CAP 4453 Robot Vision

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Credits

- Slides comes directly from:
  - Ioannis (Yannis) Gkioulekas (CMU)
  - Noah Snavely (Cornell)
  - Marco Zuliani

Short Review from last class

Last 2 classes

- Feature points
  - Correspondent points on two images

Robot Vision

9. Image warping I

How do you create a panorama?

Panorama: an image of (near) 360^o field of view.

How do you create a panorama?

Panorama: an image of (near) 360^o field of view.

1. Use a very wide-angle lens.

Wide-angle lenses

Fish-eye lens: can produce (near) hemispherical field of view.

What are the pros and cons of this?

How do you create a panorama?

Panorama: an image of (near) 360^o field of view.

1. Use a very wide-angle lens.
- Pros: Everything is done optically, single capture.
- Cons: Lens is super expensive and bulky, lots of distortion (can be dealt-with in post).

Any alternative to this?

How do you create a panorama?

Panorama: an image of (near) 360^o field of view.

1. Use a very wide-angle lens.
- Pros: Everything is done optically, single capture.
- Cons: Lens is super expensive and bulky, lots of distortion (can be dealt-with in post).

2. Capture multiple images and combine them.

Panoramas from image stitching

1. Capture multiple images from different viewpoints.

2. Stitch them together into a virtual wide-angle image.

How do we stitch images from different viewpoints?

Will standard stitching work?

1. Translate one image relative to another.
2. (Optionally) find an optimal seam.

How do we stitch images from different viewpoints?

Will standard stitching work?

1. Translate one image relative to another.
2. (Optionally) find an optimal seam.

left on top

right on top

Translation-only stitching is not enough to mosaic these images.

How do we stitch images from different viewpoints?

What else can we try?

How do we stitch images from different viewpoints?

Use image homographies.

Outline

- Linear algebra
  - Matrix addition, Matrix multiplication
  - Inverse, Pseudo Inverse
  - Least squares, SVD
- Image transformations.
- 2D transformations.
- Projective geometry 101.
- Transformations in projective geometry.
- Classification of 2D transformations.
- Determining unknown 2D transformations.
- Determining unknown image warps.

Matrix

- Array $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ of numbers with shape m by n,
  - m rows and n columns

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- A row vector is a matrix with single row
- A column vector is a matric with single column

Matrix operations

- Addition

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

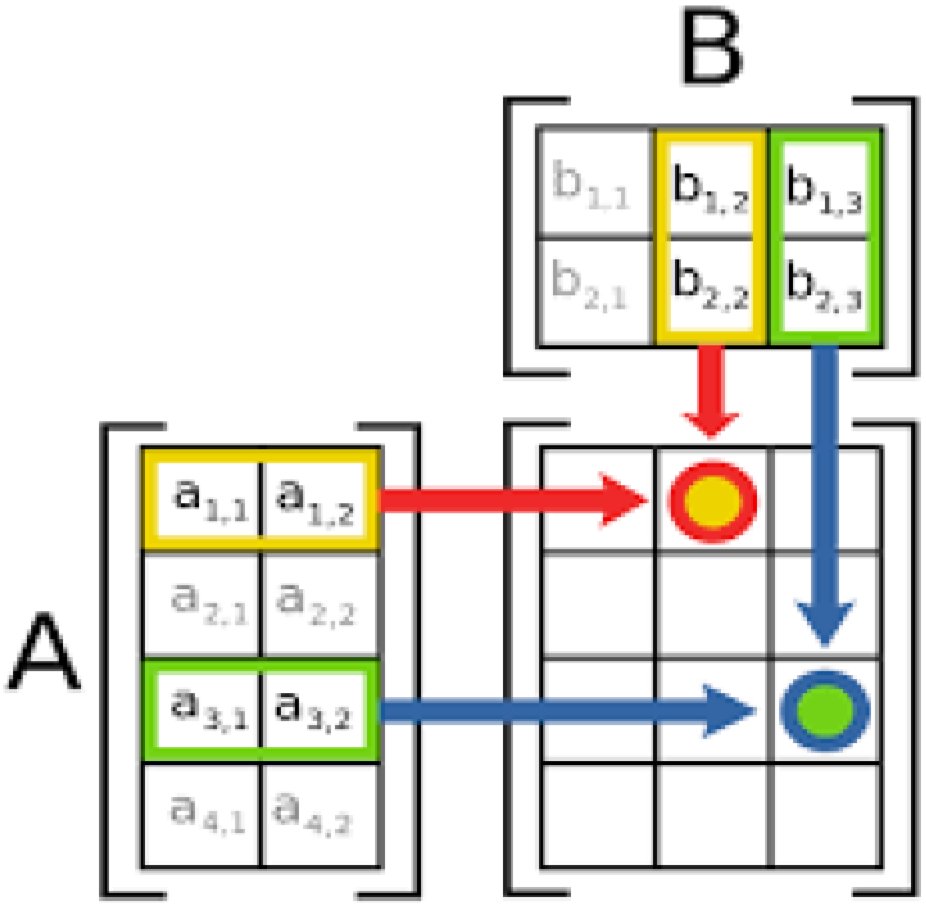
- Both matrices should have same shape, except with a scalar

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + 2 = \begin{bmatrix} a+2 & b+2 \\ c+2 & d+2 \end{bmatrix}$$

- Same with subtraction

Matrix operation

- Matrix Multiplication
  - Compatibility?
  - mxn and nxp
  - Results in mxp matrix



Matrix operation

- Transpose

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$\boldsymbol{A}^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

Special matrices

- Diagonal matrix
  - Used for row scaling

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \end{bmatrix}$$

- Identity matrix
  - Special diagonal matrix
  - 1 along diagonals

$$\boldsymbol{I}.\boldsymbol{A} = \boldsymbol{A}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Matrix operation

- Inverse
  - Given a matrix A, its inverse A^{-1} is a matrix such that

$$\mathbf{AA^{-1} = A^{-1}A = I}$$

- Inverse does not always exist
  - Singular vs non-singular
- Properties
  - $(A^{-1})^{-1} = A$
  - $(AB)^{-1} = B^{-1}A^{-1}$

PseudoInverse

$$Ax = b$$

A is not squared

$$A^T Ax = A^t b$$

$A^T A$ is squared

$$(A^T A)^{-1}(A^T A)x = (A^T A)^{-1} A^t b$$

$$x = \underbrace{(A^T A)^{-1} A^t}_{\text{PseudoInverse}} b$$

Outline

- Linear algebra
- **Image transformations**.
- 2D transformations.
- Projective geometry 101.
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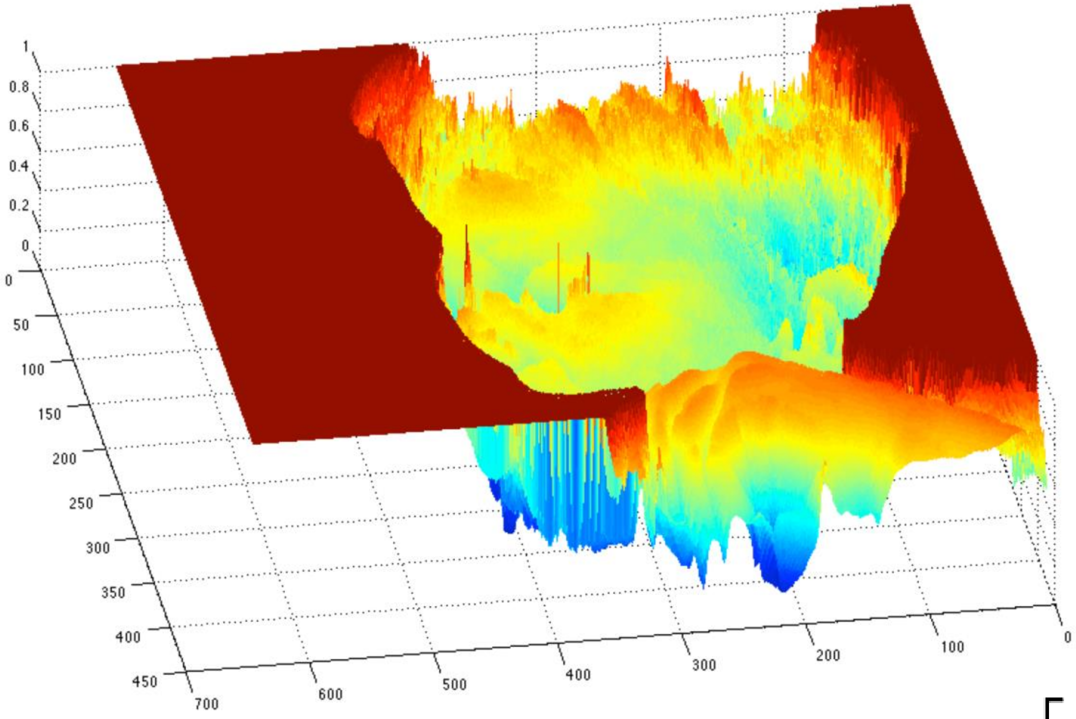
What is an image?

grayscale image

$f(\boldsymbol{x})$



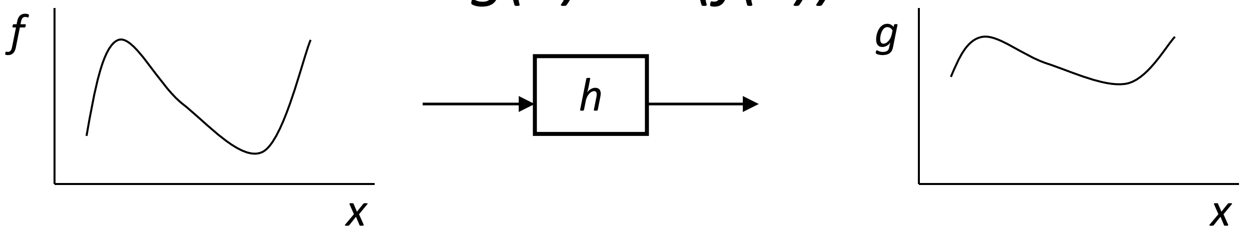
domain $\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix}$

A (grayscale) image is a 2D function.

What is the range of the image function f?

Image Warping

- image filtering: change *range* of image
  - $g(x) = h(f(x))$



- image warping: change *domain* of image
  - $g(x) = f(h(x))$

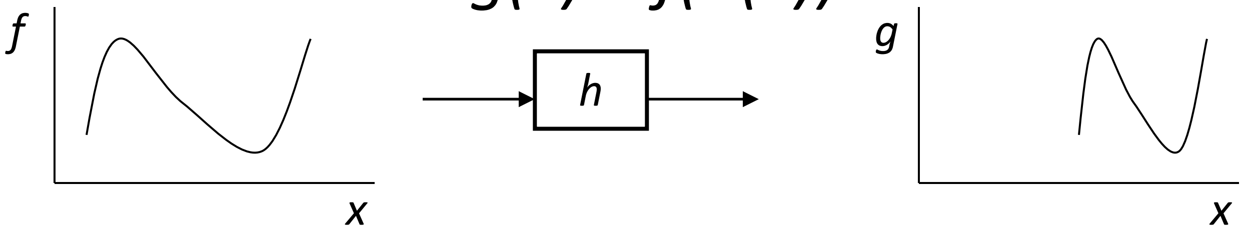


Image Warping

- image filtering: change *range* of image
  - $g(x) = h(f(x))$

f

h

g

- image warping: change *domain* of image
  - $g(x) = f(h(x))$

f

h

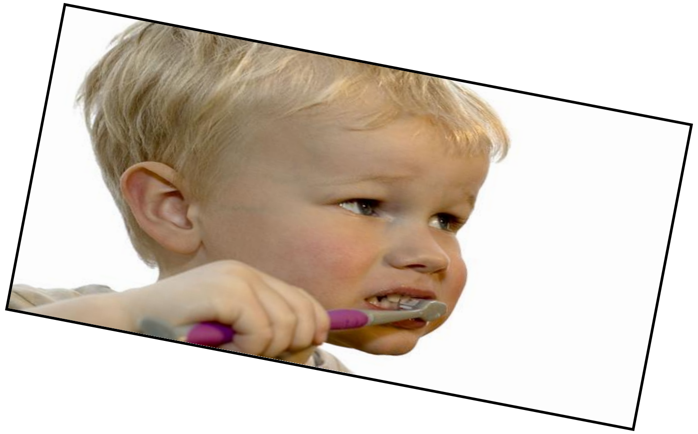
g

What types of image transformations can we do?

Filtering

changes pixel *values*

Warping



changes pixel *locations*

What types of image transformations can we do?

