accommodate various changes of PVs in a distribution network. For this reason, a distributed mode-based control strategy is provided to regulate multiple PVs' power outputs in a distribution network [6]. In that control strategy, the cooperative control theory is used to make all PVs converge to a uniform output ratio autonomously. The simulation shows good feasibility, but the strict theories are not provided. This shortcoming will be overcome in this paper, and the sufficient conditions under which the method is valid will be provided with considering several characteristics in classical distribution power systems. The minimal requirement of the communication networks under which the control strategy is valid will also be presented. The proposed control strategy has strong robustness to the communication networks and to the time-varying operating point in the distribution network. A radial distribution system with several PVs is used to verify the proposed control strategy.

This paper is organised as follows. Section 2 gives the problem formulation. Section 3 gives the control strategies and Section 4 discuss the requirement of the communication networks among the PVs. Section 5 provides the sufficient conditions under which the suggested cooperative control strategy is valid by considering several characteristics of classical distribution power networks. Simulations and conclusions are provided in Sections 6 and 7, respectively.

2 Problem formulation

2.1 Dynamical model

Consider a distribution power system with n three-phase inverter-based PVs, which use the decoupled d–q control method via phase locked loops. The dynamical model on the d-axis is shown in Fig. 1 [6, 22], and the model in the q-axis is similar.

Under the d–q frame, the terminal voltage of the ith PV generator satisfies $U_{di} = U_i$ and $U_{qi} = 0$, where U_i denotes the magnitude of the terminal voltage, so the power of the ith PV generator can be expressed by
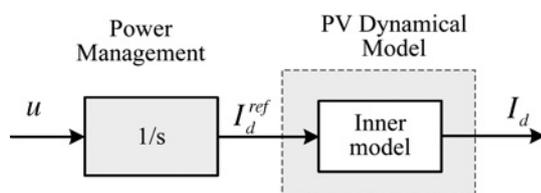
$$P_i = U_i I_{di}, \quad Q_i = -U_i I_{qi} \tag{1}$$

Fig. 1 *Simplified diagram for a PV generator*

where the subscript 'i' denotes the ith PV; I_{di} and I_{qi} are the output currents in the d-axis and q-axis, respectively; P_i and Q_i are outputs of the active and reactive power.

The dynamics of the distribution system can be denoted by the following differential–algebraic equations [6]

$$\dot{I}_{di}^{\text{ref}} = u_{1i} \tag{2}$$

$$\dot{I}_{qi}^{\text{ref}} = u_{2i} \tag{3}$$

$$P_i = U_i I_{di}^{\text{ref}}, \quad Q_i = -U_i I_{qi}^{\text{ref}} \tag{4}$$

$$0 = g(P_1, \ldots, P_n, Q_1, \ldots, Q_n, \chi, \mathbf{X}) \tag{5}$$

where (2) and (3) denote the d-loop and q-loop dynamics, which will be used to control the active and reactive power outputs, respectively; u_{1i} and u_{2i} are the input to be designed; χ is a vector of appropriate dimensions, which denotes the internal dynamics of the distribution system such as the state variables of the PV units, loads and synchronised generators; $\mathbf{X}$ is a vector which denotes the algebraic variables in the distribution network such as the voltage of buses and so on; equation (5) denotes the power flow equation of the distribution system.

It should be noted that the dynamics of χ and the inner dynamics of the PVs are not considered in system (2)–(5). Because we are only interested in the PVs' power outputs, and the dynamics of the PVs' outputs are much slower than the dynamics of other variables (e.g. the internal state variables in the PVs and the states in distribution network are diminishing much faster compared to the dynamics of the power outputs, and the dynamics of the outputs are determined by our controller), thus the dynamics of χ and the inner dynamics of the PVs can be ignored. Therefore the dynamical equations in accordance with those fast variables can be represented by the algebraic equations for simplicity. That is to say, $I_{di} = I_{di}^{\text{ref}}$ and $I_{qi} = I_{qi}^{\text{ref}}$ are satisfied and the distribution power system network can be represented by the power flow equation, as shown by the algebraic (4) and (5). The idea of simplification is similar to that in the problems of automatic generator control [23].

2.2 Problems to be solved

We are interested in regulating the output of those PVs such that some service can be provided for the distribution network. In fact, those n PVs can be considered to be a group, and their total power outputs can be controlled according to the requirement of the distribution network. As was pointed out previously, for a distribution network with many PVs, it is difficult to guarantee feasible operating conditions under varying factors, if all of the PV units run independently with traditional decentralised controls. On the other hand, the centralised mode is neither practical nor reliable, since it requires global information collection and exchange. Thus, a more practical control strategy based on the cooperative control theories will be provided to solve this problem, which makes all PVs in a designated group run at the same active and reactive power output ratios. One of the problems to be solved can be stated as follows.

Problem 1: For the system given by (2)–(5), design the controls for u_{1i} and u_{2i} ($i = 1, 2, \ldots, n$) such that at the

equilibrium points there are

$$\frac{P_1}{P_{1\,\text{max}}} = \frac{P_2}{P_{2\,\text{max}}} = \cdots = \frac{P_n}{P_{n\,\text{max}}} = \alpha_P^0 \qquad (6)$$

and

$$\frac{Q_1}{Q_{1\,\text{max}}} = \frac{Q_2}{Q_{2\,\text{max}}} = \cdots = \frac{Q_n}{Q_{n\,\text{max}}} = \alpha_Q^0 \qquad (7)$$

where $\alpha^0 = [\alpha_P^0, \alpha_Q^0]^{\text{T}}$ is the given output ratio for the active and reactive power for all PV generators; $P_{i\text{max}}$ and $Q_{i\text{max}}$ are the maximums of P_i and Q_i, respectively; all $P_{i\text{max}}$ have the same sign and so do all $Q_{i\text{max}}$.

It should be noted that in Problem 1, different weights can be considered for some PVs by multiplying their maximums with the weight coefficients in (6) and (7). In fact, in a distribution network, different weights can be considered for different services. For example, if a PV unit is far away from the concerned bus (which will be discussed in Problem 2), the weight can be chosen to be smaller when the ancillary service is the voltage regulation. Without loss of generality, the same weight will be considered for simplicity. The uniform output ratio can consider the fair utilisation for all PVs, which makes the fair ancillary duty in the future grid. Readers can refer to [6, 24] for more engineering background on Problem 1. In this paper, we will pay more attention to the basic theories for the validness of the method.

For many PVs, the distributed control mode is more practical because it only use local communication networks and can combine the positive features of both centralised and decentralised controls while limiting their disadvantages. The structure of the distributed control is shown in Fig. 2, in which the dash arrows show the information flow among PVs. The basic idea is that each PV can share its information with some others. Thus, the cooperative control strategy for each PV should be of the general form

$$u_i = w_i(s_{i0}\mathbf{y}_0, s_{i1}\mathbf{y}_1, s_{i2}\mathbf{y}_2, \ldots, s_{in}\mathbf{y}_n), \quad i = 1, 2, \ldots, n \quad (8)$$

where $\mathbf{y}_0$ denotes the output of the high level control; $\mathbf{y}_i$, $i = 1, 2, \ldots, n$, denotes the output variable of ith PV generator; $S = (s_{ij})$ is a time-variant matrix denoting the communication topology, defined as

$$S = \begin{bmatrix} s_{10}(t) & s_{11} & & \cdots & s_{1n}(t) \\ s_{20}(t) & s_{21}(t) & s_{22} & \cdots & s_{2n}(t) \\ \vdots & \vdots & & \ddots & \\ s_{n0}(t) & s_{n1}(t) & s_{n2}(t) & \cdots & s_{nn} \end{bmatrix} \in R^{n \times (n+1)} \quad (9)$$

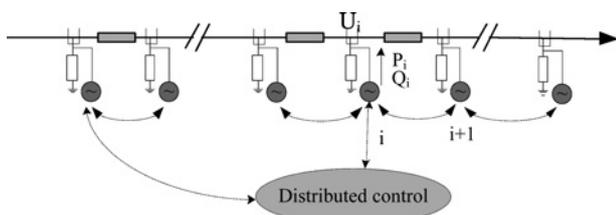where $s_{ii}(t) \equiv 1$ is satisfied for all i; $s_{ij} = 1$ if the output of the

Fig. 2 *Distributed control mode in the distribution network (dash arrows denote the information flow)*

jth PV generator is known to the ith PV generator at time t, and $s_{ij}(t) = 0$ otherwise; $s_{i0}(t) = 1$ if the ith PV can get information from the high level control (or remote control) and $s_{i0}(t) = 0$ otherwise.

It should be noted that in (9) $s_{ii}(t) \equiv 1$ will always be satisfied for each PV, which means that a PV can acquire its own information. Whether other PVs' information are used or not is completely determined by a non-zero entry in the communication matrix. In general, only a part of the neighbouring information is necessary to ensure convergence. In addition, the communication matrix is considered to be time-varying in general, not a matrix of constants. It is necessary to take this into consideration since communication equipment may malfunction or some PVs could be out of service due to environmental reasons [6]. This means that the communication matrix is piecewise continuous. Specifically, let

$$t_{\infty:0} \triangleq \{t_0, t_1, t_2, \ldots\}, S_{\infty:0} \triangleq \{S(t_0), S(t_1), \ldots\} \quad (10)$$

which means matrix S changes at time t_i $(t_0 = 0)$, that is, $S(t) = S(t_k) \in S_{\infty:0}$ for $t \in [t_k, t_{k+1})$.

Clearly, it is much difficult to handle a time-varying communication topology since the closed-loop dynamical system will become a switched dynamical system. A switchable system among several stable systems can be unstable if the switch law is not appropriate; on the other hand, a switchable system among several unstable systems can be stable if some switch laws are used. Some properties on the stability of switchable systems can be found in [19, 25] and the references therein. The sufficient condition for designing a valid communication topology which keeps the stability will be discussed later.

