Bounding the Security Number of a Graph

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Abstract

Given a graph $G$, the security number of $G$ is the cardinality of a minimum secure set of $G$, the smallest set of vertices $S \subseteq V(G)$ such that for all $X \subseteq S$, $|N[X] \cap S| \geq |N[X] - S|$. It is believed to be computationally difficult to find the security number of large graphs, so we present techniques for reducing the difficulty of both finding a secure set and determining bounds on the security number.

Keywords: secure sets, divide and conquer, cut sets

1 Introduction

A popular class of algorithms, divide and conquer, decompose a problem instance into smaller, easier to solve subinstances and then combine the subsolutions into a complete solution. We apply a similar technique, decomposing graphs into subgraphs and determine the security number of the subgraphs to determine bounds on the security number of the original graph.

Let $G = (V, E)$ be a graph. Unless otherwise defined, we follow the terminology and notation of [6].
Given a graph $G$, $S \subseteq V(G)$ is a **defensive alliance** if for all $x \in S$, $|N[x] \cap S| \geq |N[x] - S|$ [5]. Brigham et al. describe a related concept, **secure sets**, based on the idea that defensive alliances do not accurately model real-world situations [1]. In this paper we define secure sets from Theorems 10 and 11 in [1].

**Theorem 1.** A subset $X \subseteq S$ is $S$-secure if and only if $|N[X] \cap S| \geq |N[X] - S|$. A set $S$ is secure if and only if every $X \subseteq S$ is $S$-secure.

### 2 Bounds and Characterizations

The **security number** of a graph $G$, $s(G)$, is the cardinality of a minimum secure set. There exist many bounds on $s(G)$, and in this paper we show how knowledge of the bound of a subgraph of a graph can help determine a bound for the security number of the graph [4, 2]. First we present additional criteria for a set to be secure.

**Theorem 2.** Given a graph $G$, $S \subseteq V(G)$ is secure if and only if $|S| \geq |N[S] - S|$ and every $X \subseteq S$ where $|X| \leq |S| - 1 - \delta(\langle S \rangle)$ is $S$-secure.

**Proof.** If $S$ is secure, it follows from the definition of secure that $|S| \geq |N[S] - S|$ and every $X \subseteq S$ is $S$-secure, so we need only prove sufficiency. Consider the contrapositive and assume $S$ is not secure. Then there exists an $X \subseteq S$ for which $X$ is not $S$-secure. If $X$ dominates $S$, then $|N[X] \cap S| = |S|$, and since $|N[X] - S| \leq |N[S] - S|$, $|S| < |N[S] - S|$. If $X$ does not dominate $S$, then there is a vertex $v \in S - X$ with at least $\delta(\langle S \rangle)$ neighbors in $S - X$. Thus $|X| \leq |S| - 1 - \delta(\langle S \rangle)$. Since $X$ is not $S$-secure, the conclusion follows.

**Theorem 3.** Given a graph $G$, let $C \subseteq V(G)$ be a clique, then $|N[C] - C| \leq |C|$ if and only if every $S \subseteq C$ is $C$-secure.

**Proof.** Sufficiency follows from Theorem 1, so we need only prove necessity. For all $S \subseteq C$, $|N[S] \cap C| = |C|$ $\geq |N[C] - C| \geq |N[S] - C|$, so $S$ is $C$-secure.

Theorem 3 shows that a single clique forms a secure set if there are as many vertices in the clique as in its boundary. We show a similar result in Theorem 4 for a disjoint union of cliques.
Theorem 4. For a graph $G$, given $S \subseteq V(G)$, let $C_1, \ldots, C_t$ be a partition of $S$ where each $C_i, 1 \leq i \leq t$, induces a complete subgraph in $G$. If $|C_i| \geq |N[C_i] - S|$ for $1 \leq i \leq t$, then $S$ is secure.

