EXPERIMENT 5
Frequency Modulation

1 Objective

To understand the principles of frequency modulation and demodulation.

2 Theory

2.1 Introduction

A sinusoidal carrier \( c(t) = A \cos(\omega_c t + \theta_0) \) has three parameters that can be modified (modulated) according to an information signal \( f(t) \).

1. Its amplitude \( A \), which leads us to the class of systems designated as \textit{amplitude modulating} (AM) systems.

2. Its frequency \( \omega_c \), which leads us to a class of systems designated as \textit{frequency modulating} (FM) systems.

3. Its phase \( \theta_0 \), which leads us to a class of systems designated as \textit{phase modulating} (PM) systems.

We have already discussed the class of AM systems. In the sequel we focus on the class of FM and PM systems. Note that we can write that

\[
c(t) = A \cos(\omega_c t + \theta_0) = A \cos(\theta(t))
\]

where \( \theta(t) \) is often called the angle of the sinusoid. That's why FM and PM systems are sometimes referred to as \textit{angle modulating} systems.

2.2 Preliminary notions of FM and PM Systems

Consider the carrier \( c(t) = A \cos(\omega_c t + \theta_0) \). We can write

\[
c(t) = A \cos(\theta(t))
\]
where we call \( \theta(t) \) the *instantaneous phase* of the carrier. If we differentiate \( \theta(t) \) with respect to time \( t \) we get a time function that we designate by \( \omega(t) \) and we call it the *instantaneous frequency*. That is

\[
\omega(t) = \frac{d\theta}{dt}
\]  

(5.3)

It is easy to see that the above definition of instantaneous frequency makes sense if we apply it to a pure carrier \( c(t) = A \cos(\omega_c t + \theta_0) \), because then we get

\[
\omega(t) = \omega_c
\]  

(5.4)

From equation (5.3), above, we see that if you have the instantaneous phase of a sinusoid, you can compute its instantaneous frequency by differentiation. Furthermore, if you know the instantaneous frequency \( \omega(t) \) of a sinusoid you can compute its instantaneous phase by integration as follows:

\[
\theta(t) = \int_0^t \omega(\tau) d\tau + \theta_0
\]  

(5.5)

Obviously, if we start with \( \omega(t) = \omega_c \) we get

\[
\theta(t) = \omega_c t + \theta_0
\]  

(5.6)

Phase and frequency modulations are techniques that modify the instantaneous phase and frequency, respectively, of a sinusoid in a way dictated by an information signal \( f(t) \).

### 2.3 Phase Modulation

Here the information signal \( f(t) \) is placed as a linear term in the instantaneous phase of the carrier. That is

\[
\theta(t) = \omega_c t + \theta_0 + k_p f(t)
\]  

(5.7)

where \( k_p \) is a constant of the modulating device. Hence, the PM modulated signal is equal to

\[
m_p(t) = A \cos(\omega_c t + \theta_0 + k_p f(t))
\]  

(5.8)

### 2.4 Frequency Modulation

Here the information signal gets inserted as a linear term into the instantaneous frequency of the carrier. That is,

\[
\omega(t) = \omega_c + k_f f(t)
\]  

(5.9)
where \( k_f \) is a constant due to the modulator. In this case the instantaneous phase is equal to

\[
\theta(t) = \omega_t t + \theta_0 + k_f \int_0^t f(\tau) d\tau
\]

and as a result, the FM modulated signal looks like

\[
m_f(t) = A \cos \left( \omega_t t + \theta_0 + k_f \int_0^t f(u) du \right)
\]

A plot of FM and PM signals is shown in Figure 5.1.