Once Problem 1 is solved, then those PVs can be considered as a virtual generator with a larger capacity. Now, for each virtual generator, only the operating ratios need to be decided, resulting in a much simpler way to design a high-level control for the ancillary services problems, especially for large numbers of PVs. In this paper, the distributed control to be designed will satisfy that the reactive power is controlled to support the voltage of some critical bus and keep the active power consumed by loads in a concerned area or feeder to be constant. This problem can be stated as

Problem 2: Based on the solutions of Problem 1, design an additional control for (2)–(5) such that the voltage of a critical bus can be kept constant. Similarly, control the active power flow across some transmission line to be constant. Namely, at the equilibrium we have

$$P_{\text{tran}}(\cdot) = P^{\text{ref}} \qquad (11)$$

$$V_c(\cdot) = V^{\text{ref}} \qquad (12)$$

where P^{ref} and V^{ref} are the given constants; $V_c(\cdot)$ and $P_{\text{tran}}(\cdot)$ denote the voltage of the concerned bus and the active power across the concerned line (these variables are functions of the PVs' output, so $(\cdot)$ is used to denote these relationships).

Upon the solutions to Problems 1 and 2 are found, all the PVs are organised into a large group, and within the group, the PV generators are controlled to satisfy the given utilisation profile. Accordingly, the group of the PVs can be viewed as a virtual generator with a larger capacity and an aggregated output. The aggregated output of the virtual generator would be dispatched to address ancillary service

issues, which will save much long-distance communication. We will discuss how to design the control strategy and the communication matrix in the following two sections.

3 Control strategy design

It follows from (2)–(4) that

$$\dot{P}_i = \dot{U}_i I_{di}^{\text{ref}} + U_i \dot{I}_{di}^{\text{ref}} = \frac{\dot{U}_i}{U_i} P_i + U_i u_{1i} \tag{13}$$

$$\dot{Q}_i = -\dot{U}_i I_{qi}^{\text{ref}} - U_i \dot{I}_{qi}^{\text{ref}} = \frac{\dot{U}_i}{U_i} Q_i - U_i u_{2i} \tag{14}$$

Using the cooperative control theory [19], we can choose the control law for the ith PV to be

$$u_{1i} = \frac{K_0 P_{i\max}}{\text{dz}(U_i)} \left(D_{i0} \alpha_P^0 + \sum_{j=1}^{n} \frac{D_{ij} P_j}{P_{j\max}} - \frac{P_i}{P_{i\max}} \right) - \frac{\dot{U}_i P_i}{(\text{dz}(U_i))^2} \tag{15}$$

$$u_{2i} = -\frac{K_0 Q_{i\max}}{\text{dz}(U_i)} \left(D_{i0}' \alpha_Q^0 + \sum_{j=1}^{n} \frac{D_{ij}' Q_j}{Q_{j\max}} - \frac{Q_i}{Q_{i\max}} \right) + \frac{\dot{U}_i Q_i}{(\text{dz}(U_i))^2} \tag{16}$$

where $\text{dz}(U_i) = \max \{U_i, \underline{U}\}$ and $\underline{U} > 0$ is a given constant; $K_0 > 0$ is the gain

$$D_{ij} = \frac{w_{ij} s_{ij}}{\sum_{j=0}^{n} w_{ij} s_{ij}}, \quad D_{ij}' = \frac{w_{ij}' s_{ij}'}{\sum_{j=0}^{n} w_{ij}' s_{ij}'}, \quad i = 1, 2, \ldots, n \tag{17}$$

where $W = (w_{ij})$ and $W' = (w_{ij}')$ denote the weight matrix with positive elements (in this paper, those elements are set to 1 for simplicity); s_{ij} and s_{ij}' $(i, j = 1, 2, \ldots, n)$ are the entries of the communication matrix defined in (9); $\underline{U}$ denotes the lower bound of the terminal voltage of the ith PV, which guarantees the control has the ability of low voltage ride through in extreme cases.

In addition, the control whose transfer function shown in Fig. 3 is given for updating α_Q^0, and its dynamical equation can be written as [6]

$$\begin{cases} \dot{z}_0' = K_v(V_{\text{ref}} - V_c) \\ \alpha_Q^0 = z_0' \end{cases} \tag{18}$$

where V_c is the voltage of the critical bus of concern; V_{ref} is the referenced voltage of the concerned bus; α_Q^0 is the output of this control and also the input for (16).

Similar control is used for the active power control, whose transfer function is shown in Fig. 4. The corresponding dynamical equation is

$$\begin{cases} \dot{z}_0 = K_P(P_{\text{ref}} - P_{\text{tran}}) \\ \alpha_P^0 = z_0 \end{cases} \tag{19}$$

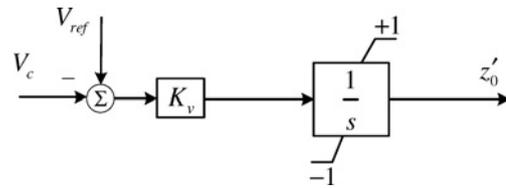where P_{tran} is the active power across the line of concern; P_{ref}

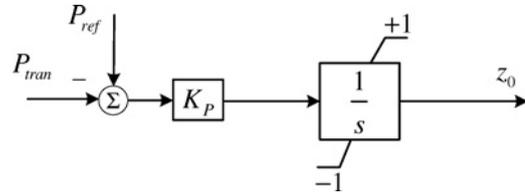**Fig. 3** *Control for the instructed value of Q*

Fig. 4 *Control for the instructed information of P*

is the referenced power across the concerned line; α_P^0 is the output and also the input for (15).

Remark: It should be noted that the active power flow P_{tran} to be controlled is related to the positive direction. In this control, P_{tran} should be an increasing function of z_0 at the power flow equilibrium, that is, increasing the output ratios of PV generators can increase P_{tran}. Otherwise, the signs of P^{ref} and P_{tran} should be changed in Fig. 4. Therefore without any loss of generality, it can be assumed that at the equilibrium P_{tran} will increase when z_0 is increased. This property will be used in the stability analysis in the next section.

Intuitively, the physical meaning of the distributed high-level control proposed above is that, if the bus voltage (or real power transmission) is lower than its reference value, it will command some of the PVs (through its local communication network) to increase their reactive (or real) power ratio and the rest of PVs will also cooperate towards the same goal under their distributed cooperative controls. Hence, an increase of α_P^0 should result in an increase of P_{tran} till P_{tran} reaches its desired reference value of P_{ref}.

To obtain the closed dynamical equations of compact form, let

$$z_0 = \alpha_P^0, \quad z_i = \frac{P_i}{P_{i\max}}, \quad z_0' = \alpha_Q^0, \quad z_i' = \frac{Q_i}{Q_{i\max}} \tag{20}$$

then by (13)–(19), the closed-loop dynamical equations can be expressed by

$$\dot{z}_0 = K_P[P_{\text{ref}} - P_{\text{tran}}(\cdot)] \tag{21}$$

$$\dot{z}_0' = K_v[V_{\text{ref}} - V_c(\cdot)] \tag{22}$$

$$\dot{z}_i = K_0 \left[-z_i + D_{i0} z_0 + \sum_{j=1,2,\ldots,n} D_{ij} z_j \right] \tag{23}$$

$$\dot{z}_i' = K_0 \left[-z_i' + D_{i0} z_0' + \sum_{j=1,2,\ldots,n} D_{ij}' z_j' \right] \tag{24}$$

$$g(z_1, \ldots, z_n, z_1', \ldots, z_n', \boldsymbol{\chi}, \mathbf{X}) = 0 \tag{25}$$

where the variables are defined previously.

The stability of system (21)–(25) is related to the features of the time-varying variables $D_{ij}(t)$ and $D'_{ij}(t)$, so the design of the communication topology is a key step for Problems 1 and 2. The rule for designing the communication topology among the PVs will be given in the next section following the network control theories.

4 Communication topology design

In this section, we will give the design rule of the communication topology that guarantees the validness of the proposed control strategy, and some definitions on positive matrix will be provided in order to describe the communication topology.

4.1 Preliminary

Some preliminary will be given for the descriptions of communication topology. Those definitions can also be found in [16, 19].

Definition 1 (Row-stochastic matrix): A matrix D is said to be row-stochastic if $D \in R_+^{n \times n}$ and $D\mathbf{1} = \mathbf{1}$, where $\mathbf{1}$ is the vector with all 1 entries.

Definition 2 (Non-negative matrix): A matrix $E \in R^{n \times n}$ is said to be non-negative if all entries in E are non-negative.

Definition 3 (Reducible and irreducible matrix): A non-negative matrix $E \in R^{r \times r}$ with $r \geq 2$ is said to be reducible if the set of its indices, $\Omega \triangleq \{1, 2, \ldots, r\}$, can be divided into two disjoint non-empty sets $S \triangleq \{i_1, i_2, \ldots, i_\mu\}$ and $S^c \triangleq \Omega/S = \{j_1, j_2, \ldots, j_v\}$ (with $\mu + v = r$) such that $e_{i_\alpha j_\beta} = 0$, where $\alpha = 1, 2, \ldots, \mu$ and $\beta = 1, 2, \ldots, v$. Matrix is said to be irreducible if it is not reducible.

Definition 4 (Canonical form of a reducible matrix): Consider matrix $E \in R^{r \times r}$ with $r \geq 2$. If E is reducible, there exists an integer $p > 1$ and a permutation matrix T such that

$$T^{\mathrm{T}} E T = \begin{bmatrix} F_{11} & 0 & \cdots & 0 \\ F_{21} & F_{22} & & \\ \vdots & & \vdots & \vdots \\ F_{p1} & F_{p2} & \cdots & F_{pp} \end{bmatrix} \triangleq F_\Delta \qquad (26)$$

where $F_{ii} \in R^{r_i \times r_i}$ is either square and irreducible sub-matrices of dimension higher than 1 or a scalar, and $\sum_{i=1}^p r_i = r$. The standard form of a reducible matrix denotes the form of (26).

Definition 5 (Lower-triangularly complete): A reducible matrix F is said to be lower triangular complete if, in its canonical form of (26) and for every $1 \leq i \leq p$, there exists at least one $j < i$ such that $F_{ij} \neq 0$.