Proof. Let $X \subseteq S$. If $X \subseteq C_i$ for some $i, 1 \leq i \leq t$, then $|N[X] \cap S| \geq |C_i| \geq |N[C_i] - S| \geq |N[X] - S|$. Hence we may assume for some $k, 2 \leq k \leq t$, that $X_i \cap C_i = \emptyset$ if and only if $k < i \leq t$. Then $|N[X] \cap S| \geq \sum_{j=1}^{k} |C_j|$. Also, $|N[X] - S| \leq \left|\bigcup_{j=1}^{k} N[C_j] - S\right| \leq \sum_{j=1}^{k} |C_j - S| \leq \sum_{j=1}^{k} |C_i|$. Thus $|N[X] \cap S| \geq |N[X] - S|$ for all $X \subseteq S$, so by Theorem 1, $S$ is secure.

In the next section we consider the security number of blocks of graphs, where we require knowledge about the structure of the blocks. If we know just the number of blocks in a graph, we find the following upper bound on security number, which depends on the following lemma.

Lemma 5. Given a graph $G$ of order $n$ with $b \geq 2$ blocks, some block containing a single cut vertex has at most $\left\lfloor \frac{n - b + 3}{2} \right\rfloor$ vertices.

Proof. When $b = 2$ there are exactly two blocks, each containing only one cut vertex. Thus one of the blocks must not have more than $\left\lfloor \frac{n + 1}{2} \right\rfloor$ vertices. When $b > 2$, there are at least two blocks, each containing only one cut vertex. Assume these two blocks each have at least $\left\lfloor \frac{n - b + 3}{2} \right\rfloor + 1$ vertices. Then the remaining $b - 2$ blocks have at most $n + b - 1 - 2(\left\lfloor \frac{n - b + 3}{2} \right\rfloor + 1) \leq n + b - 1 - 2\left(\left\lfloor \frac{n - b + 3}{2} \right\rfloor \right) - 2 \leq n + b - 1 - n + b - 2 - 2 = 2b - 5 < 2(b - 2)$ vertices. Thus at least one of the remaining $b - 2$ blocks has less than 2 vertices, a contradiction, since all blocks must have at least 2 vertices.

Theorem 6. Let $G$ be a connected graph of order $n$ with $b \geq 2$ blocks. Then $s(G) \leq \left\lfloor \frac{n - b}{2} \right\rfloor$ and equality holds when $G$ is a tree.

Proof. Among the blocks of $G$, at least two contain exactly one cut vertex, and at least one of these has $m \leq \left\lfloor \frac{n - b + 3}{2} \right\rfloor$ vertices by Lemma 5. Any set of $m - 1$ vertices in this block is a secure set for the block and, if the cut vertex is avoided (it is always possible to avoid any specific vertex in forming a secure set), a secure set for $G$. Thus, $s(G) \leq \left\lfloor \frac{n - b + 3}{2} \right\rfloor - 1 = \left\lfloor \frac{n - b}{2} \right\rfloor$. If $G$ is a tree, $b = n - 1$, so $s(G) = 1$. 

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3 Divide and Conquer

The complexity of determining the security number of a graph is believed to be high [3]. Determining the security number of small graphs, however, is easy compared to determining the security number of large graphs. Thus in some cases by examining selected subgraphs it is possible to bound the security number of a large graph. For example, consider a regular graph $G$ of degree $m$. If we find a set $S \subseteq V$ such that $⟨S⟩ ≅ K_m$, then $S$ is secure because each vertex in $S$ has one neighbor outside of $S$, so $s(G) ≤ m$. This idea is further illustrated in the next theorem.

**Theorem 7.** Given a graph $G$, $H$ a subgraph of $G$, and $S \subseteq V(H)$ a secure set of $H$, then if for all $v \in S$, $N[v] \subseteq V(H)$ then $s(G) ≤ |S|$.