Figure 5.1 Examples of frequency and phase modulation
Now let us discuss FM and PM simultaneously and get a better insight into their similarities and differences. For simplicity assume that $\theta_0 = 0$. Then,

$$m_p(t) = A \cos(\omega_c t + k_p f(t))$$  \hfill (5.12)

and

$$m_f(t) = A \cos\left(\omega_c t + k_f \int_0^t f(u) du\right)$$  \hfill (5.13)

Let us take $m_p(t)$ and find its instantaneous frequency $\omega(t)$. Indeed,

$$\omega(t) = \omega_c + k_p \frac{df(t)}{dt}$$  \hfill (5.14)

The above equation tells us that in the PM case the instantaneous frequency has a linear term proportional to the derivative of the information signal $f(t)$. So, we can say that the PM case is in reality the FM case but with an information signal being the derivative of the actual information signal. In other words, if we have a device that produces FM signals we can make it to produce PM signals by giving it as an input the derivative of the information signal. The above equation also tells us that the FM case is in reality a PM case with the modulating signal being the integral of the information signal. Also, if we have a device that produces PM, we can make it to produce FM by providing to it as an input the integral of the information signal.

Hence, what it boils down to is that we need to discuss either PM or FM and not both. We choose to focus on FM, which is used to transmit baseband analog signals, such as speech or music. PM is primarily used in transmitting digital signals.

Our primary focus in the examination of FM signals will be the analysis of its frequency characteristics. Although it has been a straightforward task to find the Fourier transform of an AM signal the same is not true for FM signals. Let us again consider the general form of an FM signal

$$m_f(t) = A \cos(\omega_c t) \cos(k_f g(t)) - A \sin(\omega_c t) \sin(k_f g(t))$$  \hfill (5.15)

Where $g(t) = \int_0^t f(u) du$. If we are able to find the FT of $\cos(k_f g(t))$ and $\sin(k_f g(t))$ we can produce the FT of the signal $m_f(t)$ without a lot of effort. The FT of $\cos(k_f g(t))$ and $\sin(k_f g(t))$ cannot be found for any $g(t)$ and in fact it has been found for few $f(t)$’s. To get a deeper insight consider $\cos(k_f g(t))$ and expand it in terms of its Taylor series.

$$\cos(k_f g(t)) = 1 - \frac{k_f^2 g^2(t)}{2!} + \frac{k_f^4 g^4(t)}{4!} - \frac{k_f^6 g^6(t)}{6!} + \cdots$$  \hfill (5.16)
If the signal \( f(t) \) is known, its Fourier transform \( F(\omega) \) is also known, and the Fourier transform of \( g(t) \) can be computed. In particular, from well known Fourier transform properties we can deduce that

\[
\begin{align*}
g^2(t) &\rightarrow G(\omega) \ast G(\omega) \\
g^4(t) &\rightarrow G(\omega) \ast G(\omega) \ast G(\omega) \ast G(\omega) \\
&\vdots \\
\end{align*}
\]

Based on the above two equations (5.16) and (5.17) we can state that to compute the Fourier transform of

\[
\cos(k_j g(t))
\]

for an arbitrary signal \( g(t) \) becomes a formidable task (we need to compute a lot of convolutions). Furthermore, it seems that the bandwidth of \( \cos(k_j g(t)) \) is infinite (note that every time we multiply a signal with itself in the time-domain we double its bandwidth in the frequency domain). Hence if the bandwidth of \( g(t) \) is \( \omega_m \), the bandwidth of \( g^2(t) \) is \( 2\omega_m \), the bandwidth \( g^4(t) \) is \( 4\omega_m \), and so on. In reality though not all the terms in the Taylor series expansion of \( \cos(k_j g(t)) \) contribute equally to the determination of the signal \( \cos(k_j g(t)) \). Notice that in the Taylor series expansion of \( \cos(k_j g(t)) \) the coefficients multiplying the powers of \( g(t) \) get smaller and smaller. This observation will leads us into the conclusion that the bandwidth of the terms \( \cos(k_j g(t)) \) and \( \sin(k_j g(t)) \) is indeed finite, and as a result the bandwidth of the FM signal \( m_j(t) \) is also finite.

To investigate the bandwidth of an FM signal thoroughly we will discriminate two cases of FM signals: The case of Narrowband FM and the case of Wideband FM.