F

Filtering

$$G(\boldsymbol{x}) = h\{F(\boldsymbol{x})\}$$

G

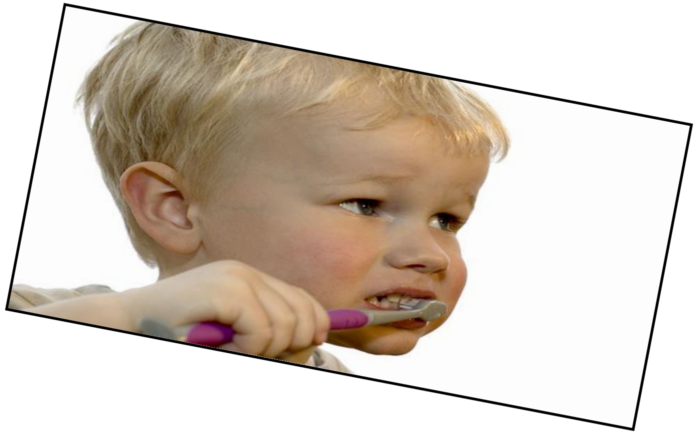
changes *range* of image function

F

Warping

$$G(\boldsymbol{x}) = F(h\{\boldsymbol{x}\})$$

G



changes *domain* of image function

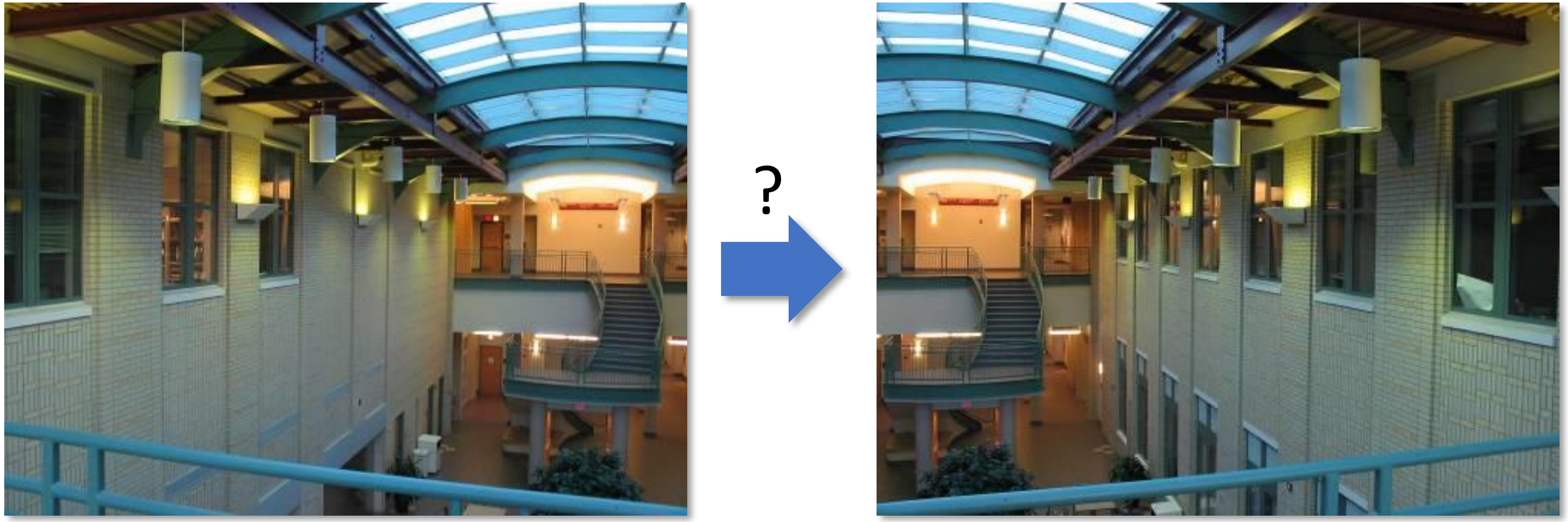
The persistence of memory by Salvador Dali

Original

Filtering operation
(Blurred)

Warping operation
(Swirled)

What is the geometric relationship between these two images?



What is the geometric relationship between these two images?

Very important for creating mosaics!

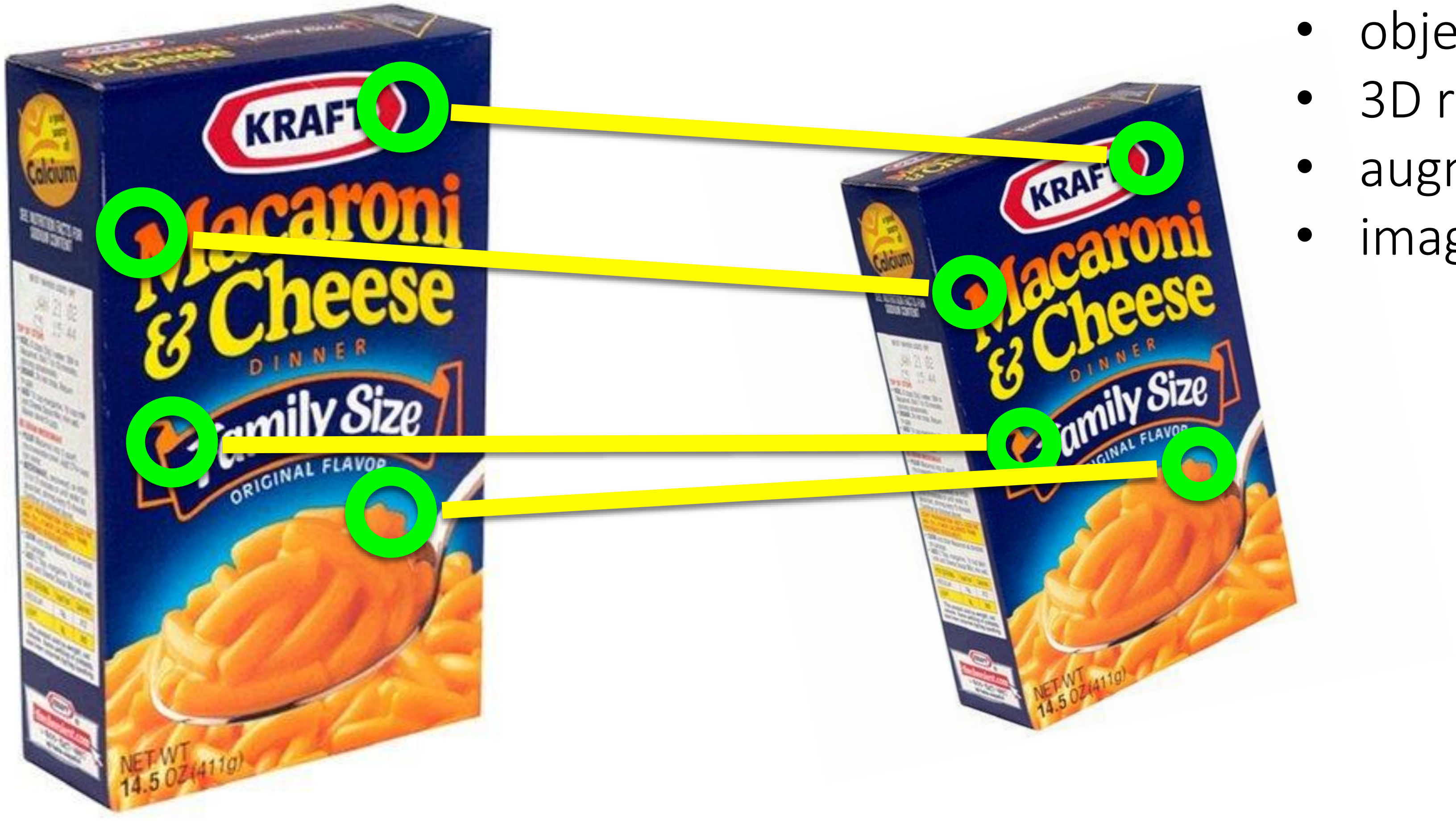
First, we need to know what this transformation is.

Second, we need to figure out how to compute it using feature matches.

Warping example: feature matching

Warping example: feature matching

Warping example: feature matching



- object recognition
- 3D reconstruction
- augmented reality
- image stitching

How do you compute the transformation?

Warping example: feature matching

Given a set of matched feature points:

$$\{\boldsymbol{x}_i, \boldsymbol{x}'_i\}$$

point in one image

point in the other image

and a transformation:

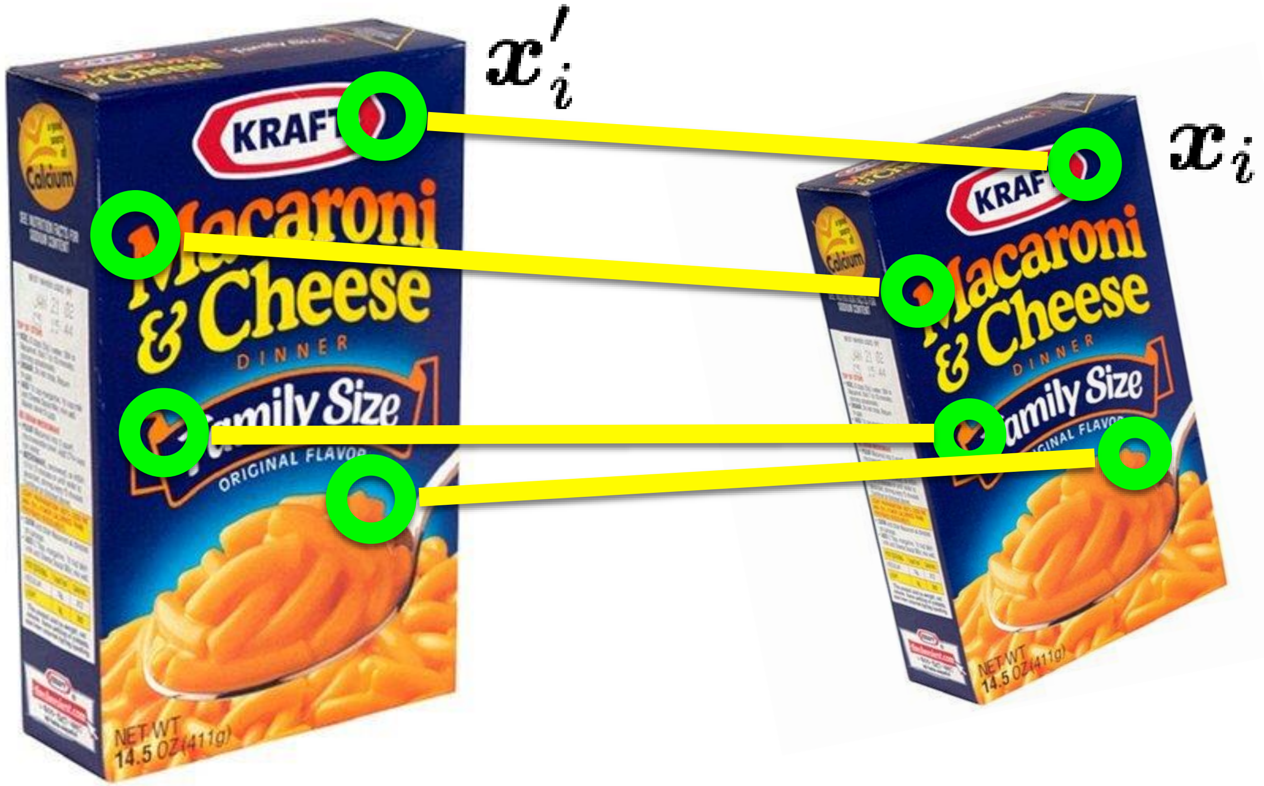
$$\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x}; \boldsymbol{p})$$

transformation function

parameters

find the best estimate of the parameters

$$\boldsymbol{p}$$



What kind of transformation functions $\boldsymbol{f}$ are there?