Definition 6 (Sequentially complete): Consider an infinite countable series of non-negative matrices $\{E_k : E_k \in R_+^{n \times n}, k \in N^+\}$, where N^+ is the set of positive integers. $E_{\infty:0}$ is said to be sequentially lower-triangular if there exists one permutation matrix that is independent of k and maps all E_k into the lower triangular canonical form of (26). Sequence $E_{\infty:0}$ is said to be sequentially lower triangularly complete if it is sequentially lower-triangular and if there exists a scalar strictly increasing sub-series $\{m_l \in N^+ : l \in N\}$ such that products $E_{m_l : m_{l-1}}$ (defined as $E_{k:l} \triangleq E_k E_{k-1} \ldots E_{l+2} E_{l+1}$, $k > l$) are all low triangularly complete and diagonally positive and that both diagonal positivity and lower-triangular completeness are uniform with respect to l. Sequence $E_{\infty:0}$ is sequentially complete if it is lower-triangularly complete and if the differences of $m_l - m_{l-1}$ are all uniformly bounded with respect to l.

The sequential completeness condition will play an important role in the proposed distributed control. It is one of the most advanced methods to characterise the properties of a time-varying non-negative matrix such as the time-varying communication matrix defined in (9). An easy approach to verify the completeness condition is provided in [16, 19]. That is, let

$$S_\Lambda(\eta) \triangleq S(t_{k_{\eta+1}-1}) \wedge S(t_{k_{\eta+1}-2}) \wedge \cdots S(t_{k_\eta}) \qquad (27)$$

where $S(t_i)$ is defined in (10); $\wedge$ denotes the operation of a binary product of two binary matrices, that is, $(S_\Lambda)_{ij} = 1$ if there exists at least one $S^t \in \{S(t_{k_{\eta+1}-1}), S(t_{k_{\eta+1}-2}), \ldots S(t_{k_\eta})\}$ such that $S_{ij}^t = 1$ is satisfied; $(S_\Lambda)_{ij} = 0$ otherwise.

If there exists an integer subsequence $\{k_\eta : \eta \in N\}$ such that $S_\Lambda(\eta)$ is lower triangularly complete and $k_\eta - k_{\eta-1}$ is uniformly bounded, then sequence $S_{\infty:0} = \{S(0), S(t_1), \ldots\}$ is sequentially complete.

An example is given for further explain this approach, which can be found in [16].

Example 1: Consider communication sequence $\{S(t_k) : k \in N\}$ defined by $S(t_k) = S_1$ if k is even and $S(t_k) = S_2$ if k is odd, where

$$S_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } S_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Thus we have

$$S_\Lambda(\eta) \triangleq S(t_k) \wedge S(t_{k+1}) = S_1 \wedge S_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$
$$(28)$$

Choose the integer subsequence $\{k_\eta : \eta \in N\}$ as $\eta = 2i$ ($i = 0, 1, \ldots,$), then $\{k_\eta : \eta \in N\} = \{0, 2, 4, \ldots\}$ and $k_\eta - k_{\eta-1}$ is uniformly bounded. In addition, it follows from Definition 5 that matrix $S_\Lambda(\eta)$ is lower-triangularly complete. Thus, by this approach $\{S(k) : k \in N\}$ is a sequentially complete sequence. □

A special case of the above method is that, if $S(t_k)$ is lower-triangularly complete for every $k \geq 0$, then its sequence is sequentially complete. This case is shown in the next example, in which the communication will also be used in the simulation.

Example 2: Consider a communication matrix which can be piecewise denoted by sequence $\{S(t_0), S(t_1), \ldots\}$ where $S(t_k)$ is randomly chosen in $\{S_0, S_1, S_2, S_3\}$ and

$$
S_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}; \quad
S_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}
$$

$$
S_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}; \quad
S_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}
$$

In those communication matrices, the elements $S_{12}(t)$, $S_{23}(t)$, $S_{31}(t)$ and $S_{50}(t)$ are time-varying, but every possible matrix is lower-triangularly complete, so the sequence $\{S(t_0), S(t_1), \ldots\}$ is sequentially complete. □

4.2 Communication topology design rule

The communication matrix is designed according to the following rule:

The communication topology among the PVs may be intermitted (time-varying), but the communication matrix S is piecewise continuous. Mathematically $S_{\infty:0} = \{S(0), S(t_1), \ldots\}$ is sequentially complete.

The completeness condition is a strict method to show the connection of a communication topology and its cumulated effects in an interval. It gives the minimal requirement of the communication network for a distributed control. From Example 1, we can observe that even none of possible matrix is a complete one, but their cumulated effects make the communication network connected. This property shows that an intermitted communication is possible to satisfy this rule, so it is very flexible for a control and it is very practically useful for a real power system.

It should be noted that the communication network can also be intuitively depicted by graphic theories, readers can refer to [26] for more details. Moreover, this property can be used to design and implement a redundant local communication network which satisfies the so-called rule of '$N-n$'. Namely, supposing that the total number of the communication channels is N, it is said that the communication network satisfies $N-n$ rule if n communication channels cannot work properly, and the communication matrix corresponding to the remaining communication channels can still be complete. However, the convergence rate of the closed-loop system depends upon connectivity of the communication network [27] (the numerical simulation will also illustrate this phenomenon in Section 6), so it is important to design a reasonably connected local communication network within certain physical and economic constraints.

5 Stability analysis for the closed-loop system

5.1 Reduced dynamical models for stability analysis

In Section 4, the communication topology and the control among all PVs are given. The remaining problem is to analyse the stability of the closed-loop dynamical system (21)–(25). The answer is positive in the local region under some trivial conditions.

It follows from (21)–(25) that the equilibrium of the dynamical–algebraic equations satisfies

$$z_1^0 = z_2^0 = \cdots = z_n^0 = \alpha_P^0, z_1'^0 = z_2'^0 = \cdots = z_n'^0 = \alpha_Q^0 \quad (29)$$

$$P_{\text{ref}} = P_{\text{tran}}^0, \qquad V_{\text{ref}} = V_c^0 \quad (30)$$

$$g(z_1^0, \ldots, z_n^0, z_1'^0, \ldots, z_n'^0, \boldsymbol{\chi}^0, \mathbf{X}'^0) = 0 \quad (31)$$

where the superscript '0' denotes the variables evaluated at the equilibrium.

If the equilibrium is a regular (non-singular) point of the power flow equation, it follows from the inversed function theorem that around the equilibrium there exist smooth functions of appropriate dimensions, say $\varphi_i'(\cdot)$, such that

$$\begin{cases} X = \varphi_1'(z_1, \ldots, z_n, z_1', \ldots, z_n') \\ \chi = \varphi_2'(z_1, \ldots, z_n, z_1', \ldots, z_n') \end{cases} \quad (32)$$

satisfies the power flow equation, that is

$$g(z_1, \ldots, z_n, z_1', \ldots, z_n', \varphi_1'(\cdot), \varphi_2'(\cdot)) = 0$$

Thus, there exist $\varphi_i(\cdot)$ ($i = 1, 2$) such that P_{tran} and V_c can be expressed as

$$P_{\text{tran}} = \varphi_1(z_1, \ldots, z_n, z_1', \ldots, z_n') \quad (33)$$

$$V_c = \varphi_2(z_1, \ldots, z_n, z_1', \ldots, z_n') \quad (34)$$

Substituting (33) and (34) into (21)–(24), the reduced ordinary dynamical equations can be expressed as

$$\dot{z}_0 = K_P[P_{\text{ref}} - \varphi_1(z_1, \ldots, z_n, z_1', \ldots, z_n')] \quad (35)$$

$$\dot{z}_0' = K_v[V_{\text{ref}} - \varphi_2(z_1, \ldots, z_n, z_1', \ldots, z_n')] \quad (36)$$

$$\dot{z}_i = K_0\left[-z_i + D_{i0}z_0 + \sum_{j=1,2,\ldots,n} D_{ij}z_j\right] \quad (37)$$

$$\dot{z}_i' = K_0\left[-z_i' + D_{i0}z_0' + \sum_{j=1,2,\ldots,n} D_{ij}'z_j'\right] \quad (38)$$

Thus, the stability is completely determined by system (35)–(38). It is noted that if z_0 and z_0' are fixed, it follows from the Theorem 5.6 in [19] that the trajectories of system (37) and (38) satisfies $|z_i - z_j| \to 0$ with $(z_i, z_i') \to (z_0, z_0')$ and $t \to \infty$ for every i and j ($i = 1, 2, \ldots, n$). In that theorem, z_0 is considered to be a virtual leader and other variables converge to the virtual leader under the condition that the communication topology is sequentially complete. From this result, one intuition is that if two small values are chosen for K_p and K_v, which implies that the variables in the virtual leader change slowly, then both z_i and z_i' will

converge to the corresponding variables in the virtual leader. In the next subsection, this intuition will be proved to be valid under some additional assumptions that are usually satisfied in power systems.

Remark: In the above derivation, we use the condition that the equilibrium is a non-singular point of the power flow equation. If the equilibrium is a singular point, it means that the equilibrium may be a bifurcation point of the power flow equation. It is prohibited in power systems, since bifurcation means a voltage collapse. Readers can refer to [23, 28] for more details. In fact, in power systems, there should be some margin between the operating point with the singular point. In a word, the assumption is very trivial in our derivation.

5.2 Basic properties for the stability analysis

For a general distribution network, the following facts can be assumed to be satisfied:

Fact 1: $V_c = \varphi_2(\cdot)$ is an increasing smooth function of the injected reactive power ($Q_i = z_i' Q_{i\max}$) and its sensitivity with respect to $Q_i = z_i' Q_{i\max}$ is much larger than that of active power $P_i = z_i P_{i\max}$, that is

$$\frac{\partial \varphi_2(\cdot)}{\partial Q_i} > 0, \quad \left| \frac{\partial \varphi_2(\cdot)}{\partial Q_i} \right| \gg \left| \frac{\partial \varphi_2(\cdot)}{\partial P_i} \right| \quad (39)$$

Fact 2: The positive direction of P_{tran} is chosen to be the increasing function of α_P^0, so $P_{\text{sum}} = \varphi_1(\cdot)$ is an increasing smooth function of the injected active power ($P_i = z_i P_{i\max}$) of the ith PV, and its sensitivity with respect to P_i is much larger than that of reactive power $Q_i = z_i' Q_{i\max}$, that is

$$\frac{\partial \varphi_1(\cdot)}{\partial P_i} > 0, \quad \left| \frac{\partial \varphi_1(\cdot)}{\partial P_i} \right| \gg \left| \frac{\partial \varphi_1(\cdot)}{\partial Q_i} \right| \quad (40)$$

Fact 3: The angles at both sides of the transmission line of concern satisfy that

$$|\sin(\delta_1 - \delta_2)| \ll |\cos(\delta_1 - \delta_2)| \quad (41)$$

where δ_i ($i = 1, 2$) denotes the angle of the concerned transmission line.

