CAP 5415: Computer Vision
Lecture 3
Periodicity, Sampling, Practicalities

Fall 2010
Foreword

• We will be doing some derivations
• The goal is to use the derivations to help you build intuition and understanding about why things behave the way they do.
• If you aren't understanding something, please ask questions
• Don't get too stressed out if something is still a bit fuzzy, you are welcome to come to office hours
What are we doing?

- Essentially, we are figuring out how to build the image out of these sinusoids of different frequency.
- Each pixel in the transformed image corresponds to the amount of a sinusoid of a particular sinusoid that is needed.
What are High Frequencies?

What if we remove the high frequencies?

Old Spectrum

New Spectrum

How will the new image look?
What are High Frequencies?

Removing the high frequencies makes the image look blurry

Old Spectrum

New Spectrum

Try building a sharp edge out of low-frequency sinusoids
What are Low Frequencies?

What if we remove the low frequencies?

Old Spectrum  
New Spectrum

How will the new image look?
What are Low Frequencies?

What if we remove the low frequencies?

How will the new image look?
Working with the DFT (Discrete Fourier Transform)

- Is the complex part bothering you yet?
- Let's look at a different representation
- Every complex number can also be represented as

\[ z = x + jy = re^{j\theta} \]

- \( r \) – magnitude (real number)
- \( \theta \) - Phase
Phase and Magnitude

- Fourier transform of a real function is complex
  - difficult to plot, visualize
  - instead, we can think of the phase and magnitude of the transform

- Phase is the phase of the complex transform
- Magnitude is the magnitude of the complex transform

Curious fact
- all natural images have about the same magnitude transform
  hence, phase seems to matter, but magnitude largely doesn’t

Demonstration
- Take two pictures, swap the phase transforms, compute the inverse
  - what does the result look like?
This is the magnitude transform of the cheetah pic
This is the phase transform of the cheetah pic.
This is the magnitude transform of the zebra pic
This is the phase transform of the zebra pic
Reconstruction with zebra phase, cheetah magnitude
Reconstruction with cheetah phase, zebra magnitude
The Fourier Transform Helps Us Analyze Convolutions

• Notation: $f(x,y)$ is the signal, $F(u,v)$ is the DFT

• If $h = f * g \leftarrow$ Convolution
  – Then $H(u,v) = F(u,v)G(u,v)$
  – Convolution in the spatial domain is multiplication in the Fourier domain
  – You'll derive this in the problem set
Back to averaging

Remember:

\[
\begin{array}{ccc}
1/9 & 1/9 & 1/9 \\
1/9 & 1/9 & 1/9 \\
1/9 & 1/9 & 1/9 \\
\end{array}
\]

\*
Back to averaging

\[
\begin{bmatrix}
1/9 & 1/9 & 1/9 \\
1/9 & 1/9 & 1/9 \\
1/9 & 1/9 & 1/9 \\
\end{bmatrix}
\]
Filtering

- We pixel-wise multiply the DFT of the input image by the DFT of the filter
- Frequencies where the magnitude of the response of the filter are near zero (black in the images) will be eliminated
Take the log to rescale brightness

Unfiltered Spectrum

Log Spectrum after 3x3 averaging

Log-Spectrum after 7x7 averaging
Back to this example

With $\sigma$ set to 1

With $\sigma$ set to 3

Input
First, the filter

Magnitude of the DFT
After Filtering

X

=
Vocabulary

Low Pass Filter:

\[
\begin{array}{ccc}
1/9 & 1/9 & 1/9 \\
1/9 & 1/9 & 1/9 \\
1/9 & 1/9 & 1/9 \\
\end{array}
\]
Vocabulary

Band-pass Filter:
Vocabulary

High-pass Filter:
Now, let's compute some DFT's
Calculate the DFT of

\[ \text{Diagram of points.} \]
Step 1:

Write out the DFT

\[ F[u] = e^{-2\pi j \frac{u_0}{N}} + e^{-2\pi j \frac{u_1}{N}} + e^{-2\pi j \frac{u(N-1)}{N}} \]

Notice that we can rewrite the last term:

\[ e^{-2\pi j \frac{u(N-1)}{N}} \rightarrow e^{-2\pi j \frac{u N}{N}} + -2\pi j \frac{-u}{N} \]
What can we say about this term?

$$e^{-2\pi j \frac{uN}{N}} + -2\pi j \frac{-u}{N}$$

Remember periodicity!
So now we have

\[ F[u] = 1 + e^{-2\pi j \frac{u}{N}} + e^{-2\pi j \frac{u}{N}} \]