Outline

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- **2D transformations.**
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2D transformations

translation

rotation

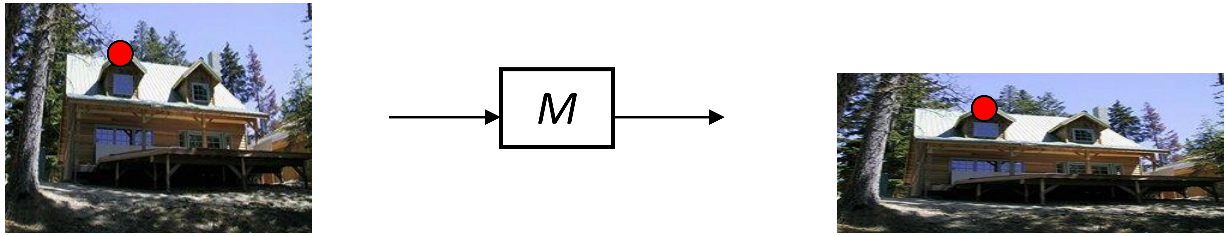
aspect

affine

perspective

cylindrical

Parametric (global) warping



p = (x,y) **p'** = (x',y')

- Transformation M is a coordinate-changing machine:

p' = *M*(p)

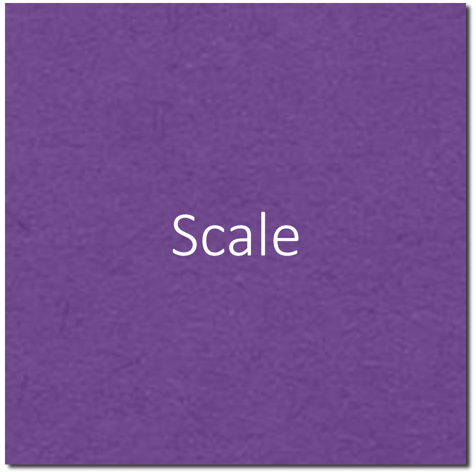
- What does it mean that *M* is global?
  - Is the same for any point p
  - can be described by just a few numbers (parameters)
- Let's consider *linear* forms (can be represented by a 2x2 matrix):

$$p' = Mp \qquad \begin{bmatrix} x' \\ y' \end{bmatrix} = \boldsymbol{M} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D planar transformations

2D planar transformations

y

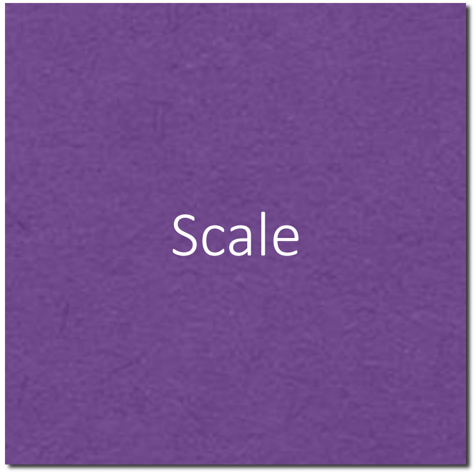


How would you implement scaling?

- Each component multiplied by a scalar
- Uniform scaling - same scalar for each component

x

2D planar transformations



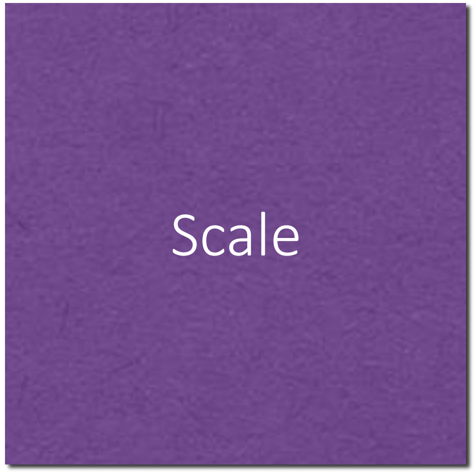
$$x' = ax$$
$$y' = by$$

What's the effect of using different scale factors?

- Each component multiplied by a scalar
- Uniform scaling - same scalar for each component

2D planar transformations

y



$$x' = ax$$

$$y' = by$$

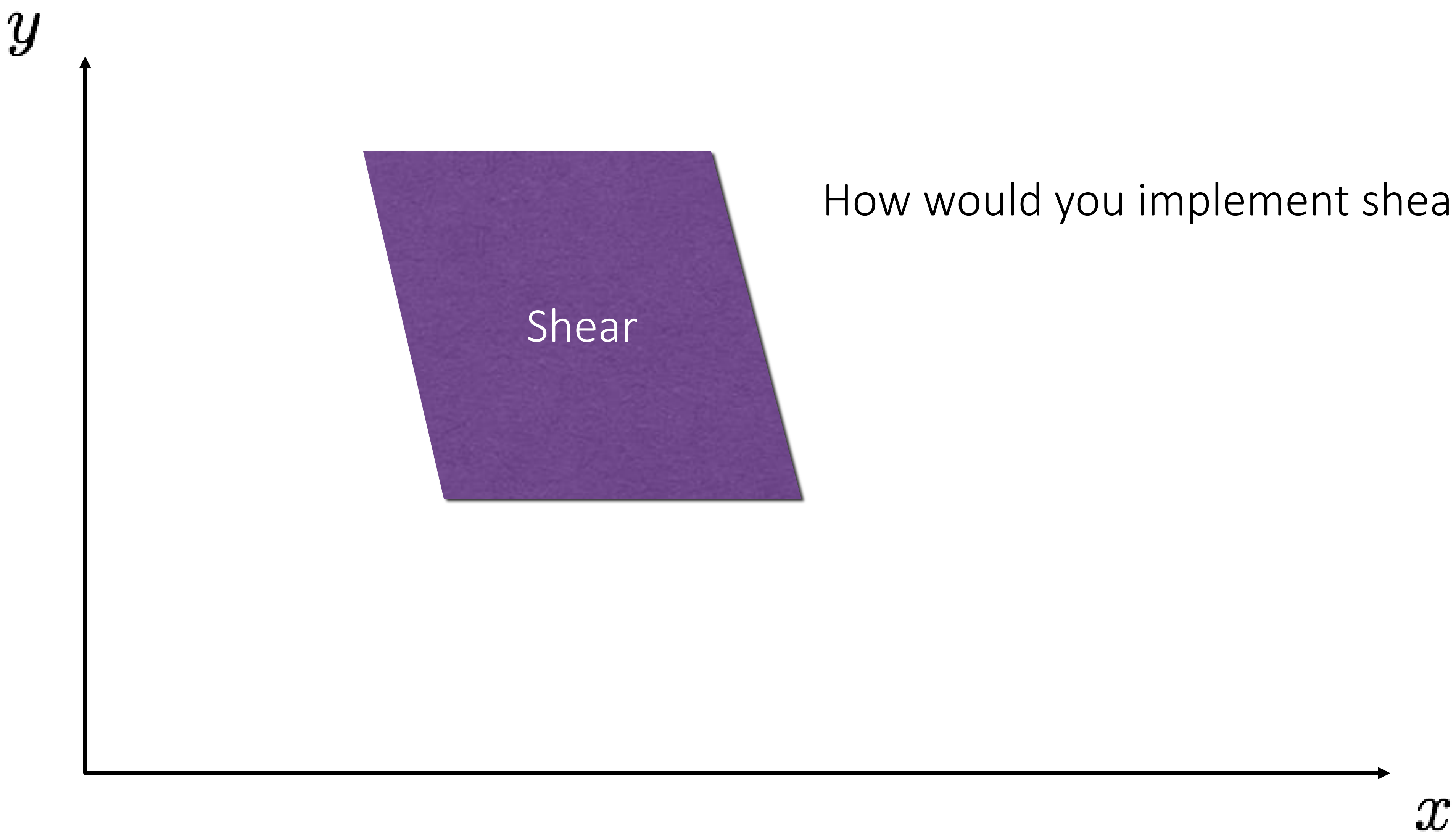
matrix representation of scaling:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}}_{\text{scaling matrix S}} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Each component multiplied by a scalar
- Uniform scaling - same scalar for each component

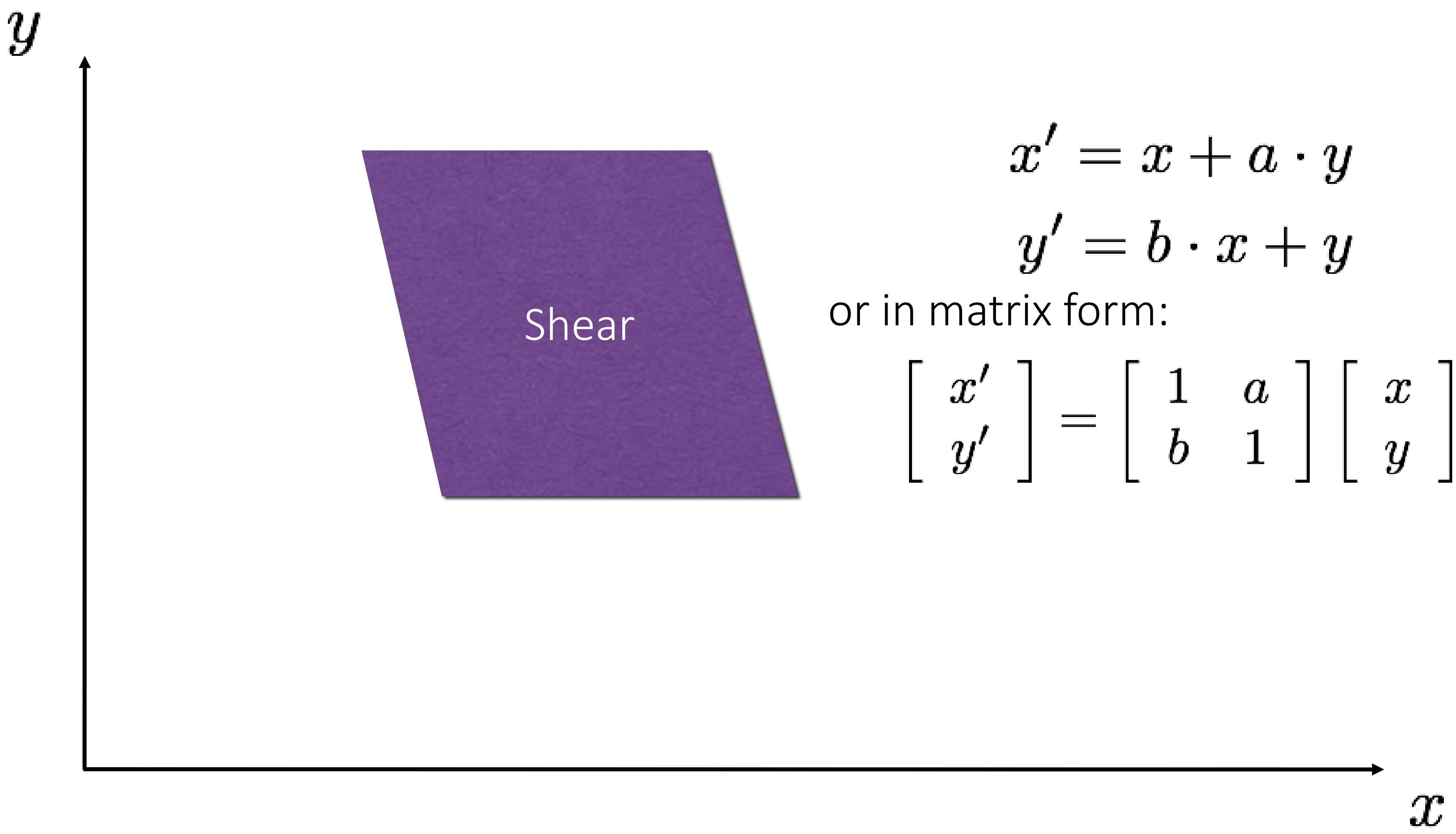
x

2D planar transformations



How would you implement shearing?