Remarks:
1. In a general power system network, the voltage at a bus is mainly determined by injected reactive power relative to the active power at the same bus, so $\varphi_i(\cdot)$ has much larger sensitivity with respect to Q_i than that with respect to P_i. That is to say, Fact 1 is usually satisfied in the neighbourhood around the equilibrium of the power flow equations.
2. Similarly, the phase angle of a bus is mainly determined by the active power, so the sensitivity of the angle with respect to the injected active power is relatively much larger than that of reactive power. In addition, the active power of the line of concern can be approximated by $P_{\text{sum}} = V_1 V_2 \sin(\delta_1 - \delta_2)/X_l$, where V_i and δ_i ($i = 1, 2$) denote the voltage and the angle of line of concern, X_l is the inductance of this line. Thus, it follows from $\sin(\delta_1 - \delta_2) \ll (\delta_1 - \delta_2)$ that P_{sum} is mainly determined by the injected active power. Namely, Fact 2 is satisfied in a neighbourhood around the equilibrium.

Next, we analyse the linearisation system of (35)–(38) based on the above facts. The linearisation system at the equilibrium (denoted by E_0) can be written as

$$\begin{cases} \dot{z}_0 = -K_P \sum_{j=1}^n (c_{1j}(z_j - \alpha_{P0}) + c'_{1j}(z'_j - \alpha_{Q0})) \\ \dot{z}'_0 = -K_v \sum_{j=1}^n (c_{2j}(z_i - \alpha_{P0}) + c'_{2j}(z'_j - \alpha_{Q0})) \\ K_0^{-1}\dot{z}_i = -z_i + D_{i0}z_0 + \sum_{j=1,2,\dots,n} D_{ij}z_j \\ K_0^{-1}\dot{z}'_i = -z'_i + D_{i0}z'_0 + \sum_{j=1,2,\dots,n} D'_{ij}z'_j \end{cases} \quad (42)$$

where

$$\begin{bmatrix} c_{1j} & c'_{1j} \\ c_{2j} & c'_{2j} \end{bmatrix} \triangleq \begin{bmatrix} \dfrac{\partial \varphi_1}{\partial z_j} & \dfrac{\partial \varphi_1}{\partial z'_j} \\ \dfrac{\partial \varphi_2}{\partial z_j} & \dfrac{\partial \varphi_2}{\partial z'_j} \end{bmatrix}_{E_0}$$

Changing the equilibrium of (42) to the origin results in a system with the same stability property as that of the following system

$$\begin{cases} \begin{bmatrix} \dot{z}_0 \\ \dot{z}'_0 \end{bmatrix} = -K_0^{-1} \begin{bmatrix} K_P c_1^{\text{T}} & K_P c_1'^{\text{T}} \\ K_V c_2^{\text{T}} & K_V c_2'^{\text{T}} \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \mathbf{z}' \end{bmatrix} \\ \begin{bmatrix} \dot{\mathbf{z}} \\ \dot{\mathbf{z}}' \end{bmatrix} = \begin{bmatrix} \mathbf{D}_{*0} & \\ & \mathbf{D}'_{*0} \end{bmatrix} \begin{bmatrix} z_0 \\ z'_0 \end{bmatrix} + \begin{bmatrix} -\mathbf{I} + \mathbf{D} & \\ & -\mathbf{I} + \mathbf{D}' \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \mathbf{z}' \end{bmatrix} \end{cases} \quad (43)$$

where $D_{*0}[D_{10}, D_{20}, \dots, D_{n0}]^{\text{T}}$, $\mathbf{D}'_{*0} = [D'_{10}, D'_{20}, \dots, D'_{n0}]^{\text{T}}$, D and D′ are the nn matrices whose entries are D_{ij} and D'_{ij}, respectively; I is the identity matrix of appropriate dimensions; $c_1[c_{11}, c_{12}, \dots, c_{1n}]^{\text{T}}$, $c'_1 = [c'_{11}, c'_{12}, \dots, c'_{1n}]^{\text{T}}$, $\mathbf{z}[z_1, z_2, \dots, z_n]^{\text{T}}$ and $\mathbf{z}' = [z'_1, z'_2, \dots, z'_n]^{\text{T}}$.

Thus, the stability analysis of system (35)–(38) is locally determined by system (43). If the gains K_p and K_v are very small, the instructed information z_0 and z'_0 is slow. In addition, the extended matrices $[\mathbf{D}_{*0}, \mathbf{D}]$ and $[\mathbf{D}'_{*0}, \mathbf{D}']$ are row-stochastic, so if those matrices are constant (i.e. the communication topology is kept to be fixed), it can be easily shown that the fast boundary-layer system is exponentially stable since its Jacobean matrix is Hurwitz (from the fact that $\rho(D)1$ and $\rho(D')1$ where $\rho(\cdot)$ denotes the radium radius). However, the method based on the eigenvalues is invalid to study the stability of system (43) if D and D′ are time-varying. The network control theory will be used to show that under some conditions similar conclusions can be drawn.

Lemma 1: Suppose Facts 1–3 are satisfied, then for (43), matrix

$$M \triangleq \left\{ -\begin{bmatrix} K_P c_1^{\text{T}} \mathbf{1} & K_P c_1'^{\text{T}} \mathbf{1} \\ K_V c_2^{\text{T}} \mathbf{1} & K_V c_2'^{\text{T}} \mathbf{1} \end{bmatrix} \right\} \in R^{2\times 2}$$

is Hurwitz, where $\mathbf{1} = [1, 1, \dots, 1]^{\text{T}}$.

Proof: As stated in Problem 1, all $P_{i\max}$ have the same sign and all $Q_{i\max}$ have the same sign. We only prove the case for $P_{i\max} > 0$ and $Q_{i\max} > 0$, but others are similar.

It follows from the definitions of c_i and c_i' ($i = 1, 2$) that there are

$$c_1^T \mathbf{1} = \sum_{j=1}^{n} \frac{\partial \varphi_1}{\partial P_j} P_{j\max}, \quad c_2'^T \mathbf{1} = \sum_{k=1}^{n} \frac{\partial \varphi_2}{\partial Q_k} Q_{k\max} \quad (44)$$

$$c_1'^T \mathbf{1} = \sum_{j=1}^{n} \frac{\partial \varphi_1}{\partial Q_j} Q_{j\max} \leq \sum_{j=1}^{n} \left| \frac{\partial \varphi_1}{\partial Q_j} \right| Q_{j\max} \quad (45)$$

$$c_2^T \mathbf{1} = \sum_{k=1}^{n} \frac{\partial \varphi_2}{\partial P_k} P_{k\max} \leq \sum_{k=1}^{n} \left| \frac{\partial \varphi_2}{\partial P_k} \right| P_{k\max} \quad (46)$$

where $\mathbf{1} = [1, 1, \ldots, 1]^T$.

It follows from (44)–(46) that

$$c_2^T \mathbf{1} \times c_1'^T \mathbf{1} \leq \sum_{k=1}^{n} \sum_{j=1}^{n} \left| \frac{\partial \varphi_1}{\partial Q_j} \right| \left| \frac{\partial \varphi_2}{\partial P_k} \right| P_{k\max} Q_{j\max} \quad (47)$$

$$c_1^T \mathbf{1} \times c_2'^T \mathbf{1} = \sum_{k=1}^{n} \frac{\partial \varphi_1}{\partial P_k} P_{k\max} \times \sum_{j=1}^{n} \frac{\partial \varphi_2}{\partial Q_j} Q_{j\max}$$
$$= \sum_{k=1}^{n} \sum_{j=1}^{n} \frac{\partial \varphi_2}{\partial Q_j} \frac{\partial \varphi_1}{\partial P_k} P_{k\max} Q_{j\max} \quad (48)$$

Taking Facts 1 and 2 into consideration, (47) and (48) leads to

$$c_1^T \mathbf{1} > 0, \quad c_2'^T \mathbf{1} > 0, \quad \frac{\partial \varphi_1}{\partial P_i} > 0 \quad \text{and} \quad \frac{\partial \varphi_2}{\partial Q_i} > 0 \quad (49)$$

and

$$c_1^T \mathbf{1} \times c_2'^T \mathbf{1} = \sum_{k=1}^{n} \sum_{j=1}^{n} \frac{\partial \varphi_2}{\partial Q_j} \frac{\partial \varphi_1}{\partial P_k} P_{k\max} Q_{j\max}$$
$$= \sum_{k=1}^{n} \sum_{j=1}^{n} \left| \frac{\partial \varphi_2}{\partial Q_j} \right| \left| \frac{\partial \varphi_1}{\partial P_k} \right| P_{k\max} Q_{j\max}$$
$$\gg \sum_{k=1}^{n} \sum_{j=1}^{n} \left| \frac{\partial \varphi_2}{\partial P_j} \right| \left| \frac{\partial \varphi_1}{\partial Q_k} \right| P_{k\max} Q_{j\max}$$
$$\geq c_2^T \mathbf{1} \times c_1'^T \mathbf{1} \quad (50)$$

It follows from (49) and (50) that both $\det(M) > 0$ and $\text{trace}(M) < 0$ are satisfied. Consequently, the conclusion of this lemma is drawn. □

Lemma 2: Consider the following time-varying system

$$\begin{cases} \dot{x} = \varepsilon(\mathbf{A}_{11} \mathbf{x} + \mathbf{A}_{12} \mathbf{z} + \mathbf{A}_{12}' \mathbf{z}') \\ \dot{z} = \mathbf{D}_{*0}(t) \mathbf{c}^T \mathbf{x} + (-\mathbf{I} + \mathbf{D}(t)) \mathbf{z} \\ \dot{z}' = \mathbf{D}_{*0}'(t) \mathbf{c}'^T \mathbf{x} + (-\mathbf{I}' + \mathbf{D}'(t)) \mathbf{z}' \end{cases} \quad (51)$$

where $x \in R^{n_1}$, $z \in R^{n_2}$ and $z' \in R^{n_3}$ are the states; $\mathbf{A}_{11} \in R^{n_1 \times n_1}$, $\mathbf{A}_{12} \in R^{n_1 \times n_2}$, $\mathbf{A}_{12}' \in R^{n_1 \times n_3}$, $\mathbf{c} \in R^{n_2}$ and $\mathbf{c}' \in R^{n_3}$, are constant; $\varepsilon > 0$ is a small constant; $\mathbf{I}$ and $\mathbf{I}'$ are identities of appropriate dimensions; $D_{*0}(t)$, $D(t)$, $\mathbf{D}_{*0}'(t)$ and $D'(t)$ are non-negative matrices with appropriate dimensions.