### 2.5 Narrowband FM

Consider again the FM signal \( m_j(t) \) given by the following equation.

\[
m_j(t) = A\cos(\omega \cdot t)\cos(k_j g(t)) - A\sin(\omega \cdot t)\sin(k_j g(t))
\]

The terms for which FT is difficult to evaluate are: \( \cos(k_j g(t)) \) and \( \sin(k_j g(t)) \). Each one of these terms has a Taylor series expansion involving infinitely many terms. Let us see what happens if each one of these terms is approximated only by their first term in the Taylor series expansion. Then,

\[
\begin{align*}
\cos(k_j g(t)) &\approx 1 \\
\sin(k_j g(t)) &\approx k_j g(t)
\end{align*}
\]

Obviously, if we make the above substitutions in equation (5.18) we get

\[
m_j(t) = A\cos(\omega \cdot t) - Ak_j g(t)\sin(\omega_j t)
\]
The advantage of the above equation is that we can evaluate its FT, and consequently the FT of \( m_f(t) \). It is not difficult to see that the bandwidth of \( m_f(t) \) is approximately equal to 2 times the bandwidth of \( f(t) \) (\( f(t) \) is the information signal). Hence, when the above approximations are accurate we are generating an FM signal whose bandwidth is approximately equal to the bandwidth of an AM signal. Since, in most cases an FM signal will occupy much more bandwidth than an AM signal, the aforementioned type of FM signal is called narrowband FM.

In the sequel, we are going to identify (quantitatively) conditions under which we can call an FM signal narrowband. These conditions will be a byproduct of our discussion of wideband FM systems.

### 2.6 Wideband FM

To illustrate the ideas of wideband FM let us start with the simplest of cases where the information signal is a single sinusoid. That is,

\[
f(t) = a \cos(\omega_m t)
\]

(5.21)

Then, the instantaneous frequency of your FM signal takes the form:

\[
\omega(t) = \omega_c + k_f a \cos(\omega_m t)
\]

(5.22)

Integrating the instantaneous frequency \( \omega(t) \) we obtain the instantaneous phase \( \theta(t) \):

\[
\theta(t) = \omega_c t + \frac{k_f a}{\omega_m} \sin(\omega_m t)
\]

(5.23)

Consequently, the FM modulated wave is

\[
m_f(t) = A \cos \left( \omega_c t + \frac{k_f a}{\omega_m} \sin(\omega_m t) \right)
\]

(5.24)

The quantity \( k_f a/\omega_m \) is denoted by \( \beta \) and it is referred to as the modulation index of the FM system. Let us now write the above expression for the signal \( m_f(t) \) in a more expanded form.

\[
m_f(t) = A \cos(\omega_c t) \cos(\beta \sin(\omega_m t)) - A \sin(\omega_c t) \sin(\beta \sin(\omega_m t))
\]

(5.25)

To simplify our discussion, from now on, we will be referring to the quantity \( \cos(\beta \sin(\omega_m t)) \) as term \( A \) and to the quantity \( \sin(\beta \sin(\omega_m t)) \) as term \( B \). Terms \( A \) and \( B \) are the real and the imaginary part of the following complex exponential function

\[
e^{i\beta \sin(\omega_m t)}
\]

(5.26)
which is a periodic function with period \( \frac{2\pi}{\omega_m} \). The above function can be expanded as an exponential Fourier series, as follows:

\[
e^{j\beta \sin \omega_m t} = \sum_{n=-\infty}^{\infty} C_n e^{j n \omega_m t}
\]

(5.27)

where the coefficients \( C_n \) are calculated by the equation:

\[
C_n = \frac{\omega_m}{2\pi} \int_{-\pi/\omega_m}^{\pi/\omega_m} e^{j(\beta \sin \omega_m t - n \omega_m t)} dt
\]

(5.28)

Let us now make a substitution of variables in the above equation. In particular, let us substitute \( \omega_m t \) with \( x \). Then,

\[
C_n = \frac{1}{2\pi} \int_{\pi}^{\pi} e^{j(\beta \sin x - nx)} dx
\]

(5.29)

The above integral cannot be evaluated in closed form, but it has been extensively calculated for various values of \( \beta \)'s and most \( n \)'s of interest. It has a special name, called the \( n \)th order Bessel function of the first kind and argument \( \beta \). This function is commonly denoted by the symbol \( J_n(\beta) \). Therefore, we can write:

\[
C_n = J_n(\beta)
\]

(5.30)