2D planar transformations



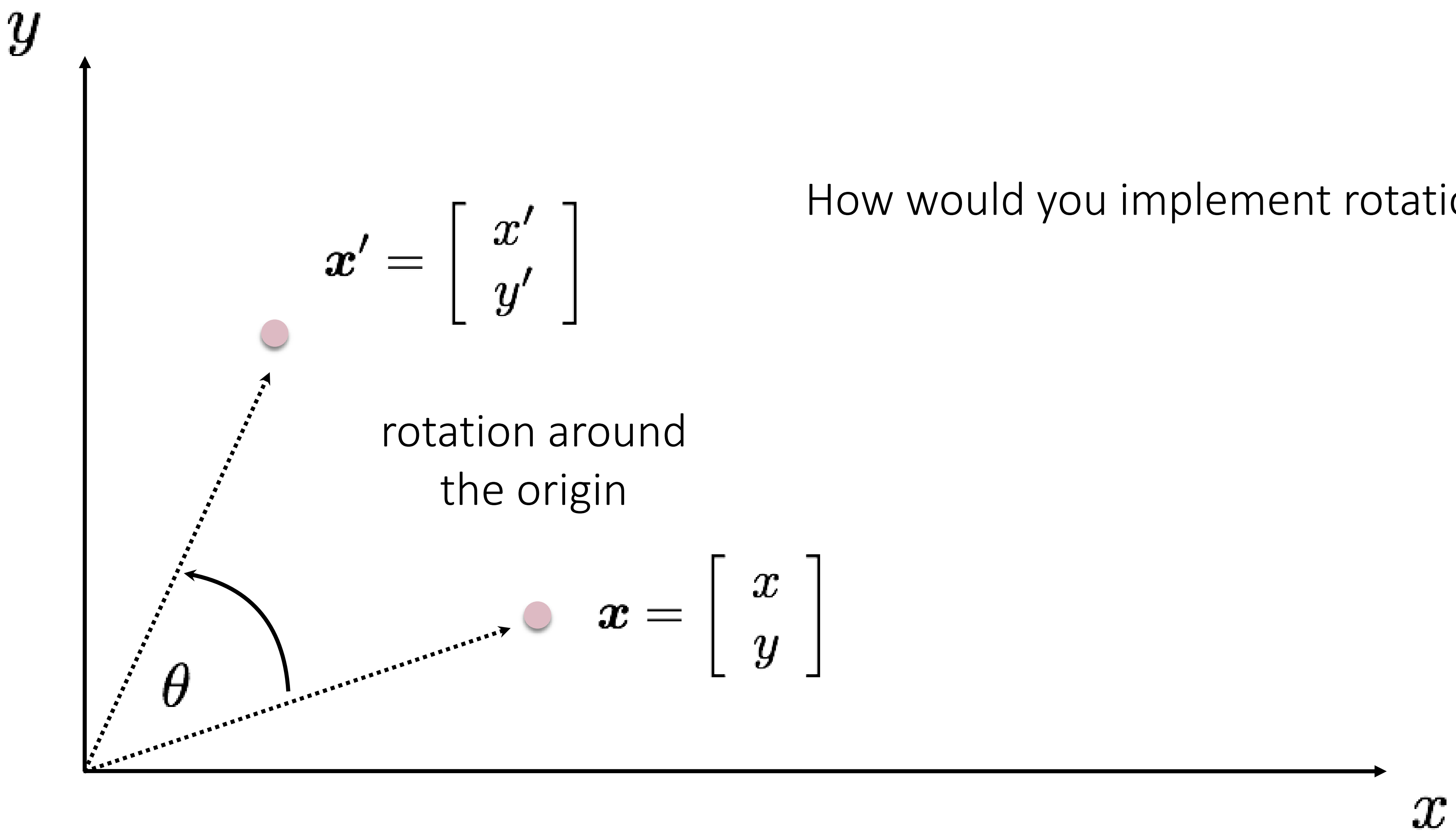
$$x' = x + a \cdot y$$

$$y' = b \cdot x + y$$

or in matrix form:

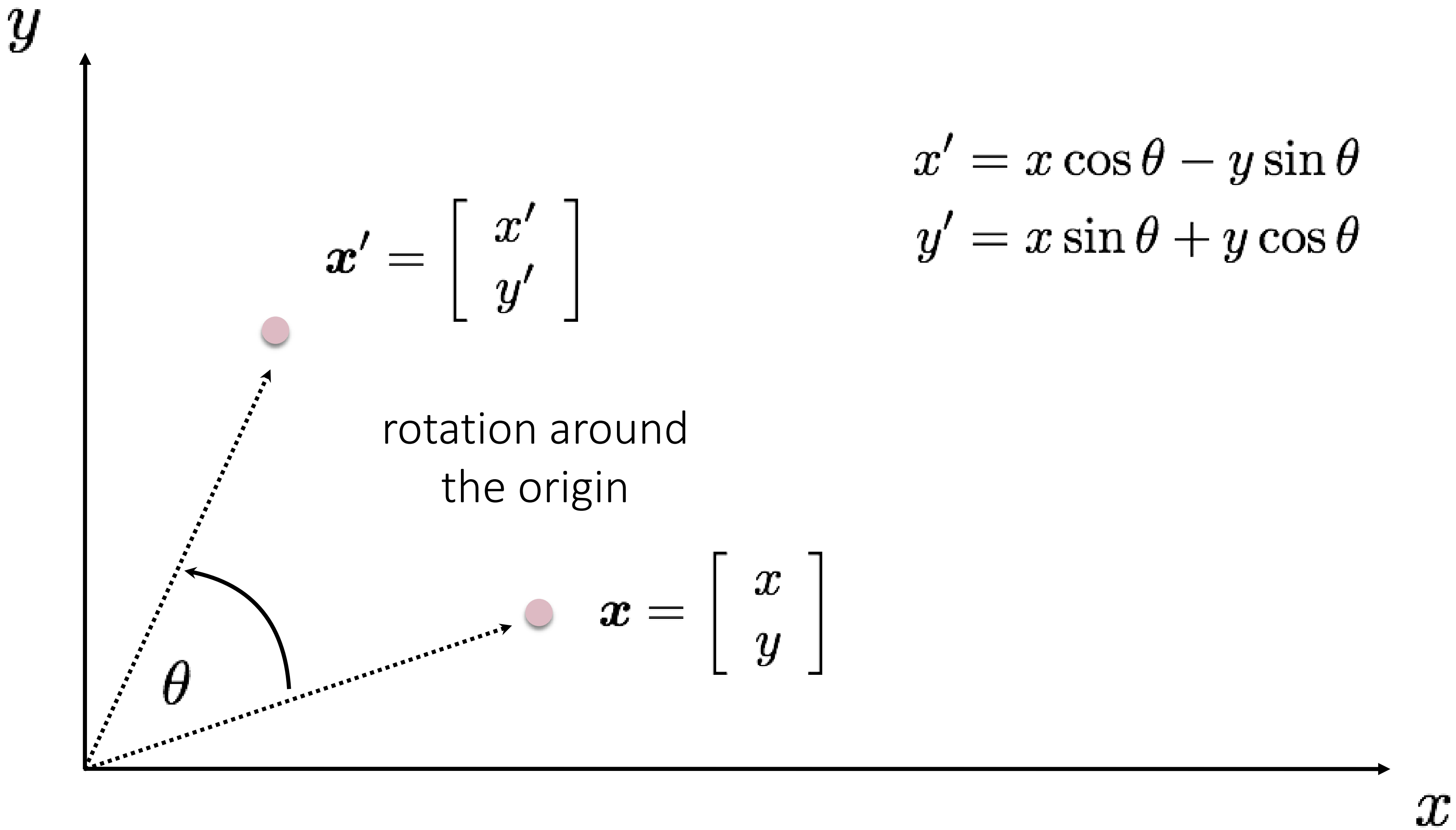
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & a \\ b & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D planar transformations



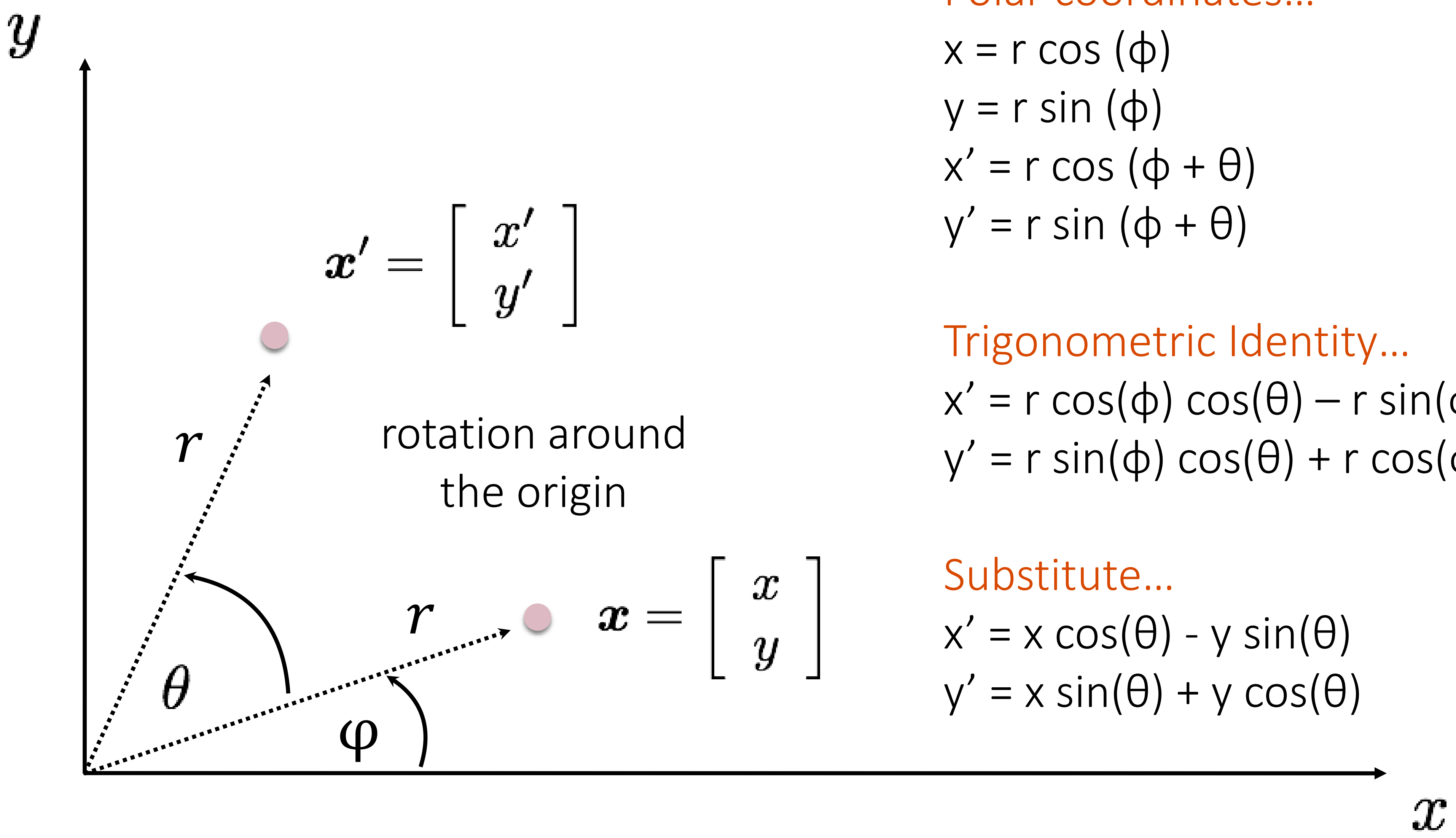
How would you implement rotation?

2D planar transformations



$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$

2D planar transformations



Polar coordinates…

x = r cos (ϕ)

y = r sin (ϕ)

x' = r cos (ϕ + θ)

y' = r sin (ϕ + θ)

Trigonometric Identity…

x' = r cos(ϕ) cos(θ) – r sin(ϕ) sin(θ)

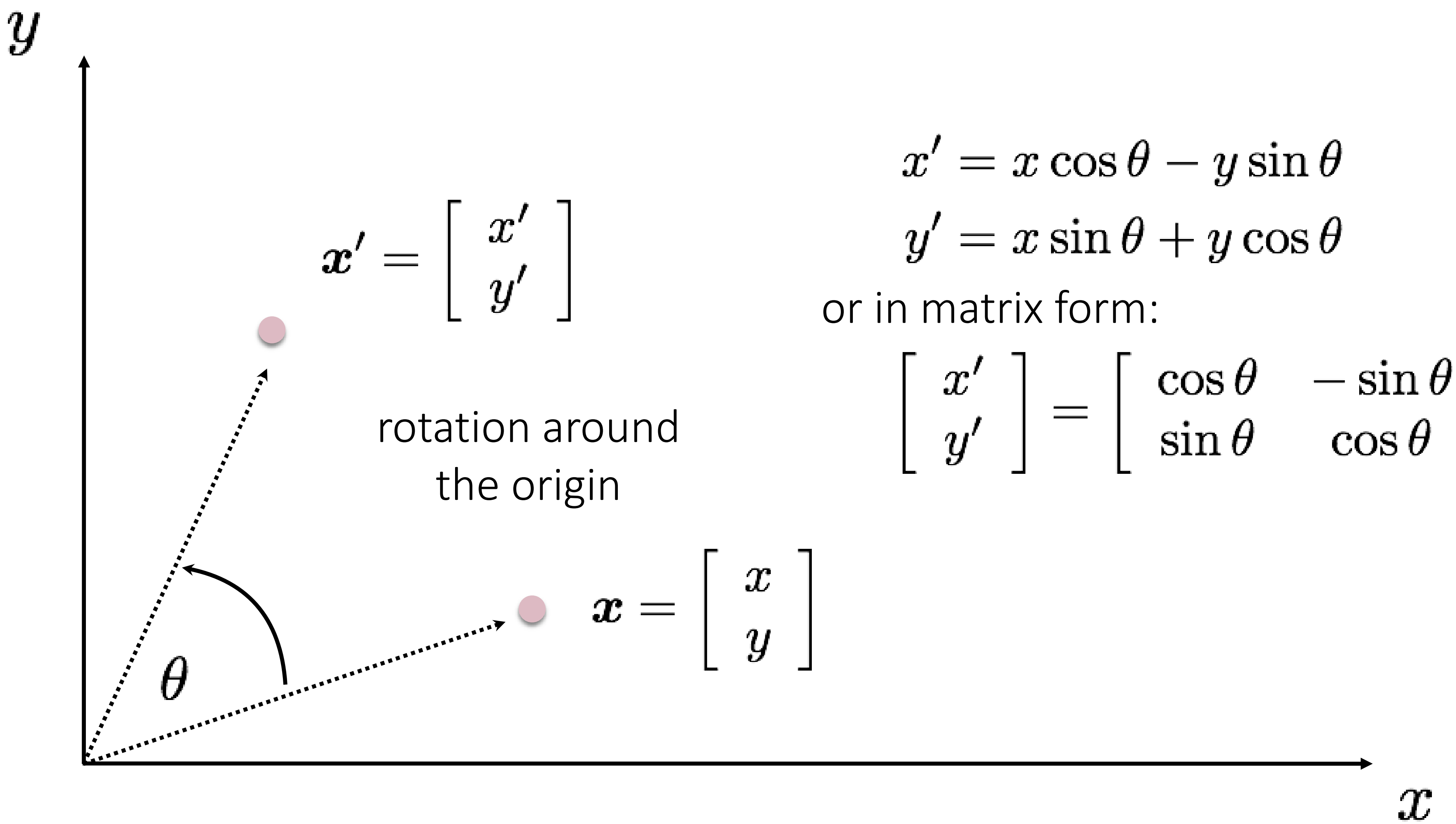
y' = r sin(ϕ) cos(θ) + r cos(ϕ) sin(θ)

Substitute…

x' = x cos(θ) - y sin(θ)

y' = x sin(θ) + y cos(θ)

2D planar transformations



$$x' = x\cos\theta - y\sin\theta$$

$$y' = x\sin\theta + y\cos\theta$$

or in matrix form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Common linear transformations

- Rotation by angle θ (about the origin)

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

What is the inverse?