Suppose that the following conditions are satisfied:

1. $\mathbf{A}_{11} + \mathbf{A}_{12} \mathbf{1} \mathbf{c}^T + \mathbf{A}_{12}' \mathbf{1} \mathbf{c}'^T$ are Hurwitz;

2. Expanded matrices $\mathbf{D}_e(t) = [\mathbf{D}_0^*(t), \mathbf{D}(t)]$ and $\mathbf{D}_e'(t) = [\mathbf{D}_{*0}'(t), \mathbf{D}'(t)]$ are row-stochastic, piecewise continuous matrices satisfying

$$\mathbf{D}_{e,k} \triangleq \mathbf{D}_e(t_k) = [\mathbf{D}_{*0}(t_k), \mathbf{D}(t_k)], \quad t \in [t_k, t_{k+1}), \quad k = 0, 1, 2 \ldots$$

$$\mathbf{D}_{e,k}' \triangleq \mathbf{D}_e'(t_k') = [\mathbf{D}_{*0}'(t_k'), \mathbf{D}'(t_k')], \quad t \in [t_k', t_{k+1}'), \quad k = 0, 1, 2 \ldots$$

and both $\{\mathbf{D}_{e,0}, \mathbf{D}_{e,1}, \ldots\}$ and $\{\mathbf{D}_{e,0}', \mathbf{D}_{e,1}', \ldots\}$ are sequentially complete sequences.

Then system (51) is uniformly asymptotically stable if $\varepsilon > 0$ is small enough.

Proof: See Appendix.

It follows from Lemmas 1 and 2 that the following result on the stability of system (43) is satisfied.

Theorem 1: Consider system (43). Suppose that the following conditions are satisfied:

1. K_P and K_v are small enough;
2. Facts 1–3 are satisfied;
3. The communication among PVs satisfies the sequential completeness condition.

Then system (43) is uniformly asymptotically stable.

Proof: Clearly, system (43) can be rewritten as the system considered in Lemma 2. It follows from the given conditions 1 and 2 and Lemma 1 that the condition 1 in Lemma 2 is satisfied. The given condition 3 implies that the condition 2 in Lemma 2 is also satisfied. Thus, system (43) is uniformly and asymptotically stable. □

It follows from this theorem that the linearised system of the closed-loop dynamical system is uniformly and asymptotically stable, so there exists a neighbourhood around the equilibrium such that if the initial states lie in this neighbourhood, both the active and reactive outputs of those PVs are asymptotically stable, that is, $[z_0, z_1, \ldots, z_n]^T \to \alpha_P^0 \mathbf{1}$ and $[z_0', z_1', \ldots, z_n']^T \to \alpha_Q^0 \mathbf{1}$ are satisfied as the time approach the infinity, where α_P^0 and α_Q^0 satisfy (11) and (12). This theorem guarantees the convergence of the designed controls, so it provides the solution and the conditions for Problems 1 and 2.

It should be noted that the first condition of Theorem 1 is used to guarantee that system (43) can be rewritten as a singular perturbation system as studied in Lemma 2. Since variables c_i' and c_i ($i = 1, 2$) are very small in a practical power network, thus the first condition indeed means that K_p and K_v are not very large.

6 Simulation

In this section, a 50 Hz radial network will be used to show the effectiveness of the proposed control strategy. The main voltage in this network is 10 KV, and the topology is shown in Fig. 5, where five PVs and six loads are connected to the low-voltage network. The case was also used in [24] and the detailed parameters are

• Every segment of the transmission line (from one bus to its neighbouring bus) is 0.85 km; The impedance is $0.443 + j0.3 \, \Omega/\text{km}$.
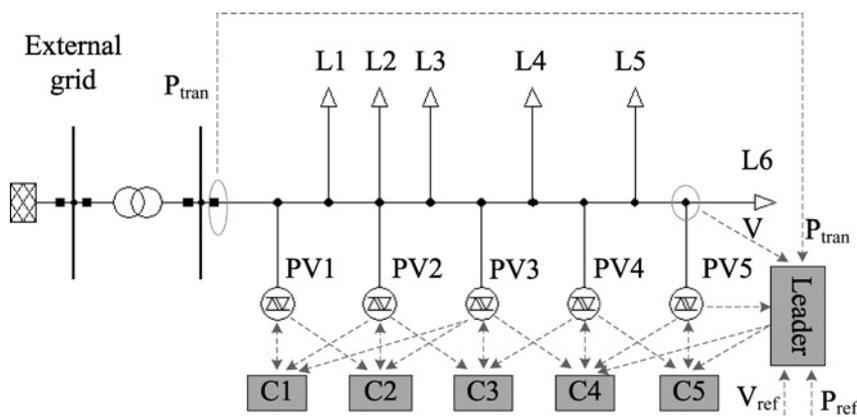
Fig. 5 *Radial system with multiple PVs (the dash arrows among controllers represent the information flow)*

• Transformer datum: the short-circuit voltage is 5%; the capacity is 1 MVA; copper loss is 5 kW.
• Spot loads (balanced) are shown in Table 1 and the constant impedance load models are considered.
• The external grid is considered to be an infinite bus whose voltage is set to 1.05 p.u.
• The communication topology among PVs is represented by the arrows among controllers in Fig. 5.
• The maximum of every PV is 0.2 MW + j0.04 MVAR and the initial output is 0.15 MW + j0.0 MVAR.
• The parameters for the distributed control are $K_0 = 20$, $K_p = K_v = 1$.

It follows from the information flow in Fig. 5 that the communication topology can be represented by the following matrix

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \qquad (52)$$

Clearly, it follows from Example 2 introduced in Section 4 that if the communication network is kept constant, then its sequence is sequentially complete (it is a special case of Example 2). In addition, a malfunction of any communication channel between PVs will not affect the completeness of the rule.

Suppose that the concerned line and bus (for Problem 2) are chosen to be the objects which are measured by the leader control, as shown in Fig. 5. The expected disturbance is that all loads decrease 25% active power and increase 20% reactive power on their normal basis at 0 s.

Figs. 6–9 plot the dynamical responses of the proposed distributed control and the PVs outputs. It follows from that

Table 1 Load information

Load	Active power, KW	Reactive power, KVAR
L1	346.28	92.34
L2	364.50	58.32
L3	473.85	97.20
L4	394.88	63.18
L5	413.10	121.5
L6	273.38	77.76

the distributed control guarantees that the active power outputs of PVs converge to the uniform ratio. Similar conclusion can be drawn for reactive power from Fig. 7. Thus, the requirement of Problem 1 is satisfied.

In addition, it follows from Fig. 8 that the active power across the concerned line can converge to the desired value and the voltage of the concerned bus can also converge to the desired value. Thus, the requirements in Problem 2 are also satisfied.
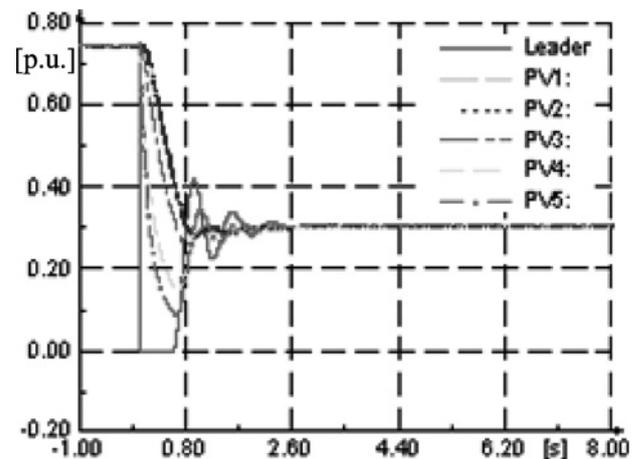

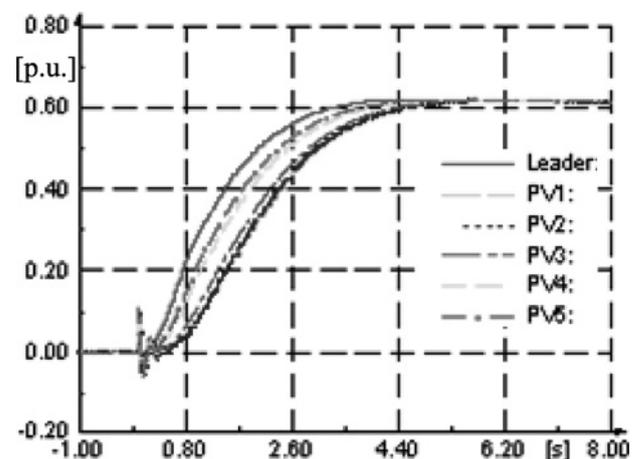**Fig. 6** *Active power output ratios*

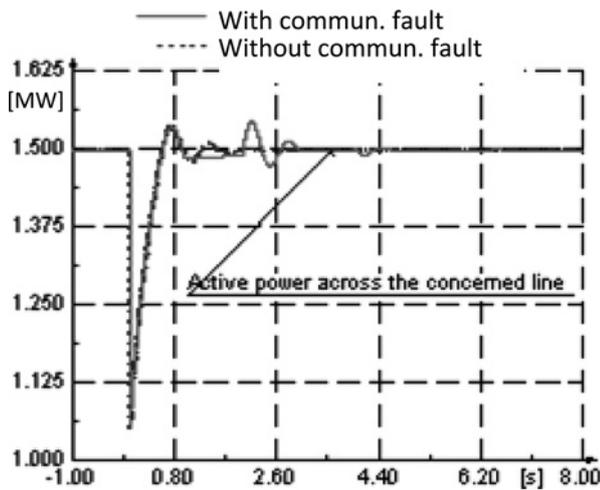**Fig. 7** *Reactive power output ratios*

