Example of Chaotic Iteration

1 The Dataflow Equations as a Function

Consider the Reaching Definitions (RD) analysis for the program shown at the beginning of section 1.3.1 of our textbook [1], which is also given in Figure 1.2. For purposes of using the Chaotic Iteration algorithm (from section 1.7), we formulate the dataflow equations for this program as a function

$$F : (\mathcal{P}(\mathbf{Var}_\star \times \mathbf{Lab}_\star^?))^{12} \to (\mathcal{P}(\mathbf{Var}_\star \times \mathbf{Lab}_\star^?))^{12},$$

The function F is defined pointwise by the following

$$F(\vec{RD}) = (F_1(\vec{\mathsf{RD}}), F_2(\vec{\mathsf{RD}}), \ldots, F_{12}(\vec{RD})), \tag{1}$$

where the F_i are defined as follows.

$$\begin{aligned}
F_1(\mathsf{RD}_1, \mathsf{RD}_2, \ldots, \mathsf{RD}_{12}) &= \{(x, ?) \mid x \in \mathbf{Var}_\star\}\\
F_2(\mathsf{RD}_1, \mathsf{RD}_2, \ldots, \mathsf{RD}_{12}) &= (\mathsf{RD}_1 \setminus \{(\mathtt{y}, \ell) \mid \ell \in \mathbf{Lab}_\star^?\}) \cup \{(\mathtt{y}, 1)\}\\
F_3(\mathsf{RD}_1, \mathsf{RD}_2, \ldots, \mathsf{RD}_{12}) &= \mathsf{RD}_2\\
F_4(\mathsf{RD}_1, \mathsf{RD}_2, \ldots, \mathsf{RD}_{12}) &= (\mathsf{RD}_3 \setminus \{(\mathtt{z}, \ell) \mid \ell \in \mathbf{Lab}_\star^?\}) \cup \{(\mathtt{z}, 2)\}\\
F_5(\mathsf{RD}_1, \mathsf{RD}_2, \ldots, \mathsf{RD}_{12}) &= \mathsf{RD}_4 \cup \mathsf{RD}_{10}\\
F_6(\mathsf{RD}_1, \mathsf{RD}_2, \ldots, \mathsf{RD}_{12}) &= \mathsf{RD}_5\\
F_7(\mathsf{RD}_1, \mathsf{RD}_2, \ldots, \mathsf{RD}_{12}) &= \mathsf{RD}_6\\
F_8(\mathsf{RD}_1, \mathsf{RD}_2, \ldots, \mathsf{RD}_{12}) &= (\mathsf{RD}_7 \setminus \{(\mathtt{z}, \ell) \mid \ell \in \mathbf{Lab}_\star^?\}) \cup \{(\mathtt{z}, 4)\}\\
F_9(\mathsf{RD}_1, \mathsf{RD}_2, \ldots, \mathsf{RD}_{12}) &= \mathsf{RD}_8\\
F_{10}(\mathsf{RD}_1, \mathsf{RD}_2, \ldots, \mathsf{RD}_{12}) &= (\mathsf{RD}_9 \setminus \{(\mathtt{y}, \ell) \mid \ell \in \mathbf{Lab}_\star^?\}) \cup \{(\mathtt{y}, 5)\}\\
F_{11}(\mathsf{RD}_1, \mathsf{RD}_2, \ldots, \mathsf{RD}_{12}) &= \mathsf{RD}_6\\
F_{12}(\mathsf{RD}_1, \mathsf{RD}_2, \ldots, \mathsf{RD}_{12}) &= (\mathsf{RD}_{11} \setminus \{(\mathtt{y}, \ell) \mid \ell \in \mathbf{Lab}_\star^?\}) \cup \{(\mathtt{y}, 6)\}
\end{aligned}$$

To understand the correspondence of these functions to the dataflow equations in section 1.3.1, note that if $1 \leq \ell \leq 6$, then $F_{2\cdot\ell-1}(\mathbf{fix}(F)) = \mathsf{RD}_{\mathit{entry}}(\ell)$ and $F_{2\cdot\ell}(\mathbf{fix}(F)) = \mathsf{RD}_{\mathit{exit}}(\ell)$, where $\mathbf{fix}(F)$ is the fixed-point of F, as calculated below.

2 Chaotic Iteration Calculation

In the following, we calculate the dataflow information for the example program in section 1.3.1 (Figure 1.2), using the Chaotic Iteration algorithm. We represent the steps of the Chaotic Iteration algorithm using the symbol $\rightsquigarrow$. This calculation makes use of the following abbreviations.

$$\begin{aligned}
\mathit{x?y?z?} &= \{(\mathtt{x}, ?), (\mathtt{y}, ?), (\mathtt{z}, ?)\}\\
\mathit{y1} &= \{(\mathtt{x}, ?), (\mathtt{y}, 1), (\mathtt{z}, ?)\}\\
\mathit{y1z2} &= \{(\mathtt{x}, ?), (\mathtt{y}, 1), (\mathtt{z}, 2)\}\\
\mathit{y1z4} &= \{(\mathtt{x}, ?), (\mathtt{y}, 1), (\mathtt{z}, 4)\}\\
\mathit{y5z4} &= \{(\mathtt{x}, ?), (\mathtt{y}, 5), (\mathtt{z}, 4)\}\\
\mathit{y15z24} &= \{(\mathtt{x}, ?), (\mathtt{y}, 1), (\mathtt{y}, 5), (\mathtt{z}, 2), (\mathtt{z}, 4)\}\\
\mathit{y15z4} &= \{(\mathtt{x}, ?), (\mathtt{y}, 1), (\mathtt{y}, 5), (\mathtt{z}, 4)\}\\
\mathit{y6z24} &= \{(\mathtt{x}, ?), (\mathtt{y}, 6), (\mathtt{z}, 2), (\mathtt{z}, 4)\}
\end{aligned}$$