Now you can write this in terms of sin and cos

Quick Exercise: Show that

\[ e^{2\pi j \frac{u}{N}} + e^{-2\pi j \frac{u}{N}} = 2 \cos \left( 2\pi \frac{u}{N} \right) \]
DFT of this signal

Shifted, so 0 is at the center
Periodicity and the DFT

Assume you’ve computed the DFT and you want to compute a value in the original signal

Inverse DFT (or Synthesis Equation)

\[
x[n] = \sum_{k=0}^{N-1} X[k] e^{2\pi j \frac{kn}{N}}
\]
Periodicity and the DFT

Inverse DFT (or Synthesis Equation)

\[ x[n] = \sum_{k=0}^{N-1} X[k] e^{2\pi j \frac{kn}{N}} \]

What if we try to compute \( x[N+1] \)?

Using a DFT representation assumes that the signal is periodic.
Sampling

What's the wrong way to do this?
What's the right way?
Why?
Let's talk about the wrong way first

You want half as many pixels?
Take every other pixel
Really fast!
Analyzing what's going on

Fortunately, we have almost all of the pieces that we need to analyze this problem.

First, worksheet time
Convolution Solution

We need to compute the DFT of

\[ h[n] = f[n]g[n] \]

First Step, write DFT Equation:

\[ H[u] = \sum_{n=0}^{N-1} f[n]g[n]e^{-2\pi j \frac{un}{N}} \]
Write $g[n]$ in terms of the Inverse DFT

$$H[u] = \sum_{n=0}^{N-1} f[n] e^{-2\pi j \frac{un}{N}} \frac{1}{N} \sum_{m=0}^{N-1} G[m] e^{2\pi j \frac{mn}{N}}$$

Do some arithmetic

$$H[u] = \frac{1}{N} \sum_{m=0}^{N-1} G[m] \sum_{n=0}^{N-1} f[n] e^{-2\pi j \frac{n(u-m)}{N}}$$
\[ H[u] = \frac{1}{N} \sum_{m=0}^{N-1} G[m] \sum_{n=0}^{N-1} f[n] e^{-2\pi j \frac{n(u-m)}{N}} \]

- Rewrite as a convolution

\[ H[u] = \frac{1}{N} \sum_{m=0}^{N-1} G[m] F[u - m] \]

- Important fact: multiplying signals is the same as circular convolution of their DFT's!
Sampling

What is sampling?

Same thing as multiplying the signal (image) by a periodic impulse train
What does happen in the Fourier Domain?

The spectrum of the signal is convolved with an impulse train.
What does happen in the Fourier Domain?

If the signal has components with too high a frequency, then the copies will overlap.

Effectively, you are introducing high-frequency content that doesn't exist.
Aliasing

- This is called aliasing
  - Bad high frequencies accidentally introduced by sampling

- Two ways to solve
  - Take more samples
    - Spreads out the impulse train in the Fourier Domain
  - Make sure spectra don't overlap
    - Eliminate high-frequencies by blurring
Practical Example of Aliasing

http://www.michaelbach.de/ot/mot_strob/index.html
In Images, Aliasing is often called “Jaggies”
Practical consequences from my own work

Goal: Zoom up low-resolution images
Getting the blurry image

To verify the method is working, we generate the low-resolution image from the full resolution image

Makes a big difference in the result

Result from properly-antialiased image

Result from \textit{improperly}-antialiased image
What's going on?
The system is enhancing the unwanted aliasing!

Result from properly-antialiased image

Result from *improperly*-antialiased image
Aliasing in Graphics

- Aliased
- Anti-Aliased

UCF UCF
Practicalities of working with the DFT

If use `fft2` in MATLAB or Python on this image, I get

I get
Wait!

That doesn't look like what I've been showing you.

Remember periodicity.
What if you like low-frequencies in the center?

Both MATLAB and Numerical Python provide functions called `fftshift` and `ifftshift`.

These shift the image so that the low frequencies are at the center.
Back to the image sharpening example

Task: Design a filter to make things look sharper

How?

Remember, high-frequencies correspond to the “sharp stuff” in images

If we boost the high frequencies, the image should look sharper
As a reminder

Operations

1 convolution

1 subtraction over the whole image

As an equation:

$$\mathcal{I} \ast f + 2 \left( \mathcal{I} - \mathcal{I} \ast f \right)$$
In terms of DFT

This filter boosts the high-frequencies in the image