As a result,

\[
e^{j\beta \sin \omega_m t} = \sum_{n=-\infty}^{\infty} J_n(\beta)e^{j n \omega_m t}
\]

(5.31)

often called the Bessel Jacobi equation. If we evaluate the real and imaginary parts of the right hand side of equation (5.31) we will be able to calculate term A and term B, respectively. It turns out that if we substitute these values for term A and term B in the original equation for \( m_j(t) \) (see equation (5.25)) we will end up with

\[
m_j(t) = \sum_{n=-\infty}^{\infty} J_n(\beta)\cos(\omega_c + n \omega_m)t
\]

(5.32)

The property of the Bessel function coefficients (actually Property 1) that led us to the above results is listed below. Some additional properties of the Bessel function coefficients are also listed.

1. \( J_n(\beta) \) is real valued.

2. \( J_n(\beta) = J_{-n}(\beta) \) for \( n \) even.
3. \( J_n(\beta) = -J_{-n}(\beta) \) for \( n \) odd.

4. \( \sum_{n=-\infty}^{\infty} J_n^2(\beta) = 1 \)

The advantage of equation (5.32) compared to the original equation (5.24) that defined the FM signal \( m_f(t) \) is that now, through equation (5.32), we can compute the FT of the signal \( m_f(t) \). It will consist of an infinite sequence of impulses located at positions \( \omega_c + n\omega_m \), where \( n \) is an integer. In reality though, no matter how big \( \beta \) is, the significant \( J_n(\beta) \)'s will be only for indices \( n \leq \beta + 1 \). Hence, the approximate bandwidth of your FM signal, when the information signal is of sinusoidal nature, is given by the following equation.

\[
B = 2(\beta + 1)\omega_m
\]  

(5.33)

In Figure 5.2 various plots of the Bessel function coefficients \( J_n(\beta) \) are shown. As we can see these plots verify our claim, above, that \( J_n(\beta) \) become small for indices \( n > \beta + 1 \). In Figures 5.3 we show the FT of signals \( m_f(t) \) for various \( \beta \) values.

![Figure 5.2 Plot of Bessel function of the first kind, \( J_n(\beta) \)](image-url)
Figure 5.3: Magnitude line spectra for FM waveforms with sinusoidal modulation
(a) for constant $\omega_m$; (b) for constant $\Delta\omega$

In Table 5.1 the values of the Bessel function coefficients $J_n(\beta)$ are shown for various $\beta$ values. We can use the values of the Table 5.1 to evaluate the bandwidth of the signal $m_f(t)$ as follows. We are still operating under the assumption that the information signal is of sinusoidal nature. As a result, expression (5.32) is a valid representation of our signal $m_f(t)$. Let us now impose the criterion that for the evaluation of the bandwidth of the signal $m_f(t)$ we are going to exclude all terms of the infinite sum with index $n_{\text{max}}$, such that $|J_n(\beta)| < 0.01$ for $n > n_{\text{max}}$. This criterion is often called the 1% criterion for the evaluation of bandwidth. If we find that $n_{\text{max}}$ is the minimum index $n$ that does not violate the 1% criterion then we can claim that the approximate bandwidth of our signal, according to the 1% criterion, is:

$$B = 2n_{\text{max}}\omega_m$$

(5.34)
It is worth mentioning that the evaluation of bandwidth based on equation (5.33) corresponds to the bandwidth of your FM signal according to a 10% criterion.

The above procedure followed for the evaluation of the FT of \( m_f(t) \), can be extended to the cases where the information signal \( f(t) \) is a sum of sinusoidal signals, or a periodic signal. In particular, if

\[
f(t) = a_1 \cos(\omega_1 t) + a_2 \cos(\omega_2 t)
\]

then the phase of our FM signal \( m_f(t) \) is provided by the following equation.

\[
\theta(t) = \omega_c t + \frac{k_f a_1}{\omega_1} \sin(\omega_1 t) + \frac{k_f a_2}{\omega_2} \sin(\omega_2 t)
\]

Omitting the details, we arrive at a representation of the signal \( m_f(t) \) such that

\[
m_f(t) = A \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} J_n(\beta_1)J_k(\beta_2) \cos(\omega_c + n\omega_1 + k\omega_2)t
\]

where \( \beta_1 = a_1 k_f / \omega_1 \) and \( \beta_2 = a_2 k_f / \omega_2 \). As we can see from the above equation, we now have impulses at \( \omega_c \pm n\omega_1, \omega_c \pm k\omega_2 \) as well as \( \omega_c \pm n\omega_1 \pm k\omega_2 \).