For rotations:

$$\mathbf{R}^{-1} = \mathbf{R}^T$$

2D planar and linear transformations

$$\boldsymbol{x}' = f(\boldsymbol{x}; p)$$

$$\downarrow$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \boldsymbol{M} \begin{bmatrix} x \\ y \end{bmatrix}$$

parameters p point $\boldsymbol{x}$

2D planar and linear transformations

Scale

$$\mathbf{M} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$

Flip across y

$$\mathbf{M} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Rotate

$$\mathbf{M} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Flip across origin

$$\mathbf{M} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

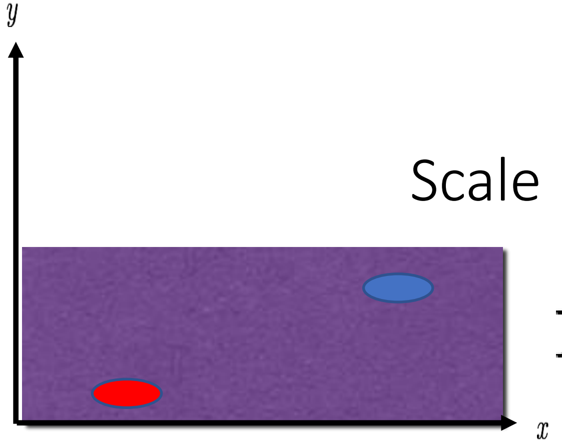
Shear

$$\mathbf{M} = \begin{bmatrix} 1 & s_x \\ s_y & 1 \end{bmatrix}$$

Identity

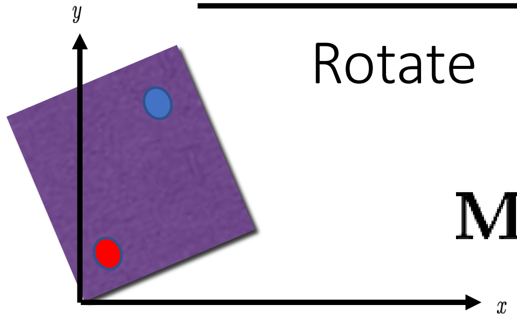
$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2D planar transformations



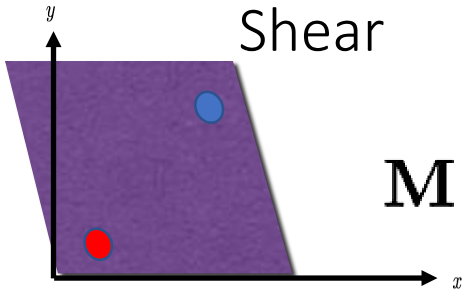
Scale

$$\mathbf{M} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$



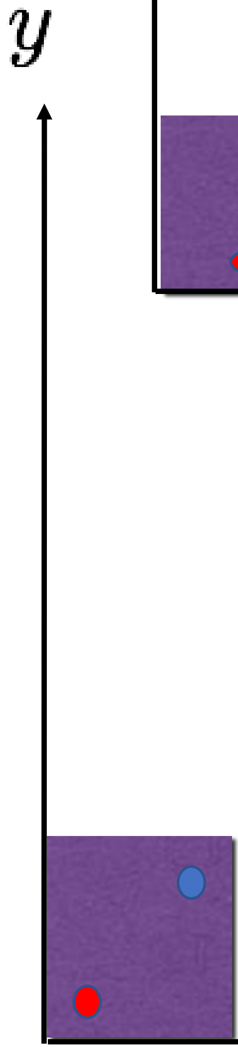
Rotate

$$\mathbf{M} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



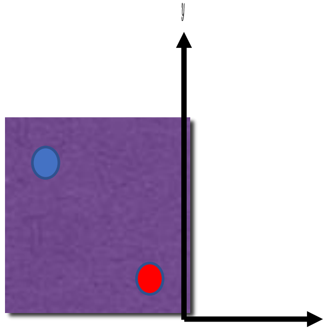
Shear

$$\mathbf{M} = \begin{bmatrix} 1 & s_x \\ s_y & 1 \end{bmatrix}$$



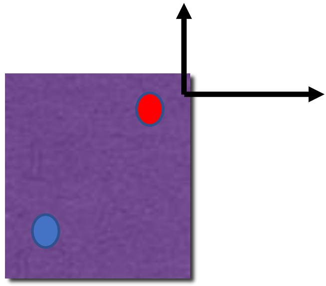
Flip across y

$$\mathbf{M} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$



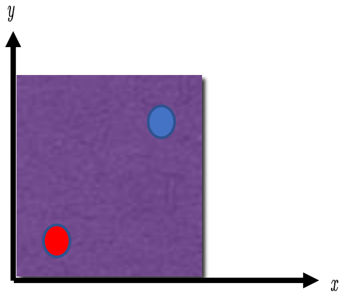
Flip across origin

$$\mathbf{M} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$



Identity

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



All 2D Linear Transformations

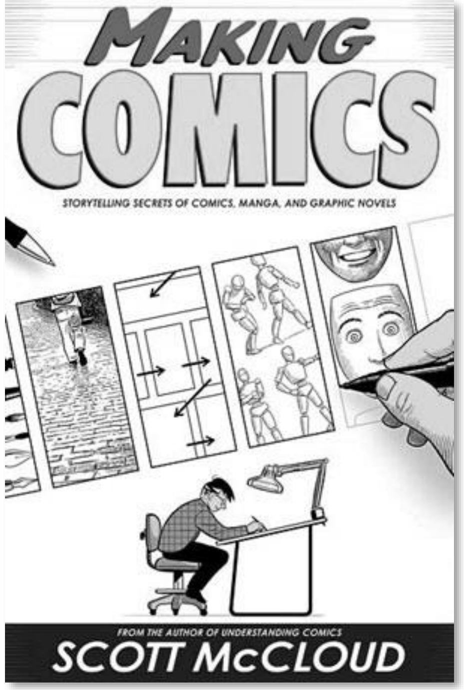
- Linear transformations are combinations of …
  - Scale,
  - Rotation,
  - Shear, and
  - Mirror

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Properties of linear transformations:
  - Origin maps to origin
  - Lines map to lines
  - Parallel lines remain parallel
  - Ratios are preserved
  - Closed under composition

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} i & j \\ k & l \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

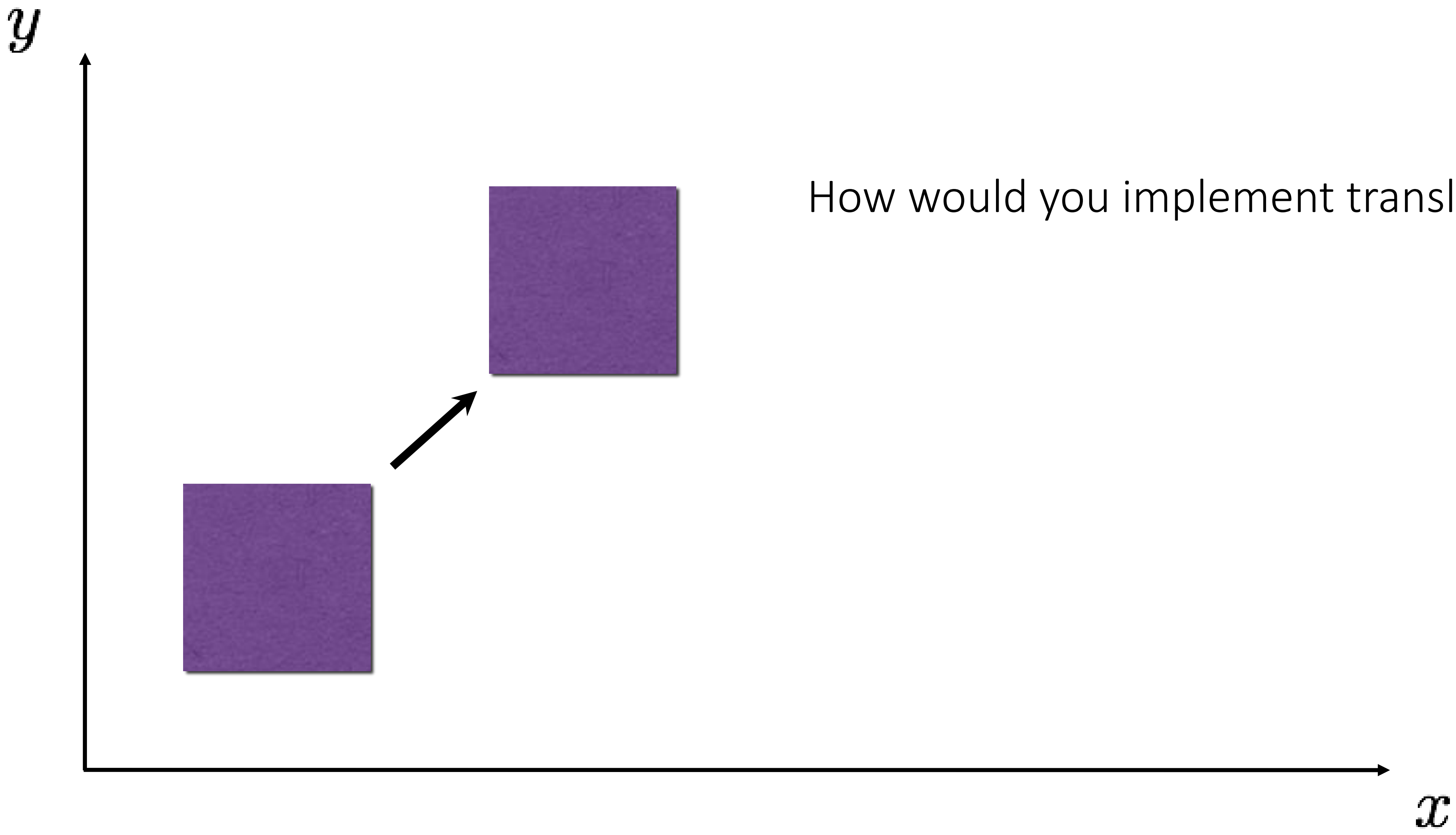
What is the geometric relationship between these two images?



?