The calculation at each step always uses the definition of the lowest index i such that $\mathsf{RD}_i \neq F_i(\vec{\mathsf{RD}})$.

$$\vec{\mathsf{RD}} = \vec{\emptyset}$$

$\rightsquigarrow$ $\langle$by definition of F_1, $\vec{\emptyset}_1 \neq F_1(\vec{\emptyset})\rangle$
$\vec{\mathsf{RD}} = (x?y?z?, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset)$
$\rightsquigarrow$ $\langle$by definition of $F_2\rangle$
$\vec{\mathsf{RD}} = (x?y?z?, y1, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset)$
$\rightsquigarrow$ $\langle$by definition of $F_3\rangle$
$\vec{\mathsf{RD}} = (x?y?z?, y1, y1, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset)$
$\rightsquigarrow$ $\langle$by definition of $F_4\rangle$
$\vec{\mathsf{RD}} = (x?y?z?, y1, y1, y1z2, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset)$
$\rightsquigarrow$ $\langle$by definition of F_5, note $\mathsf{RD}_{10} = \emptyset\rangle$
$\vec{\mathsf{RD}} = (x?y?z?, y1, y1, y1z2, y1z2, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset)$
$\rightsquigarrow$ $\langle$by definition of $F_6\rangle$
$\vec{\mathsf{RD}} = (x?y?z?, y1, y1, y1z2, y1z2, y1z2, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset)$
$\rightsquigarrow$ $\langle$by definition of $F_7\rangle$
$\vec{\mathsf{RD}} = (x?y?z?, y1, y1, y1z2, y1z2, y1z2, y1z2, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset)$
$\rightsquigarrow$ $\langle$by definition of $F_8\rangle$
$\vec{\mathsf{RD}} = (x?y?z?, y1, y1, y1z2, y1z2, y1z2, y1z2, y1z4, \emptyset, \emptyset, \emptyset, \emptyset)$
$\rightsquigarrow$ $\langle$by definition of $F_9\rangle$
$\vec{\mathsf{RD}} = (x?y?z?, y1, y1, y1z2, y1z2, y1z2, y1z2, y1z4, y1z4, \emptyset, \emptyset, \emptyset)$
$\rightsquigarrow$ $\langle$by definition of $F_{10}\rangle$
$\vec{\mathsf{RD}} = (x?y?z?, y1, y1, y1z2, y1z2, y1z2, y1z2, y1z4, y1z4, y5z4, \emptyset, \emptyset)$
$\rightsquigarrow$ $\langle$by definition of $F_5\rangle$
$\vec{\mathsf{RD}} = (x?y?z?, y1, y1, y1z2, y15z24, y1z2, y1z2, y1z4, y1z4, y5z4, \emptyset, \emptyset)$
$\rightsquigarrow$ $\langle$by definition of $F_6\rangle$
$\vec{\mathsf{RD}} = (x?y?z?, y1, y1, y1z2, y15z24, y15z24, y1z2, y1z4, y1z4, y5z4, \emptyset, \emptyset)$
$\rightsquigarrow$ $\langle$by definition of $F_7\rangle$
$\vec{\mathsf{RD}} = (x?y?z?, y1, y1, y1z2, y15z24, y15z24, y15z24, y1z4, y1z4, y5z4, \emptyset, \emptyset)$
$\rightsquigarrow$ $\langle$by definition of $F_8\rangle$
$\vec{\mathsf{RD}} = (x?y?z?, y1, y1, y1z2, y15z24, y15z24, y15z24, y15z4, y1z4, y5z4, \emptyset, \emptyset)$
$\rightsquigarrow$ $\langle$by definition of $F_9\rangle$
$\vec{\mathsf{RD}} = (x?y?z?, y1, y1, y1z2, y15z24, y15z24, y15z24, y15z4, y15z4, y5z4, \emptyset, \emptyset)$
$\rightsquigarrow$ $\langle$by definition of $F_{11}\rangle$
$\vec{\mathsf{RD}} = (x?y?z?, y1, y1, y1z2, y15z24, y15z24, y15z24, y15z4, y15z4, y5z4, y15z24, \emptyset)$
$\rightsquigarrow$ $\langle$by definition of $F_{12}\rangle$
$\vec{\mathsf{RD}} = (x?y?z?, y1, y1, y1z2, y15z24, y15z24, y15z24, y15z4, y15z4, y5z4, y15z24, y6z24)$

At this point no more steps are possible, so the fixed point of F is as in the last formula.

References

[1] Flemming Nielson, Hanne Riis Nielson, and Chris Hankin. *Principles of Program Analysis*. Springer-Verlag, second printing edition, 2005.