Most of the aforementioned discussion regarding the FT and the bandwidth of an FM signal \( m_f(t) \) is based on the assumption that the information signal \( f(t) \) is a sinusoid, a sum of sinusoids, or a periodic signal. We want to be able to derive a formula for the bandwidth of an FM signal \( m_f(t) \) for an arbitrary information signal \( f(t) \). Let us revisit the approximate FM signal bandwidth formula (5.33) derived for a sinusoidal information signal

\[
B = 2(\beta + 1)\omega_m = 2(ak_f + \omega_m)
\]

The second equality in (5.38) is obtained by substituting the value of \( \beta \) with its equal. Now let us pay a closer look at the two terms involved in the evaluation of the approximate bandwidth \( B \). The first term \( ak_f \) is the maximum frequency deviation of the instantaneous frequency \( \omega(t) \) from the carrier frequency \( \omega_c \); it is often denoted by \( \Delta\omega \). The second term \( \omega_m \) is the maximum frequency content of the information signal \( f(t) \). Keeping these two clarifications in mind, we now define the approximate bandwidth of an FM signal \( f(t) \) to be equal to

\[
B = 2(\Delta\omega + \omega_m)
\]
where $\Delta \omega$ is the maximum frequency deviation from the carrier frequency, and $\omega_m$ is the maximum frequency content of the information signal $f(t)$. It is not difficult to show that for an arbitrary information signal $f(t)$, $\Delta \omega = k_f \max_i |f(t)|$. Furthermore, to find $\omega_m$ we first need to compute the FT of the information signal $f(t)$. Hence, based on the above equation we can claim that the approximate bandwidth of an FM signal is computable even for the case of an arbitrary info signal. It is worth pointing out that the bandwidth formula given above has not been proven to be true for FM signals produced by arbitrary info signals; but it has been verified experimentally in a variety of cases. Equation (5.39) is referred to as Carson's formula (rule) for the evaluation of the bandwidth of an FM signal, and from this point on it can be applied freely, independently of whether the information signal is of sinusoidal nature or not.

One of the ramifications of Carson's rule is that we can increase the bandwidth of an FM signal at will, by increasing the modulation constant $k_f$, or equivalently, by increasing the peak frequency deviation $\Delta \omega$. One of the advantages of increasing the bandwidth of the FM signal is that larger bandwidths result in FM signals that exhibit better tolerances to noise. Unfortunately the peak frequency deviation of an FM signal is constrained by other considerations, such as a limited overall bandwidth that needs to be shared by a multitude of FM users. For example, in commercial FM the peak frequency deviation is specified to be equal to 75 KHz.

Let us now say a word about PM. From our previous discussions the form of a PM signal produced by a sinusoidal modulating signal is as follows:

$$m_p(t) = A \cos(\omega_c t + k_p a \cos(\omega_m t))$$  \hspace{1cm} (5.40)

and the instantaneous frequency is equal to

$$\omega(t) = \omega_c - k_p a \omega_m \sin \omega_m t$$  \hspace{1cm} (5.41)

Hence,

$$\Delta \omega = ak_p \omega_m$$  \hspace{1cm} (5.42)

That is $\Delta \omega$ depends on $\omega_m$. This is considered a disadvantage compared to commercial FM, where $\Delta \omega$ is fixed. Carson's rule is also applicable for PM systems but to find the peak frequency deviation of a PM system you need to find the maximum, with respect to time, of $|\dot{f}(t)|$, where $\dot{f}(t)$ is the time derivative of $f(t)$. Actually, for PM systems...
\[
\Delta \omega = k_p \max \left| f'(t) \right|
\]  

(5.43)