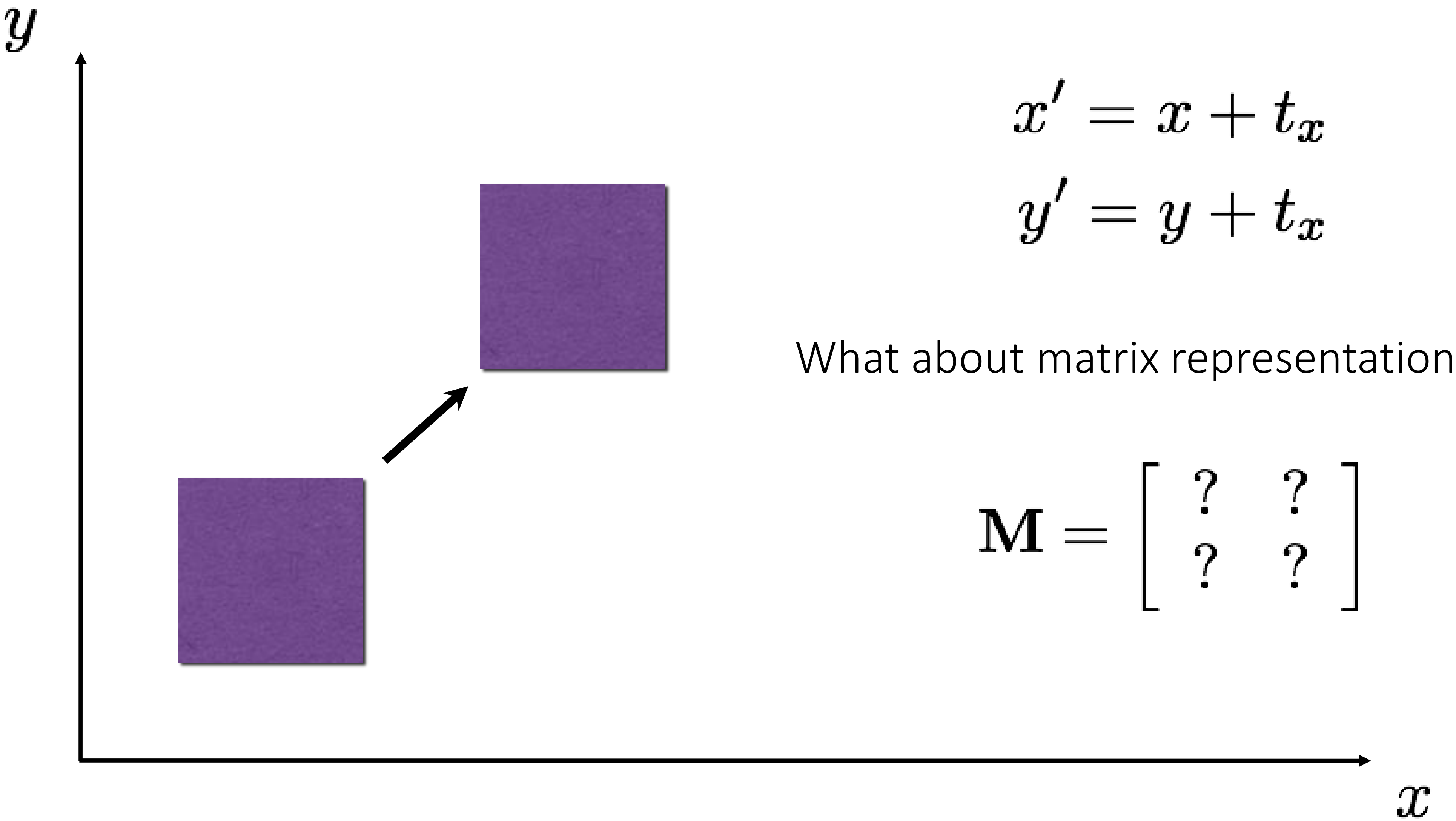
Answer: Similarity transformation (translation, rotation, uniform scale)

2D translation



How would you implement translation?

2D translation



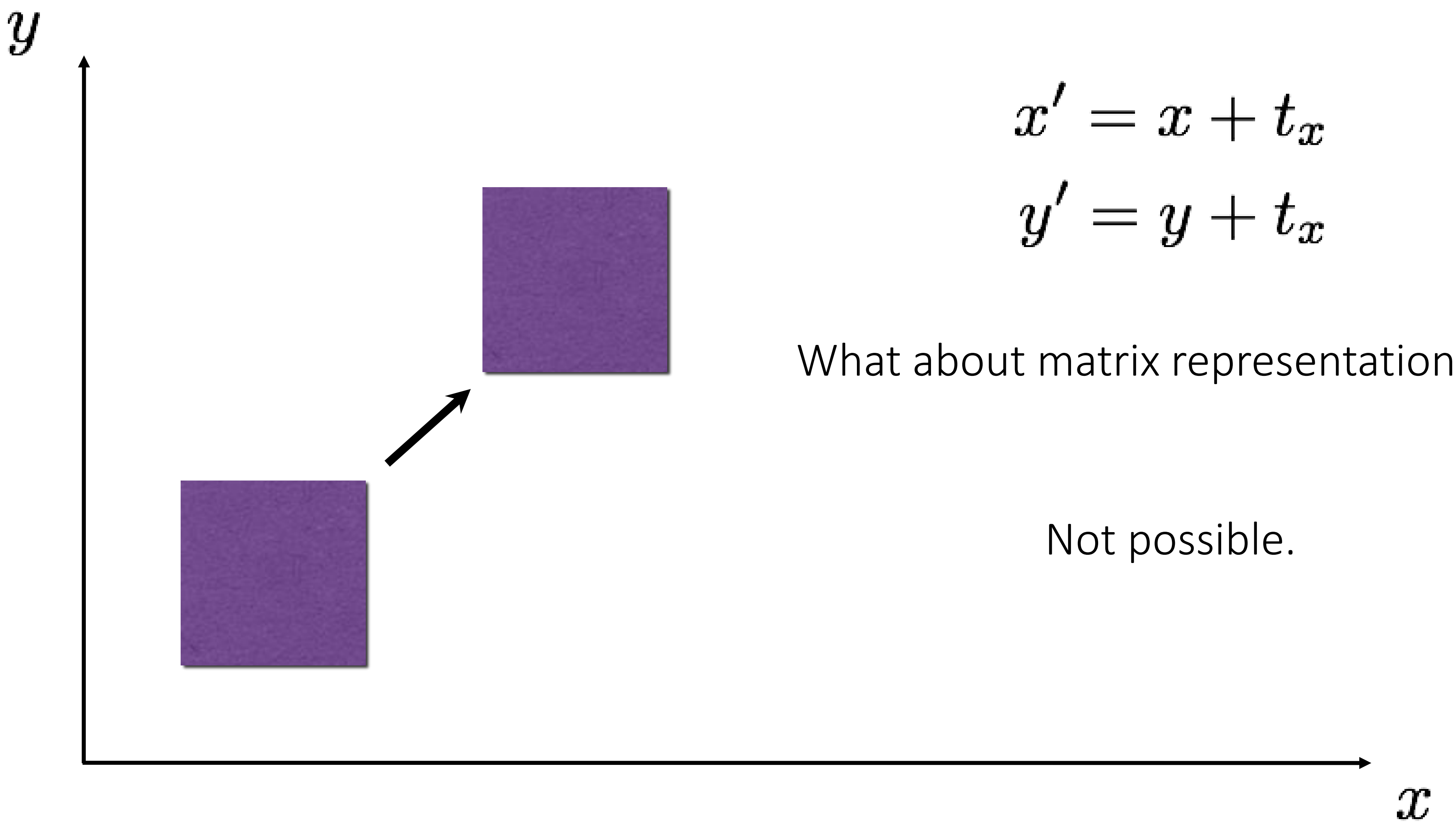
$$x' = x + t_x$$

$$y' = y + t_x$$

What about matrix representation?

$$\mathbf{M} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$$

2D translation



$$x' = x + t_x$$
$$y' = y + t_x$$

What about matrix representation?

Not possible.

Outline

- Linear algebra
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Homogeneous coordinates

heterogeneous coordinates | homogeneous coordinates

$$\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

add a 1 here

- Represent 2D point with a 3D vector

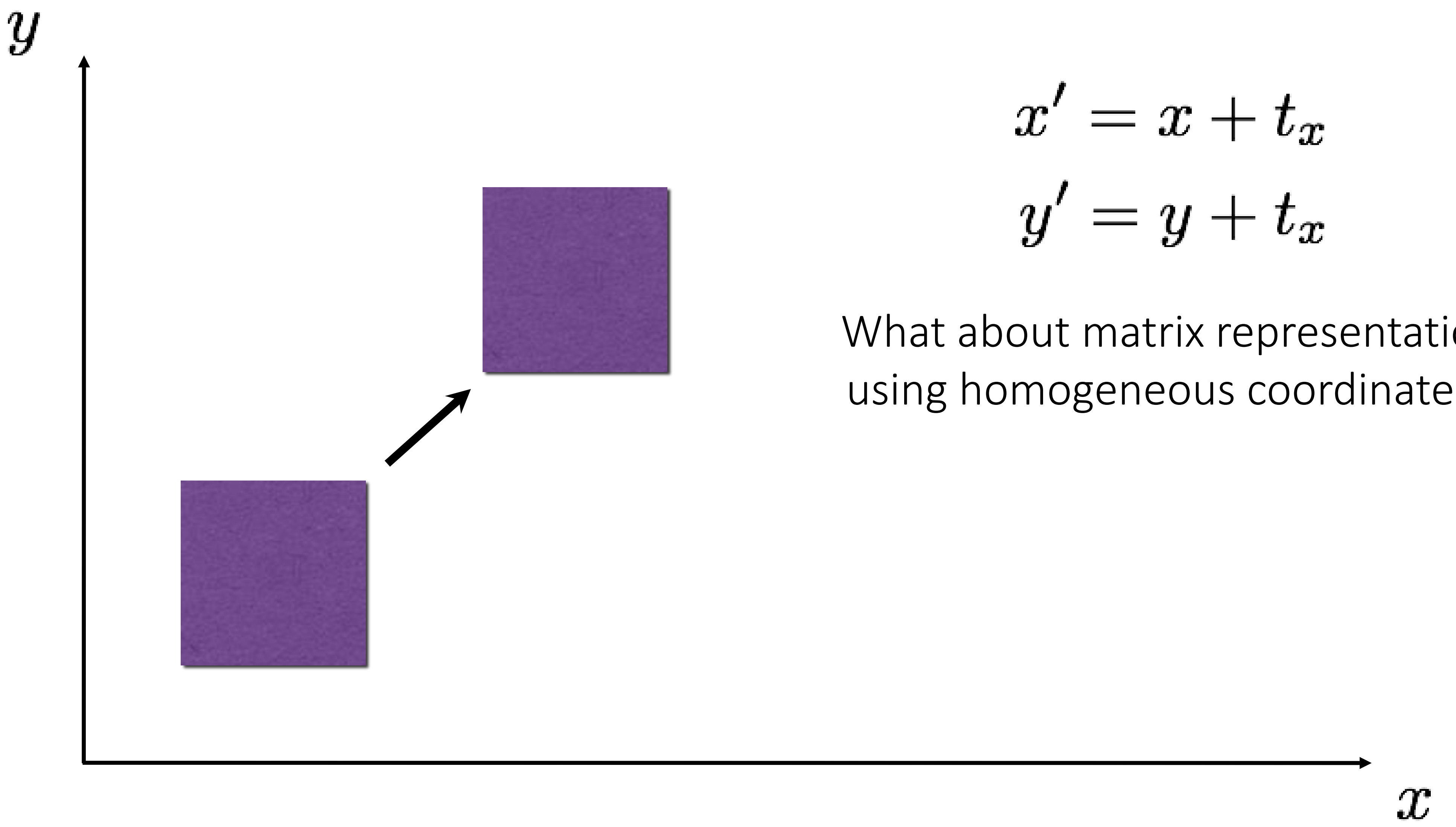
Homogeneous coordinates

heterogeneous coordinates	homogeneous coordinates

$$\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} ax \\ ay \\ a \end{bmatrix}$$