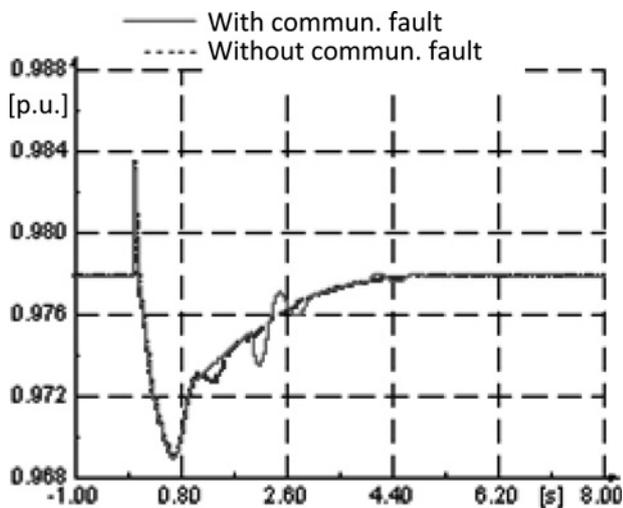**Fig. 8** *Active power across the concerned line*

Fig. 9 *Voltage of the concerned bus*

It also follows from Figs. 8 and 9 that even if two entries ($s_{40}(t)$ and $s_{50}(t)$) in the communication matrix are time-varying, the convergence of the proposed control is also guaranteed. Fig. 10 shows the values of $s_{40}(t)$ and $s_{50}(t)$. Note that when $s_{40}(t)$ and $s_{50}(t)$ are zero, the
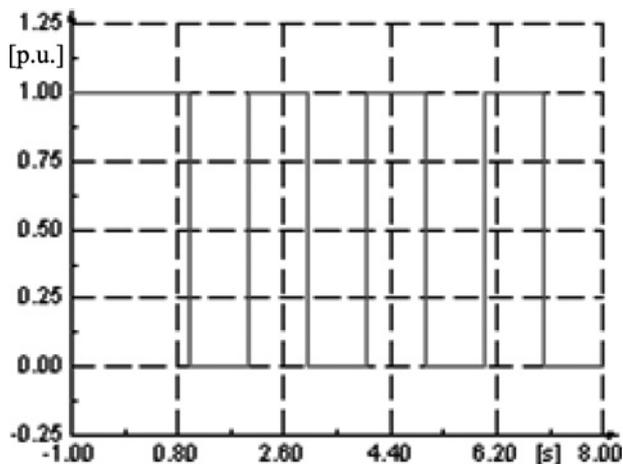

Fig. 10 *Time-varying entries $s_{40}(t)$ and $s_{50}(t)$ in the communication matrix*

communication topology is not connected (i.e. it is not lower-triangularly complete), but the communicative effect makes the communication matrix complete. Thus, this communication topology satisfies the completeness condition provided in Section 4. Clearly, Figs. 8 and 9 imply that the convergence rate is related to the communication topology.

7 Conclusion

A distributed control scheme is provided for the power output control for a group of PVs in distribution networks, which guarantees all PVs to run at the same output ratio and autonomously adjust to a new point in accordance with the ancillary service requirement when some disturbances occur. The proposed control requires only intermittent information sharing among neighbouring PV generators, and allows topologies of local communication networks to be time-varying. The minimal requirement of the communication topology is given based on the sequential completeness (of the matrix sequence) and a simple method to verify the completeness condition is shown as well. As long as the communication networks meet the minimum information exchange requirement, the proposed control ensures convergence by considering several trivial assumptions in classical power systems. Numerical simulations based on a classical distribution power system network show the validity of the proposed method. The proposed methodology is also applicable to distribution networks with different types of DGs including solar, wind and ocean-energy power generators.

8 Acknowledgments

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10 Appendix

To prove the Lemma 2, we first give two lemmas as follows:

Lemma 3: Suppose that $D(t_k)$ is row-stochastic matrix for every k. The sequence $\{D(t_0),\ D(t_1),\ D(t_2),\ \ldots\}$ is sequentially complete, then there exist a constant $\mu \in (0,\ 1)$ and an integer $\kappa > 0$ such that

$$\lambda(H_{k+\kappa:k}) = \lambda\left(\prod_{i=k}^{k+\kappa} H_i\right) \leq \mu, \quad \forall k \geq 0 \tag{53}$$

$$\lambda(E) \leq 1, \quad \delta(E) \leq 1 \tag{54}$$

$$\delta(EF) \leq \lambda(E)\delta(F), \quad \lambda(EF) \leq \lambda(E)\lambda(F) \tag{55}$$

are satisfied uniformly for all $k > 0$, where

$$H_i \triangleq e^{-(I+D_i)(t_{i+1}-t_i)} \tag{56}$$

$$\lambda(E) = 1 - \min_{1 \leq i_1, i_2 \leq n} \sum_{j=1}^{n} \min\left(e_{i_1 j},\ e_{i_2 j}\right) \tag{57}$$

$$\delta(E) = \max_{1 \leq j \leq n} \max_{1 \leq i_1, i_2 \leq n} |e_{i_1 j} - e_{i_2 j}| \tag{58}$$

where $E = (e_{i,j})$, both E and F are row-stochastic matrices with appropriate dimensions.

Moreover, there exists a $\Delta T_0 > 0$ such that

$$t_{k+\kappa} - t_k \geq \Delta T_0 \tag{59}$$

is uniformly satisfied for every $k \geq 0$.

Proof: The results of (53)–(55) can be found in [19] (Lemma 4.41). Expression (53) implies that (59) is satisfied, otherwise, there is $t_{(k+1)\kappa} - t_{k\kappa} \to 0$, so $t_{i+1} - t_i \to 0$ and thus $H_i \to I$ for all i. Thus, it follows from the definition in (57) that $\lambda(H_{k+\kappa:k}) \to 1 > \mu$ is satisfied. It is contradictory to (53). □

Lemma 4: Consider a time-varying system as follows

$$\begin{cases} \dot{\mathbf{x}} = \varepsilon\mathbf{A}_{11}\mathbf{x} + \varepsilon\mathbf{A}_{12}\mathbf{z} \\ \dot{\mathbf{z}} = \mathbf{D}_{*0}(t)\mathbf{c}^{\mathrm{T}}\mathbf{x} + (-\mathbf{I} + \mathbf{D}(t))\mathbf{z} \end{cases} \tag{60}$$

where $\mathbf{x} \in R^{n_1}$, $\mathbf{z} \in R^{n_2}$ are the states; $\mathbf{c} \in R^{n_1 \times 1}$, $\mathbf{A}_{11} \in R^{n_1 \times n_1}$ and $\mathbf{A}_{12} \in R^{n_1 \times n_2}$ are constants; $\varepsilon > 0$ is a constant; $\mathbf{I} \in R^{n_2 \times n_2}$ is the identity; $\mathbf{D}_{*0}(t) \in R^{n_2 \times 1}$ and $\mathbf{D}(t) \in R^{n_2 \times n_2}$ are non-negative piecewise continuous.

Suppose that the following conditions are satisfied:

1. $\mathbf{A}_{11} + \mathbf{A}_{12}\mathbf{1}\mathbf{c}^{\mathrm{T}}$ is Hurwitz;
2. The expanded matrix $\mathbf{D}_e = [\mathbf{D}_{*0},\ \mathbf{D}]$ is row-stochastic, piecewise continuous, which satisfies

$$\mathbf{D}_{e,k} \triangleq \mathbf{D}_e(t_k) = [\mathbf{D}_{*0}(t_k), \mathbf{D}(t_k)], \quad t \in [t_k, t_{k+1}), \quad k = 0, 1, 2, \ldots$$

and sequence $\{D_{e,0},\ D_{e,1},\ \ldots\}$ is sequentially complete.

Then there exists an $\varepsilon_0 > 0$ such that system (60) is uniformly asymptotically stable for every $\varepsilon \in (0, \varepsilon_0)$.

Proof: Perform the following coordinate transformation.

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{z}_f \end{bmatrix} = \begin{bmatrix} \mathbf{I} & 0 \\ -\mathbf{1}\mathbf{c}^{\mathrm{T}} & \mathbf{I} \end{bmatrix}\begin{bmatrix} \mathbf{x} \\ \mathbf{z} \end{bmatrix}$$

Since the expanded matrix $\mathbf{D}_e = [\mathrm{D}_{*0},\ \mathrm{D}]$ is row-stochastic, there is

$$\mathbf{D}_{*0}\mathbf{c}^{\mathrm{T}}\mathbf{x} + (-\mathbf{I} + \mathbf{D})\mathbf{1}\mathbf{c}^{\mathrm{T}}\mathbf{x} = (\mathbf{D}_e\mathbf{1} - \mathbf{1})\mathbf{c}^{\mathrm{T}}\mathbf{x} = \mathbf{0}$$

Thus, in the coordinate $[\mathbf{x}, \mathbf{z}_f]$, system (60) can be rewritten as

$$\dot{\mathbf{x}} = \varepsilon(\mathbf{A}_{11} + \mathbf{A}_{12}\mathbf{1}\mathbf{c}^{\mathrm{T}})\mathbf{x} + \varepsilon\mathbf{A}_{12}\mathbf{z}_f \tag{61}$$

$$\dot{\mathbf{z}}_f = (-\mathbf{I} + \mathbf{D})\mathbf{z}_f + \varepsilon\mathbf{w} \tag{62}$$

where $\mathbf{w} = -\mathbf{1}\mathbf{c}^{\mathrm{T}}[(\mathbf{A}_{11} + \mathbf{A}_{12}\mathbf{1}\mathbf{c}^{\mathrm{T}})\mathbf{x} + \mathbf{A}_{21}\mathbf{z}_f]$.