**Proof.** When $N[v] \subseteq V(H)$ for all $v \in S$, $N[X] \subseteq V(H)$ for all $X \subseteq S$, $N[X]$ is identical in $G$ and $H$. Therefore, $N[X] \cap S$ is the same in $G$ and $H$. That is, if $|N[X] \cap S| ≥ |N[X] − S|$ in $H$, then $|N[X] \cap S| ≥ |N[X] − S|$ in $G$. Thus, if $S$ is secure in $H$, $S$ is secure in $G$, so $s(G) ≤ |S|$.

Consider the subgraphs formed by removing a disconnecting set of vertices. The next result follows immediately from Theorem 7.

**Corollary 8.** Given a graph $G$ with disconnecting set $S \subseteq V(G)$, $⟨V(G) − S⟩$ is a collection of connected components $C_1, \ldots, C_m$. Let $R_1 \subseteq C_1, \ldots, R_m \subseteq C_m$ be secure sets of $⟨C_1⟩, \ldots, ⟨C_m⟩$. If $R_i \cap S = \emptyset$ for at least one $i, 1 \leq i \leq m$, then $s(G) ≤ \min\{|R_i| \text{ such that } R_i \cap S = \emptyset\}$.

By considering blocks, we can get equality for the above bound in some cases.

**Theorem 9.** Let $G$ be a graph with a single cut vertex $v$ and blocks $H$ and $K$. Suppose $R$ and $S$ are minimum secure sets in $H$ and $K$, respectively. If $v \notin R \cup S$, then $s(G) = \min\{s(H), s(K)\}$.

**Proof.** From Corollary 8, $s(G) ≤ \min\{s(H), s(K)\}$, thus we need only show that $s(G) ≥ \min\{s(H), s(K)\}$, which we prove by contradiction. Assume there exists a secure set $T \subseteq V(G)$ such that $|T| = s(G) < \min\{s(H), s(K)\}$. Since $R$ and $S$ are minimal, $T$ cannot be entirely contained within any block, thus $T$ must contain $v$, vertices from $H − v$, and $K − v$, and so $s(G) ≥ \min\{s(H), s(K)\}$. Thus, $s(G) = \min\{s(H), s(K)\}$. 


vertices from \( K - v \). Since \( T - V(H) \) is not secure in \( K \) by assumption, \( T - V(H) \) must be \( T \)-secure in \( K \), thus \( T - V(H) \cup \{v\} \) is secure in \( K \). Similarly, \( T - V(K) \) must be \( T \)-secure in \( H \) and \( T - V(K) \cup \{v\} \) is secure in \( H \). Hence, \(|T - V(H) \cup \{v\}| \geq s(K)\) and \(|T - V(K) \cup \{v\}| \geq s(H)\). Thus, \(|T - V(H) \cup \{v\}| + |T - V(K) \cup \{v\}| \geq s(H) + s(K)\). On the other hand, \(|T - V(H) \cup \{v\}| + |T - V(K) \cup \{v\}| = |T| + 1 \leq \min\{s(H), s(K)\}\). That is, \(\min\{s(H), s(K)\} \geq s(H) + s(K)\), a contradiction since this implies \(s(H)\) or \(s(K)\) is non-positive. Thus \(T\) must not exist, so \(s(G) \geq \min\{s(H), s(K)\}\), therefore \(s(G) = \min\{s(H), s(K)\}\).

We can extend the previous results to include graphs with multiple blocks. Consider a graph \( G \) with cut vertices \( v_1, v_2, ..., v_m, m \geq 2 \) and blocks \( B_1, ..., B_b, b \geq m + 1 \). If no \( S_j \) contains a cut vertex, where \( S_j \) is a minimum secure set of \( \langle B_j \rangle \), then \(s(G) = \min\{s(B_1), ..., s(B_b)\}\). If there exists at least one \( S_j \) that contains no cut vertex, then \(s(G) \leq |S_j|\).

In this paper we considered only cut sets to select subgraphs to bound the security number of graphs. It remains an open area of research to find other ways of selecting subgraphs.

References