- Represent 2D point with a 3D vector
- 3D vectors are only defined up to scale

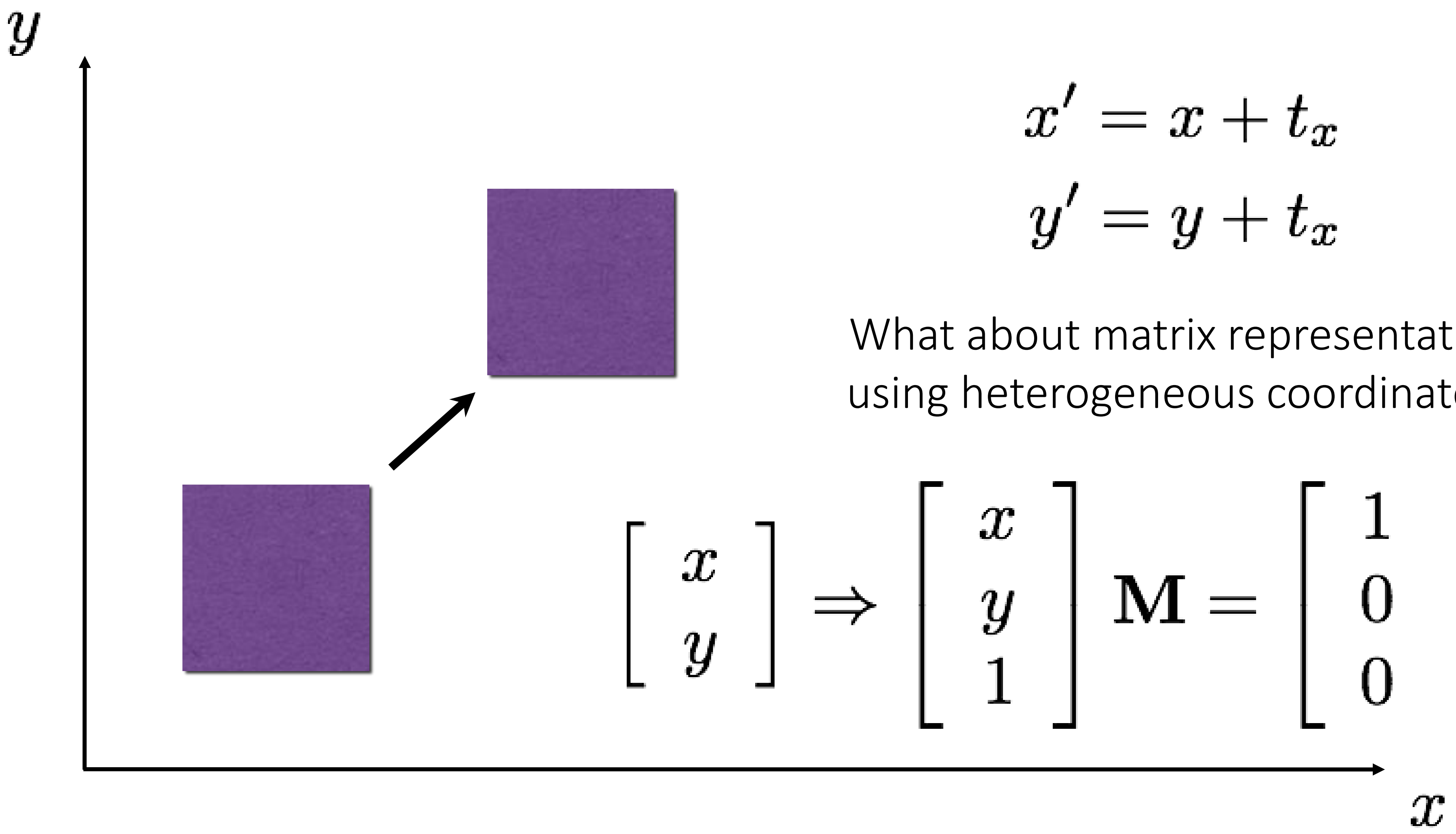
2D translation



$$x' = x + t_x$$
$$y' = y + t_x$$

What about matrix representation using homogeneous coordinates?

2D translation



$$x' = x + t_x$$

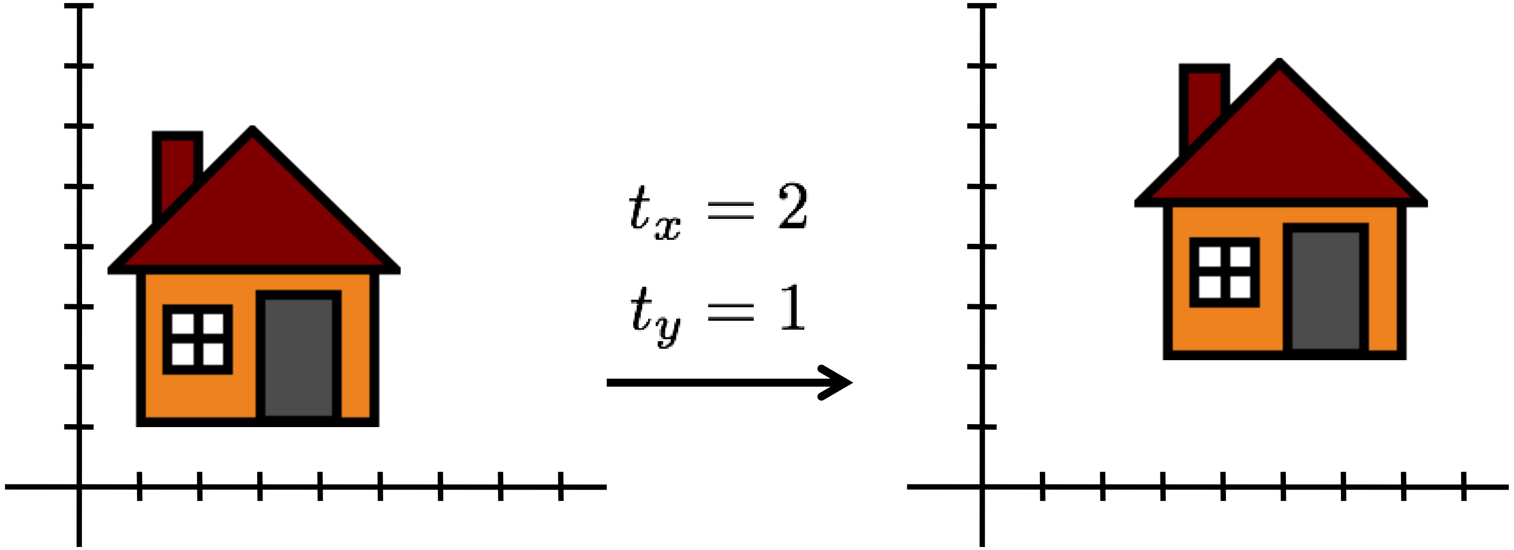
$$y' = y + t_x$$

What about matrix representation using heterogeneous coordinates?

$$\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \quad \mathbf{M} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

2D translation using homogeneous coordinates

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix}$$



Homogeneous coordinates

Conversion:

- heterogeneous → homogeneous

$$\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- homogeneous → heterogeneous

$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} \Rightarrow \begin{bmatrix} x/w \\ y/w \end{bmatrix}$$

- scale invariance

$$\begin{bmatrix} x & y & w \end{bmatrix}^\top = \lambda \begin{bmatrix} x & y & w \end{bmatrix}^\top$$

Special points:

- point at infinity (x,y)

$$\begin{bmatrix} x & y & 0 \end{bmatrix}$$

- undefined

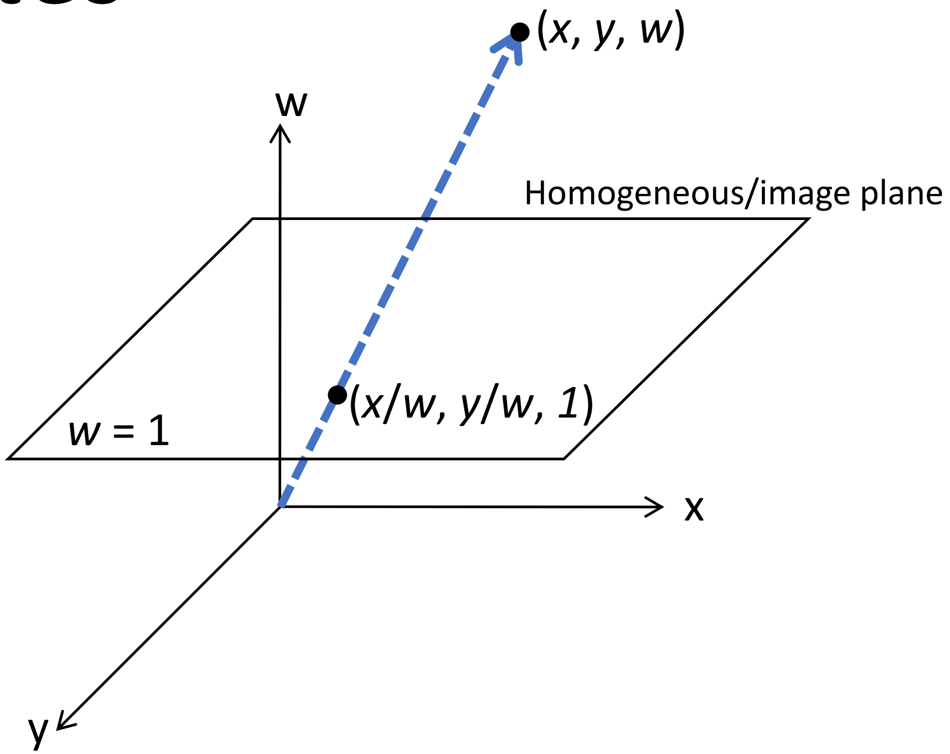
$$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

Homogeneous coordinates

Trick: add one more coordinate:

$$(x, y) \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

homogeneous image coordinates



Converting *from* homogeneous coordinates

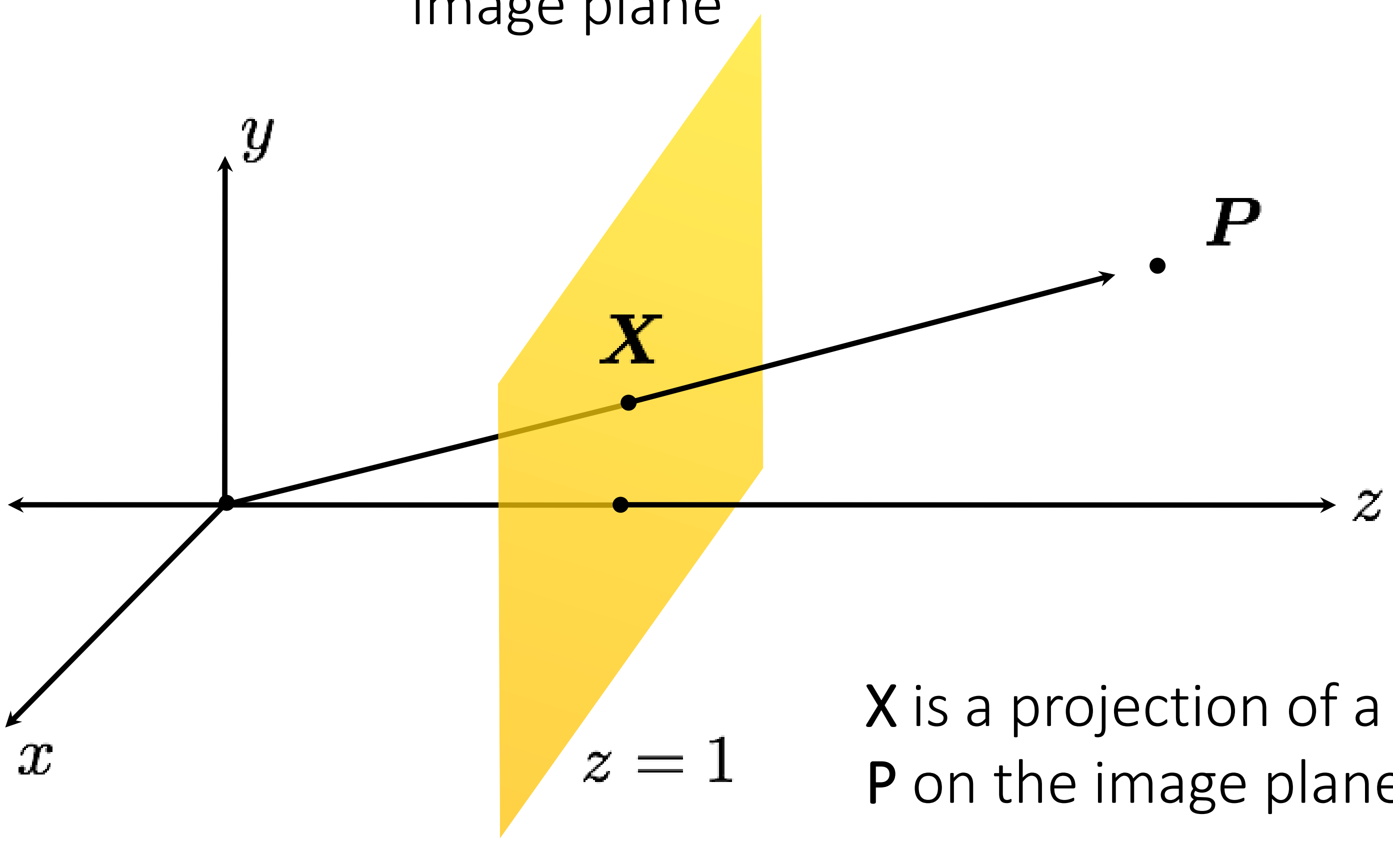
$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} \Rightarrow (x/w, y/w)$$

Projective geometry

image point in pixel coordinates $\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix}$

$$\Updownarrow$$

image point in homogeneous coordinates $\boldsymbol{X} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$



X is a projection of a point P on the image plane

What does scaling X correspond to?

Outline

- Linear algebra
- Image transformations.
- 2D transformations.
- Projective geometry
- **Transformations in projective geometry.**
- Classification of 2D transformations.
- Determining unknown 2D transformations.
- Determining unknown image warps.