Let

$$\tilde{\mathbf{D}}_k \triangleq \tilde{\mathbf{D}}(t_k) = \begin{bmatrix} 1 & 0 \\ \mathbf{D}_{*0}(t_k) & \mathbf{D}(t_k) \end{bmatrix}, \quad \tilde{\mathbf{I}} \triangleq \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{I} \end{bmatrix},$$

$$\tilde{\mathbf{w}} \triangleq \begin{bmatrix} 0 \\ \mathbf{w} \end{bmatrix}, \quad \tilde{\mathbf{z}}_f \triangleq \begin{bmatrix} z_{f,0} \\ \mathbf{z}_f \end{bmatrix}$$

The solution of system (62) can be calculated by

$$\begin{cases} \dot{\tilde{\mathbf{z}}}_f = (-\tilde{\mathbf{I}} + \tilde{\mathbf{D}})\tilde{\mathbf{z}}_f + \varepsilon\tilde{\mathbf{w}} \\ \tilde{\mathbf{z}}_{f,0}(0) = 0 \end{cases} \tag{63}$$

The sequential completeness of sequence $\{\mathbf{D}_{e,0}, \mathbf{D}_{e,1}, \ldots\}$ implies that $\{\tilde{\mathbf{D}}_0, \tilde{\mathbf{D}}_1, \ldots\}$ is sequentially complete. In addition, every $\tilde{\mathbf{D}}_k$ is row-stochastic. Thus it follows from Lemma 3 that there exists a constant $\mu \in (0, 1)$ and an integer $\kappa > 0$ such that

$$\lambda\left(\prod_{i=k}^{k+\kappa} e^{(-\tilde{\mathbf{I}} + \tilde{\mathbf{D}}_i)(t_{i+1} - t_i)}\right) \leq \mu, \quad \forall k \geq 0 \tag{64}$$

where λ is defined in Lemma 3.
   Let

$$N = [1/\varepsilon] \tag{65}$$

$$y(m) \triangleq \max_{t \in [t_{mN\kappa}, t_{(m+1)N\kappa})} \max_{1 \leq i_1, i_2 \leq n} \{\tilde{\mathbf{z}}_{f,i_1}(t) - \tilde{\mathbf{z}}_{f,i_2}(t)\}, \quad m \geq 0 \tag{66}$$

$$y'(m) \triangleq \max_{t \in [t_{mN\kappa}, t_{(m+1)N\kappa})} ||x(t)||_\infty, \quad m \geq 0 \tag{67}$$

where $[1/\varepsilon]$ is the maximum integer less than $1/\varepsilon$.
   Clearly, there exists two positive constants β_1 and β_2 such that

$$\begin{aligned} ||\varepsilon\tilde{\mathbf{w}}||_\infty &\leq \varepsilon\beta_1||\mathbf{x}||_\infty + \varepsilon\beta_2||\mathbf{z}_f||_\infty \\ &= \varepsilon\beta_1||\mathbf{x}||_\infty + \varepsilon\beta_2||\tilde{\mathbf{z}}_{f,i} - \tilde{\mathbf{z}}_{f,0}\mathbf{1}||_\infty \\ &\leq \varepsilon\beta_1||\mathbf{x}||_\infty + \varepsilon\beta_2 \max_{0 \leq i,j \leq n}\{|\tilde{\mathbf{z}}_{f,i} - \tilde{\mathbf{z}}_{f,j}|\} \\ &\leq \varepsilon\beta_1 y'(l) + \varepsilon\beta_2 y(l) \end{aligned} \tag{68}$$

Consider $t \in [t_{lN\kappa}, t_{(l+1)N\kappa})$. The solution of (63) satisfies

$$\begin{aligned} \tilde{\mathbf{z}}_f(t) &= \varphi(t, 0)\tilde{\mathbf{z}}_f(0) + \int_0^t \varphi(t, \tau)\tilde{\mathbf{w}}(\tau)\mathrm{d}\tau \\ &= \varphi(t, t_{lN\kappa})\prod_{i=0}^{l-1}\prod_{j=0}^{N-1} \varphi(t_{(iN+j+1)\kappa}, t_{(iN+j)\kappa})\mathbf{z}_f(0) \\ &\quad + \int_{t_{lN\kappa}}^t \varphi(t, \tau)\tilde{\mathbf{w}}(\tau)\mathrm{d}\tau \\ &\quad + \sum_{i=0}^{l-1}\sum_{j=0}^{N-1}\int_{t_{(iN+j)\kappa}}^{t_{(iN+j+1)\kappa}} \varphi(t, \tau)\tilde{\mathbf{w}}(\tau)\mathrm{d}\tau \end{aligned} \tag{69}$$

where $\varphi(t, 0) = e^{(-\tilde{\mathbf{I}} + \tilde{\mathbf{D}})t}$.
   Substituting expression (68) into (69) and considering expressions (54), (55) and (64), for every $t \in [t_{lN\kappa}, t_{(l+1)N\kappa})$

there is

$$\max_{i,j}\{\tilde{\mathbf{z}}_{f,i}(t) - \tilde{\mathbf{z}}_{f,j}(t)\}$$

$$\leq \mu^{lN}\|\tilde{\mathbf{z}}_f(0)\|_\infty + \int_{t_{lN\kappa}}^t \lambda(\varphi(t, \tau))\|\tilde{\mathbf{w}}(\tau)\|_\infty\mathrm{d}\tau$$

$$+ \sum_{i=0}^{l-1}\sum_{j=0}^{N-1}\int_{t_{(iN+i)\kappa}}^{t_{(iN+j+1)\kappa}} \lambda(\varphi(t_{lN\kappa}, \tau))||\tilde{\mathbf{w}}(\tau)||_\infty\mathrm{d}\tau$$

$$\leq \mu^{Nl}||\tilde{\mathbf{z}}_f(0)||_\infty + \int_{t_{lN\kappa}}^t \lambda(\varphi(t, \tau))||\tilde{\mathbf{w}}(\tau)||_\infty\mathrm{d}\tau$$

$$+ \sum_{i=0}^{l-1}\sum_{j=0}^{N-1}\int_{t_{(iN+j)\kappa}}^{t_{(iN+j+1)\kappa}} \lambda(\varphi(t_{lN\kappa}, t_{(iN+j+1)\kappa}))||\tilde{\mathbf{w}}(\tau)||_\infty\mathrm{d}\tau$$

$$\leq \mu^{Nl}||\tilde{\mathbf{z}}_f(0)||_\infty + \varepsilon(\beta_1 y'(l) + \beta_2 y(l))$$

$$+ \varepsilon\kappa\sum_{i=0}^{l-1}\left[(\beta_1 y'(i) + \beta_2 y(i))\sum_{j=0}^{N-1}\mu^{(l-i)N-j-1}\right]$$

$$\leq \mu^{Nl}||\tilde{\mathbf{z}}_f(0)||_\infty + \varepsilon(\beta_1 y'(l) + \beta_2 y(l))$$

$$+ \frac{\varepsilon\kappa\mu}{1 - \mu}\sum_{i=0}^{l-1}[\mu^{N(l-i-1)}(\beta_1 y'(i) + \beta_2 y(i))] \tag{70}$$

Since the expression (70) is satisfied for all $t \in [t_{lN\kappa}, t_{(l+1)N\kappa})$, it follows that

$$\begin{aligned} y(l) &\leq \mu^{Nl}||\tilde{\mathbf{z}}_f(0)||_\infty + \varepsilon(\beta_1 y'(l) + \beta_2 y(l)) \\ &\quad + \frac{\varepsilon\kappa\mu}{1 - \mu}\sum_{i=0}^{l-1}[\mu^{N(l-i-1)}(\beta_1 y'(i) + \beta_2 y(i))] \\ &\triangleq a_0\bar{\mu}^l + \varepsilon(\beta_1 y'(l) + \beta_2 y(l)) \\ &\quad + \varepsilon\alpha_3\sum_{i=0}^{l-1}[\bar{\mu}^{(l-i-1)}(\beta_1 y'(i) + \beta_2 y(i))] \end{aligned} \tag{71}$$

where $\bar{\mu} = \mu^N$, $a_0 = ||\tilde{\mathbf{z}}_f(0)||_\infty$ and $\alpha_3 = [(\kappa\mu)/(1 - \mu)]$.
   Next we perform the similar skills for the trajectories of system (61).
   For every $t \in [t_{lN\kappa}, t_{(l+1)N\kappa})$, the trajectories of system (61) satisfy

$$\begin{aligned} \mathbf{x}(t) &= e^{\varepsilon(\mathbf{A}_{11}+\mathbf{A}_{12}\mathbf{1c}^\mathrm{T})t}\mathbf{x}(0) + \varepsilon\int_0^t e^{\varepsilon(\mathbf{A}_{11}+\mathbf{A}_{12}\mathbf{1c}^\mathrm{T})(t-\tau)}\mathbf{A}_{12}\mathbf{z}_f(\tau)\mathrm{d}\tau \\ &= e^{\varepsilon(\mathbf{A}_{11}+\mathbf{A}_{12}\mathbf{1c}^\mathrm{T})t}\mathbf{x}(0) + \varepsilon\int_{t_{lNk}}^t e^{\varepsilon(\mathbf{A}_{11}+\mathbf{A}_{12}\mathbf{1c}^\mathrm{T})(t-\tau)}\mathbf{A}_{12}\mathbf{z}_f(\tau)\mathrm{d}\tau \\ &\quad + \sum_{i=0}^{l-1}\sum_{j=0}^{N-1}\int_{t_{(iN+j)\kappa}}^{t_{(iN+j+1)\kappa}} e^{\varepsilon(\mathbf{A}_{11}+\mathbf{A}_{12}\mathbf{1c}^\mathrm{T})(t-\tau)}\mathbf{A}_{12}\mathbf{z}_f(\tau)\mathrm{d}\tau \end{aligned} \tag{72}$$

Since $\mathbf{A}_{11} + \mathbf{A}_{12}\mathbf{1c}^\mathrm{T}$ is Hurwitz, there exist constants $\beta_3 > 0$ and $\beta_4 > 0$ such that

$$||e^{(\mathbf{A}_{11}+\mathbf{A}_{12}\mathbf{1c}^\mathrm{T})t}||_\infty \leq \beta_3 e^{-\beta_4 t} \tag{73}$$

is satisfied for every $t \geq 0$.
   It follows from Lemma 3 that there exists a constant $\Delta T_0 > 0$ such that $t_{l\kappa} - t_{i\kappa} \geq (l - i)\Delta T_0$ is satisfied. In

addition, it follows from the definition in expression (65) that $\varepsilon N \geq 1$ is satisfied. Therefore there exists a constant μ' defined as

$$\mu' \triangleq e^{-\beta_4 \Delta T_0} \in (0, 1)$$

such that

$$||e^{\varepsilon(\mathbf{A}_{11} + \mathbf{A}_{12}\mathbf{1}\mathbf{c}^T)(t_{lN\kappa} - t_{iN\kappa})}||_\infty$$
$$\leq \beta_3 e^{-\varepsilon\beta_4(t_{lN\kappa} - t_{iN\kappa})}$$
$$\leq \beta_3 e^{-\varepsilon N\beta_4(l-i)\Delta T_0} \leq \beta_3 \mu'^{l-i} \qquad (74)$$

is satisfied for every pair of $\{i, l\}$ ($l \geq i$).