One last comment to conclude our discussion of angle modulation systems. It can be shown that the power of an FM or PM signal of the form

\[
m(t) = A \cos(\theta(t))
\]

(5.44)

is equal to \(\frac{A^2}{2}\)

### 3 Pre-lab Questions

1. Provide a paragraph, where you compare AM and FM modulation (advantages, disadvantages)

2. Why is the use of FM more preferred than PM? Explain your answer. (Hint: Compare the frequency deviations in both cases)

3. Give a short qualitative justification of the fact that FM has more noise immunity than AM.

### 4 Simulation

We are going to do the simulation in Simulink of Matlab. To run the Simulink, enter the ‘simulink’ command in the Matlab command window. It should look as it is shown in the following. Open a new window using the left icon.
Construct the following block diagram for the FM modulation and demodulation simulation. The following blocks will be necessary for your simulation. The paths of the blocks have been given as well.
Just drag and drop the blocks you need for your simulation in your work window. Click the left mouse button, hold and drag to connect the blocks by wire. The parameters for the individual module have been given as an example. To set the parameters of a block just double click on it.

<table>
<thead>
<tr>
<th><strong>Block</strong></th>
<th><strong>Path</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>Signal Generator</td>
<td>Simulink -&gt; Source</td>
</tr>
<tr>
<td>Discrete time VCO</td>
<td>Communications Block set -&gt; Comm Sources-&gt; Controlled Sources</td>
</tr>
<tr>
<td>Phase Lock Loop</td>
<td>Communications Block set -&gt; Synchronization</td>
</tr>
<tr>
<td>Scope</td>
<td>Simulink -&gt; Sink</td>
</tr>
</tbody>
</table>
♦ **Signal Generator**

Wave form: sine

Amplitude: 1

Frequency: 1000

♦ **Discrete time VCO**

![Block Parameters: Discrete-Time VCO](image)

Generate a discrete-time output signal whose frequency changes in response to the amplitude variations of the input signal. The input signal must be a scalar.

- **Output amplitude:** 1
- **Oscillation frequency (Hz):** 10000
- **Input sensitivity:** 5000
- **Initial phase (rad):** 0
- **Sample time:** 0.00001
Set the simulation stop time at say 0.002 from the simulation -> parameters menu.

Run the simulation from the simulation -> start menu. Then double click on the scopes to see the time domain signals.

Do the simulation with square wave input signal. Is the demodulated signal the faithful reproduction of the input signal? Why?

5 Implementation

In linear FM (frequency modulation), the instantaneous frequency of the output is linearly dependent on the voltage at the input. Zero volts at the input will yield a sinusoid at center frequency \( f_c \) at the output. The equation for a FM signal is:

\[
m_f(t) = \cos(\omega_c t + g(t))
\]  

(5.45)

Where \( \omega_c t + g(t) = \phi_f(t) \) (instantaneous phase) and the instantaneous angular frequency is:
\[ \omega_f(t) = \omega_c + \frac{dg(t)}{dt} \]  

(5.46)

In linear FM the instantaneous frequency can be approximated by a straight line.

\[ \omega_f(t) = \omega_c + k_f f(t) \]  

(5.47)

Where \( k_f \) is a constant and \( f(t) \) is the input signal.

\[ \phi_f(t) = \int \left( \omega_c + k_f f(t) \right) dt = \omega_c t + 2\pi k_f \int f(t) dt \]  

(5.48)

\[ m_f(t) = \cos \left( \omega_c t + 2\pi k_f \int f(t) dt \right) \]  

(5.49)

Note that the amplitude of an FM signal never varies.

For a sinusoid input and positive \( k_f \), the modulated FM waveform will relate to the input as shown in figure 5.1.

Depending on the VCO, \( k_f \) can be positive or negative and \( f_c \) can also vary and is a function of external timing resistor and capacitor values. The relationship between input voltage and output frequency at any given point in time is:

\[ f_{out} = f_c + k_f V_{in} \]  

(5.50)

where \( k_f \) is positive for figure 5.1 and has units of Hz/Volt.

