Technical Report:

Robust Object Pose Estimation via Statistical Manifold Modeling

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Proposition 1. Given Equation 2, the distance between two object poses O_i^j and O_m^n can be computed as,

$$D\left(O_i^j, O_m^n\right) = E_X(D_{KL}(P_i^j(\boldsymbol{A}|\boldsymbol{X})||P_m^n(A|\boldsymbol{X})) + D_{KL}(P_i^j(\boldsymbol{X})||P_m^n(\boldsymbol{X}))$$

Proof:
Suppose that we have two PDFs, $P(A,X)$ and $Q(A,X)$,

$$D_{KL}(P(A,X)||Q(A,X)) \equiv \iint P(A,X)\log\frac{P(A,X)}{Q(A,X)}dXdA$$
$$= \iint P(A|X)P(X)\log\frac{P(A|X)P(X)}{Q(A|X)Q(X)}dXdA$$
$$= \iint P(A|X)P(X)\log\frac{P(A|X)}{Q(A|X)}dXdA + \iint P(A|X)P(X)\log\frac{P(X)}{Q(X)}dXdA$$
$$= \int P(X)\left\{\int P(A|X)\log\frac{P(A|X)}{Q(A|X)}dA\right\}dX + \int\left\{\int P(A|X)dA\right\}P(X)\log\frac{P(X)}{Q(X)}dX$$
$$= E_X\left\{\int P(A|X)\log\frac{P(A|X)}{Q(A|X)}dA\right\} + \int P(X)\log\frac{P(X)}{Q(X)}dX$$
$$= E_X\{D_{KL}(P(A|X)||Q(A|X))\} + D_{KL}(P(X)||Q(X))$$

Combining the above derivation and Equation (2), i.e.,
$D\left(O_i^j, O_m^n\right) = D_{KL}(P_i^j(A,X)||P_m^n(A,X))$, then we have,

$$D\left(P_i^j(O), P_m^n(O)\right) = E_X(D_{KL}(P_i^j(\boldsymbol{A}|\boldsymbol{X})||P_m^n(A|\boldsymbol{X})) + D_{KL}(P_i^j(\boldsymbol{X})||P_m^n(\boldsymbol{X}))$$

Proposition 2. If $A_1, \dots, A_K$ are independent given **X**, and a one-to-one part correspondence $f_{map}\left(Q_k^{i,j}\right) = Q_k^{m,n}$ exists, then the following equation holds,

$$E_X\left\{D_{KL}(P_i^j(\boldsymbol{A}|\boldsymbol{X})||P_m^n(\boldsymbol{A}|\boldsymbol{X}))\right\} = E_X(\sum_{k=1}^{K} D_{KL}(P_i^j(A_k|\boldsymbol{X})||P_m^n(A_k|\boldsymbol{X})))$$

where $P_i^j(A_k|\boldsymbol{X})$ is the conditional PDF of part $A_k$given $\boldsymbol{X}$**.**

Proof:

In the proof of Proposition 1, if we further assume that A and X are independent, then we have $D_{KL}(P(A,X)||Q(A,X)) = E_X\{D_{KL}(P(A|X)||Q(A|X))\} + D_{KL}(P(X)\|Q(X)\big) = D_{KL}(P(A)||Q(A)) + D_{KL}(P(X)||Q(X))$.
Based on this observation, for two PDFs $P(X)$ and $Q(X)$ and $X = (X_1, X_2, \dots, X_n)$, where X_i and X_j are independent, we can obtain $D_{KL}(P(X)||Q(X)) = \sum_{i=1}^{n} D_{KL}(P(X_i)||Q(X_i))$.
Since $A_1, \dots, A_K$ are independent given **X**, then,

$$E_X\{D_{KL}(P_i^j(\boldsymbol{A}|\boldsymbol{X})||P_m^n(\boldsymbol{A}|\boldsymbol{X}))\} = E_X(\sum_{k=1}^{K} D_{KL}(P_i^j(A_k|\boldsymbol{X})||P_m^n(A_k|\boldsymbol{X})))$$