Therefore it can follow from (72) and (74) that for all $t \in [t_{lN\kappa}, t_{(l+1)N\kappa})$, there is

$$||\mathbf{x}(t)||_\infty \leq ||e^{\varepsilon(\mathbf{A}_{11} + \mathbf{A}_{12}\mathbf{1}\mathbf{c}^T)t}||_\infty ||\mathbf{x}(0)||_\infty$$
$$+ \varepsilon \int_{t_{lNk}}^t ||e^{\varepsilon(\mathbf{A}_{11} + \mathbf{A}_{12}\mathbf{1}\mathbf{c}^T)(t-\tau)}||_\infty ||\mathbf{A}_{12}\mathbf{z}_f(\tau)||_\infty d\tau$$
$$+ \varepsilon \sum_{i=0}^{l-1}\sum_{j=0}^{N-1} \int_{t_{(iN+j)\kappa}}^{t_{(iN+j+1)\kappa}} ||e^{\varepsilon(\mathbf{A}_{11} + \mathbf{A}_{12}\mathbf{1}\mathbf{c}^T)(t-\tau)}||_\infty$$
$$\times ||\mathbf{A}_{12}\mathbf{z}_f(\tau)||_\infty d\tau$$
$$\leq ||e^{\varepsilon(\mathbf{A}_{11} + \mathbf{A}_{12}\mathbf{1}\mathbf{c}^T)t}||_\infty ||\mathbf{x}(0)||_\infty$$
$$+ \varepsilon \int_{t_{lNk}}^t ||e^{\varepsilon(\mathbf{A}_{11} + \mathbf{A}_{12}\mathbf{1}\mathbf{c}^T)(t-\tau)}||_\infty ||\mathbf{A}_{12}\mathbf{z}_f(\tau)||_\infty d\tau$$
$$\times ||e^{\varepsilon(\mathbf{A}_{11} + \mathbf{A}_{12}\mathbf{1}\mathbf{c}^T)(t_{lNk} - t_{(iN+j+1)\kappa})}||_\infty ||\mathbf{A}_{12}\mathbf{z}_f(\tau)||_\infty d\tau$$
$$\leq \beta_3 \mu'^l ||\mathbf{x}(0)||_\infty + \varepsilon y(l)\beta_3 ||\mathbf{A}_{12}||_\infty$$
$$+ \varepsilon\beta_3 \kappa ||\mathbf{A}_{12}||_\infty \sum_{i=0}^{l-1}\left[y(i)e^{-\varepsilon\beta_4\Delta T_0[(l-i)N-1]}\sum_{j=0}^{N-1} e^{\varepsilon\beta_4\Delta T_0 j}\right]$$
$$\leq \beta_3 \mu'^l ||\mathbf{x}(0)||_\infty + \varepsilon y(l)\beta_3 ||\mathbf{A}_{12}||_\infty$$
$$+ \frac{\varepsilon\mu'\beta_3\kappa||\mathbf{A}_{12}||_\infty}{1-\mu'} \sum_{i=0}^{l-1}[y(i)\mu'^{\varepsilon[(l-i)N-1]}\mu'^{\varepsilon(-N+1)}]$$
$$\triangleq a_0' \mu'^l + \alpha_2' y(l) + \varepsilon\alpha_3' \sum_{i=0}^{l-1} \mu'^{l-i-1} y(i)$$
$$\qquad (75)$$

where $a_0' = \beta_3 ||\mathbf{x}(0)||_\infty$, $\alpha_2' = \beta_3 ||\mathbf{A}_{12}||_\infty$, $\alpha_3' = \{(\mu'\beta_3\kappa ||\mathbf{A}_{12}||_\infty)/(1 - \mu')\}$.
Since (75) is satisfied for all $t \in [t_{lN\kappa}, t_{(l+1)N\kappa})$, we have

$$y'(l) \leq a_0' \mu'^l + \varepsilon\alpha_2' y(l) + \varepsilon\alpha_3' \sum_{i=0}^{l-1} \mu'^{l-i-1} y(i) \qquad (76)$$

Since $\varepsilon > 0$ is considered to be small, N is a large integer and $\bar{\mu} < \mu'$ is satisfied. Thus, simultaneously considering expressions (71) and (76), we have

$$\max\{y(l), y'(l)\} \leq \max\{a_0, a_0'\}\mu'^l + \varepsilon\max\{\beta_1 + \beta_2, \alpha_2'\}$$
$$\max\{y(l), y'(l)\} + \varepsilon\max\{\alpha_3\beta_1 + \alpha_3\beta_2, \alpha_3'\}$$
$$\times \sum_{i=0}^{l-1} \mu'^{l-i-1}\max\{y(i), y'(i)\}$$

i.e.

$$\mu'^{-l}\bar{y}(l) \leq \bar{a}_0 + \varepsilon\mu'^{-1}\bar{M}_2 \sum_{i=0}^{l-1} \mu'^{-i}\bar{y}(i) \qquad (77)$$

where $\bar{y}(i) = \max\{y(i), y'(i)\}$, $\bar{a}_0 = (1 - \varepsilon\bar{M}_1)^{-1}\max\{a_0, a_0'\}$, $\bar{M}_1 = \max\{\beta_1 + \beta_2, \alpha_2'\}$, $\bar{M}_2 = (1 - \varepsilon\bar{M}_1)^{-1}\max\{\alpha_3\beta_1 + \alpha_3\beta_2, \alpha_3'\}$.

The Gronwall–Bellman inequality [29] states that if

$$x(n) \leq b_0 + \sum_{i=0}^{l-1} f(i)x(i)$$

is satisfied for all, where $b_0 > 0$ and $f(i) \geq 0$, then

$$x(n) \leq b_0 \prod_{i=0}^{n-1} (1 + f(i))$$

is satisfied for all n.

Thus, applying the Gronwall–Bellman inequality into (77), we have

$$\bar{y}(l) \leq \bar{a}_0 \mu'^l \left(\frac{1 + \varepsilon\mu'^{-1}\bar{M}_2}{1 - \varepsilon\bar{M}_1}\right)^l \qquad (78)$$

Clearly, it follows from (78) that if $\varepsilon > 0$ is small enough then $\bar{y}(l) \to 0$ is satisfied as $l \to \infty$. Therefore $y(l)$ and $y'(l)$ are both convergent to zero and system (60) is asymptotically stable in turn. $\qquad\square$

This lemma shows that under the sequentially complete condition, a singularly perturbed dynamical system can be asymptotically stable although it is not an autonomous system. Based on this result, we prove Lemma 2 as follows.

Proof of Lemma 2: Because the proof steps are similar to those used in the Proof of Lemma 3, we only give some critical steps.

Perform a coordinate transformation as

$$\begin{bmatrix} \mathbf{x} \\ \bar{\mathbf{z}}_f \end{bmatrix} = \begin{bmatrix} \mathbf{I} & 0 \\ -\mathbf{1}\bar{\mathbf{c}}^T & \bar{\mathbf{I}} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \bar{\mathbf{z}} \end{bmatrix}$$

where $\bar{\mathbf{z}}_f = [\mathbf{z}_f^T, \mathbf{z}_f'^T]^T$, $\bar{\mathbf{c}} = [\mathbf{c}^T, \mathbf{c}'^T]^T$, $\bar{\mathbf{z}} = [\mathbf{z}^T, \mathbf{z}'^T]^T$.

In the new coordinate, system (51) can be rewritten as

$$\begin{cases} \dot{\mathbf{x}} = \varepsilon(\mathbf{A}_{11}\mathbf{x} + \bar{\mathbf{A}}_{12}\bar{\mathbf{z}}_f) + \varepsilon\bar{\mathbf{A}}_{12}\bar{\mathbf{z}}_f \\ \dot{\bar{\mathbf{z}}}_f = (-\bar{\mathbf{I}} + \bar{\mathbf{D}}(t))\bar{\mathbf{z}}_f + \varepsilon\bar{\mathbf{w}} \end{cases} \qquad (79)$$

where $\bar{\mathbf{A}}_{12} = [\mathbf{A}_{12}, \mathbf{A}_{12}']$, $\bar{\mathbf{D}} = \text{diag}\{\mathbf{D}, \mathbf{D}'\}$, the expression of $\bar{\mathbf{w}}$ is similar to that in (62).

The condition that $\{\mathbf{D}_{e,0}, \mathbf{D}_{e,1}, \ldots\}$ and $\{\mathbf{D}_{e,0}', \mathbf{D}_{e,1}', \ldots\}$ are both sequentially complete cannot lead to the conclusion that the expanded matrix $\bar{\mathbf{D}}(t)$ is sequentially complete. However, it follows from Lemma 3 that the similar results as stated in (53) and (59) can be obtained. Consequently, the result shown in (77) can be derived by similar steps used in the Proof of Lemma 3, thus the system is asymptotically stable in turn.