Let \( f(t) = A \cos \omega_m t \), then

\[ m_f(t) = \cos \left( \omega_c t + \frac{ak_f}{f_m} \sin \omega_m t \right) \]  

(5.51)

The peak frequency deviation from \( \omega_c \) is \( ak_f 2\pi \) (radian/sec), and the total peak-to-peak deviation is \( 2(ak_f 2\pi) \). The modulation index \( \beta \) for this signal is:

\[ \beta = \frac{ak_f}{f_m} \]  

(5.52)

Note that \( \beta \) will vary for each frequency component of the signal.
5.1 Spectrum of FM

The spectrum of an FM signal is described by Bessel functions. As shown in section 2.6, for a single frequency, constant amplitude message, the spectrum of \( m_f(t) \) is:

\[
m_f(f) = \sum J_n(\beta) \cos(\omega_c t + n \omega_m t)
\]  \hspace{1cm} (5.53)

where \( \frac{ak_f}{f_m} = \beta \) is the FM modulation index and \( J_n(\beta) = \frac{1}{2\pi} \int e^{i\beta \sin \beta - J_n(\beta)} d\theta \), which is the \( n^{th} \) order Bessel function evaluated at \( \beta \). Therefore, for each frequency at the input there are infinite number of spectral components at the output with the amplitude or each component determined by the modulation index \( \beta \).

The amplitude of the higher order terms will decrease, on the average, such that there will be a limited bandwidth where most of the energy is concentrated.

5.2 VCO (FM modulator)

A voltage-controlled oscillator (VCO) converts the voltage at its input to a corresponding frequency at its output. This is accomplished by a variable reactor (varactor) where the reactance varies with the voltage across it. The varactor is part of a timing circuit, which sets the VCO output frequency. There are two limitations for most VCO's:

1. The input voltage must be small (usually there is an attenuating circuit at the input).
2. The bandwidth is limited for a linear frequency-to-voltage relationship.

5.3 PLL (FM demodulator)

The phase-locked loop (PLL) is used to demodulate FM. The phases of the input and feedback signals are compared and the PLL works to make the phase difference between the two signals equal to zero. Figure 5.4 shows a basic block diagram of a PLL.
The VCO in the feedback is an FM modulator, and the center frequency \( \omega_c \) can be set equal to the center frequency of \( m_f(t) \).

The equation for \( m_f(t) \) is

\[
m_f(t) = \cos(\omega_c t + 2\pi k_f \int f(\alpha) d\alpha)
\] (5.54)

where \( k_f \int f(\alpha) d\alpha = \theta_1 t \) and the equation of \( e(t) \) is

\[
e(t) = \cos(\omega_c t + k_d \int F(\alpha) d\alpha)
\] (5.55)

where \( k_d \int f(\alpha) d\alpha = \theta_2 t \)

Since each signal has the same center frequency \( \omega_c \), the phase comparator compares the instantaneous values of \( \theta_1(t) \) and \( \theta_2(t) \). The difference in phase is transformed into a DC voltage level proportional to the phase difference and then amplified to yield \( F(t) \). This voltage is the input for the VCO in the feedback. As a result, the difference between \( \theta_1(t) \) and \( \theta_2(t) \) will be made smaller. This process occurs continually, such that \( \theta_1(t) = \theta_2(t) \) at all times.

Substituting for \( \theta_1(t) = \theta_2(t) \)

\[
k_d \int F(\alpha) d\alpha = k_f \int f(\alpha) d\alpha \text{ and } F(t) = \frac{k_f}{k_d} f(t).
\] (5.56)
where $f(t)$ was the original message signal and $F(t)$ is the demodulated output.

6 Equipment

Oscilloscope, Tektronix DPO 4034
Function Generator, Tektronix AFG 3022B
Triple Output Power Supply, Agilent E3630A
Bring a USB Flash Drive to store your waveforms.

7 Procedure

7.1 Modulation

a. Build the FM modulator shown in Figure 5.5(a).

b. Determine the constant $k_f$ from the following:

(1) Use your triple output power supply to apply the input voltages specified in the following table and record the output frequency:

<table>
<thead>
<tr>
<th>$V_{in}$ (Volts)</th>
<th>$f_{out}$ (KHz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>+1.0</td>
<td></td>
</tr>
<tr>
<td>+2.0</td>
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</tr>
<tr>
<td>+3.0</td>
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<td>+4.0</td>
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<tr>
<td>+5.0</td>
<td></td>
</tr>
<tr>
<td>+6.0</td>
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</tbody>
</table>

(2) Plot $f_i$ vs. $V_{in}$. Draw the best straight line through these points. The slope of this line is $k_f$. Note that $k_f$ has units of Hz/Volts. What is the measured value of $k_f$?
(3) According to the XR2206 data sheet, the expected voltage-to-frequency conversion gain is \( k_f = -\frac{0.33}{R C_1} \text{ Hertz/Volts} \).

Calculate the expected value of \( k_f \). Assuming the resistors have a tolerance of \( \pm 10\% \) and the capacitors have a tolerance of \( \pm 20\% \), how do the measured and calculated values compare?

c. Add input coupling capacitor \( C_2 \) to the circuit as shown in Figure 5.5 (b). Set the modulation frequency to \( f_m = 2 \text{ KHz} \). Fill in the following table using the following equations:

\[
\Delta f_{\text{peak}} = \beta f_m = \alpha k_f \quad (5.57)
\]

\[
|V_{\text{in}}| = \frac{\Delta f_{\text{peak}}}{k_f} \quad (5.58)
\]

<table>
<thead>
<tr>
<th>( \beta ) (modulation index)</th>
<th>Calculated ( \Delta f_{\text{peak}} ) (Hz)</th>
<th>Calculated ( V_{\text{in}} )</th>
<th>Measured ( \Delta f_{\text{peak}} ) (Hz)</th>
</tr>
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<tr>
<td>12.3</td>
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</tbody>
</table>
Figure 5.5 (a) VCO Determination of modulation constant $K_f$

Figure 5.5 (b) Generation of FM signal
d. Use the $V_{in}$ found as the amplitude of the input signal and find $\Delta f_{peak}$ on the oscilloscope. Compare $\Delta f_{peak}$ calculated and measured. Display only the FM signal on the oscilloscope and trigger on the rising edge of the waveform such that you see the following Figure 5.6 (it will look like a ribbon):

$$f_{max} = \frac{1}{t_{min}}$$

$$f_{min} = \frac{1}{t_{max}}$$

$$\Delta f_{peak} = \frac{(f_{max} - f_{min})}{2}$$

![Figure 5.6: Frequency components of the modulated signal](image)

This "ribbon" displays all frequencies in the FM signal at once. The minimum and maximum frequencies can be easily detected and directly measured. Recall that $\Delta f_{peak}$ is only half of the peak-to-peak frequency swing. What parameters determine the bandwidth of an FM signal?

e. View the frequency domain waveform to obtain modulation indices of 2.4, 5.52, 8.65. These are zero carrier amplitude indices. Include calculations to verify your results in your lab report.

7.2 Demodulation

a. Build the FM demodulator utilizing the LM565N PLL shown in Figure 5.7. Connect the output of the FM modulator shown in Figure 5.5 (b) to the input of the FM demodulator.

b. Apply a sinusoidal message signal and observe the demodulated message signal. Sketch both waveforms. How do they compare?

Hint for the Tektronix DPO 4034 Oscilloscope:
An FM demodulator is typically followed by a filter to block the carrier. Setting the oscilloscope Trigger Coupling to HF REJ will help sync on the modulation frequency.

c. Apply a square wave message signal and observe the demodulated message signal. Sketch both waveforms. How do they compare? Why is the demodulated sinusoidal message more faithfully reproduced than the demodulated square wave?
Phase Lock Loop Pin Assignment

Figure 5.7. PLL as an FM Demodulator
Table 5.1

Bessel Functions of the first kind, \( J_n(x) \)

<table>
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<tr>
<th>( x )</th>
<th>( J_0 )</th>
<th>( J_1 )</th>
<th>( J_2 )</th>
<th>( J_3 )</th>
<th>( J_4 )</th>
<th>( J_5 )</th>
<th>( J_6 )</th>
<th>( J_7 )</th>
<th>( J_8 )</th>
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