2D transformations in heterogeneous coordinates

Re-write these transformations as 3x3 matrices:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

translation

$$\begin{bmatrix} \boldsymbol{x}' \\ \boldsymbol{y}' \\ 1 \end{bmatrix} = \begin{bmatrix} & & \\ & ? & \\ & & \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ 1 \end{bmatrix}$$

scaling

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} & & \\ & ? & \\ & & \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

rotation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} & & \\ & ? & \\ & & \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

shearing

2D transformations in heterogeneous coordinates

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$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

translation

$$\begin{bmatrix} \boldsymbol{x}' \\ \boldsymbol{y}' \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{s_x} & 0 & 0 \\ 0 & \boldsymbol{s_y} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ 1 \end{bmatrix}$$

scaling

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} & & \\ & ? & \\ & & \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

rotation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} & & \\ & ? & \\ & & \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

shearing

2D transformations in heterogeneous coordinates

Re-write these transformations as 3x3 matrices:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

translation

$$\begin{bmatrix} \boldsymbol{x'} \\ \boldsymbol{y'} \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{s_x} & 0 & 0 \\ 0 & \boldsymbol{s_y} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ 1 \end{bmatrix}$$

scaling

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} & & \\ & ? & \\ & & \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

rotation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & \beta_x & 0 \\ \beta_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

shearing

2D transformations in heterogeneous coordinates

Re-write these transformations as 3x3 matrices:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

translation

$$\begin{bmatrix} \boldsymbol{x'} \\ \boldsymbol{y'} \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{s_x} & 0 & 0 \\ 0 & \boldsymbol{s_y} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ 1 \end{bmatrix}$$

scaling

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

rotation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & \beta_x & 0 \\ \beta_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

shearing

Matrix composition

Transformations can be combined by matrix multiplication:

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 & tx \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} sx & 0 & 0 \\ 0 & sy & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

p' = ? ? ? p

Matrix composition

Transformations can be combined by matrix multiplication:

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 & tx \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} sx & 0 & 0 \\ 0 & sy & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

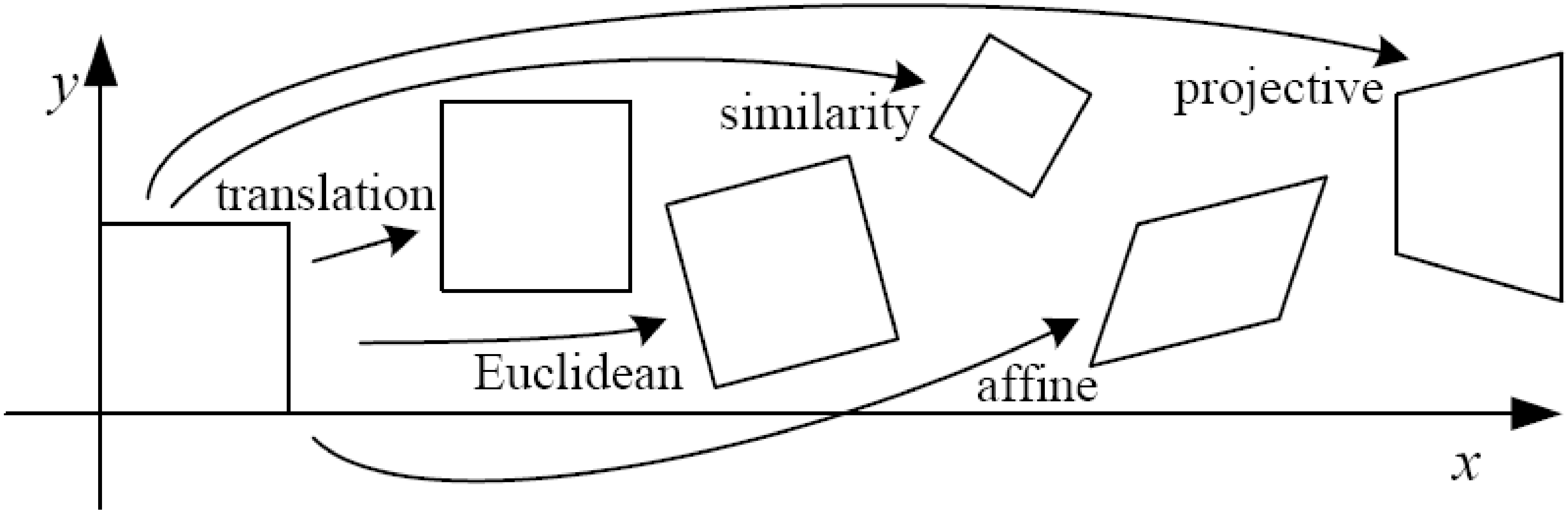
p' = translation(t_x,t_y) rotation(θ) scale(s,s) **p**

Does the multiplication order matter?

Outline

- Linear algebra
- Image transformations.
- 2D transformations.
- Projective geometry
- Transformations in projective geometry.
- **Classification of 2D transformations.**
- Determining unknown 2D transformations.
- Determining unknown image warps.

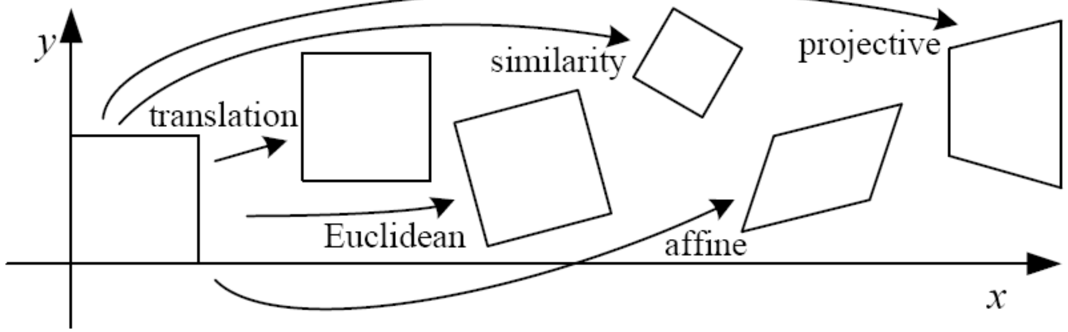
Classification of 2D transformations



Classification of 2D transformations

Name	Matrix	# D.O.F.
translation	$\begin{bmatrix} I \mid t \end{bmatrix}$	?
rigid (Euclidean)	$\begin{bmatrix} R \mid t \end{bmatrix}$	?
similarity	$\begin{bmatrix} sR \mid t \end{bmatrix}$	?
affine	$\begin{bmatrix} A \end{bmatrix}$	?
projective	$\begin{bmatrix} \tilde{H} \end{bmatrix}$	?

2D image transformations



Name	Matrix	# D.O.F.	Preserves:	Icon
translation	$\begin{bmatrix} I \mid t \end{bmatrix}_{2\times3}$	2	orientation $+\cdots$	
rigid (Euclidean)	$\begin{bmatrix} R \mid t \end{bmatrix}_{2\times3}$	3	lengths $+\cdots$	
similarity	$\begin{bmatrix} sR \mid t \end{bmatrix}_{2\times3}$	4	angles $+\cdots$	
affine	$\begin{bmatrix} A \end{bmatrix}_{2\times3}$	6	parallelism $+\cdots$	
projective	$\begin{bmatrix} \tilde{H} \end{bmatrix}_{3\times3}$	8	straight lines	

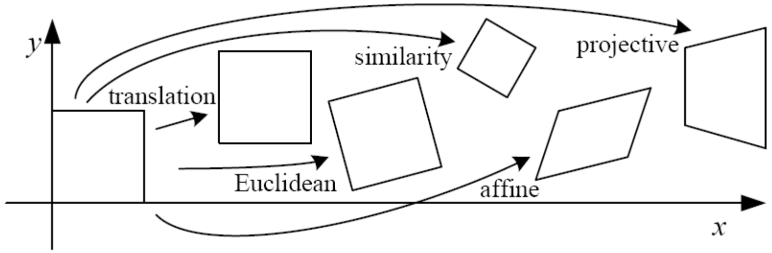
These transformations are a nested set of groups

- Closed under composition and inverse is a member

Classification of 2D transformations

Translation: $$\begin{bmatrix} 1 & 0 & t_1 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{bmatrix}$$

How many degrees of freedom?

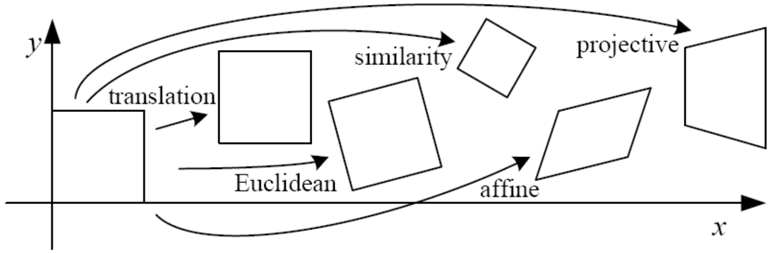


Classification of 2D transformations

Euclidean (rigid):
rotation + translation

$$\begin{bmatrix} r_1 & r_2 & r_3 \\ r_4 & r_5 & r_6 \\ 0 & 0 & 1 \end{bmatrix}$$

Are there any values that are related?

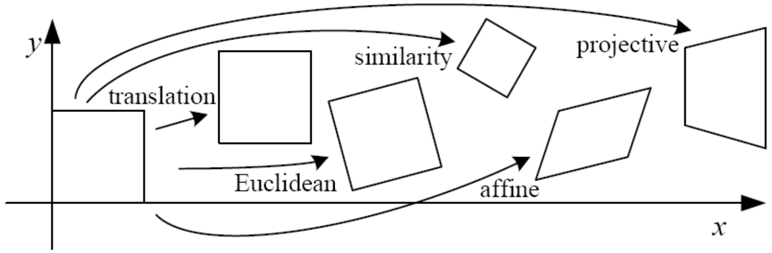


Classification of 2D transformations

Euclidean (rigid):
rotation + translation

$$\begin{bmatrix} \cos\theta & -\sin\theta & r_3 \\ \sin\theta & \cos\theta & r_6 \\ 0 & 0 & 1 \end{bmatrix}$$

How many degrees of freedom?

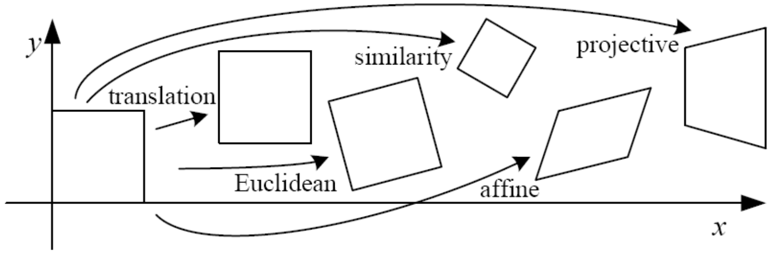


Classification of 2D transformations

which other matrix values will change if this increases?

Euclidean (rigid): rotation + translation

$$\begin{bmatrix} \cos\theta & -\sin\theta & r_3 \\ \sin\theta & \cos\theta & r_6 \\ 0 & 0 & 1 \end{bmatrix}$$

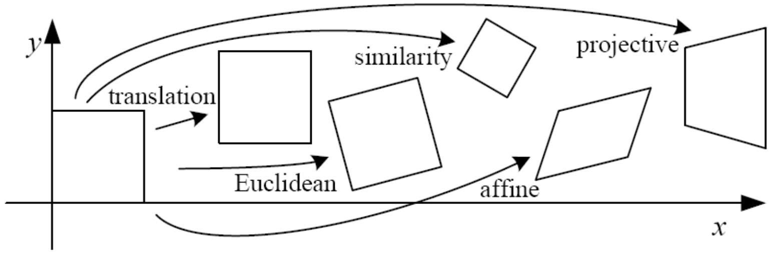


Classification of 2D transformations

what will happen to the image if this increases?

Euclidean (rigid): rotation + translation

$$\begin{bmatrix} \cos\theta & -\sin\theta & r_3 \\ \sin\theta & \cos\theta & r_6 \\ 0 & 0 & 1 \end{bmatrix}$$

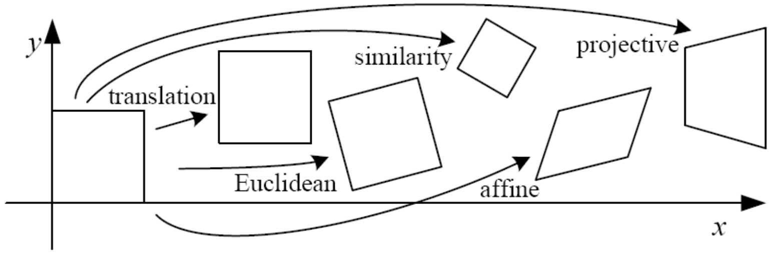


Classification of 2D transformations

what will happen to the image if this increases?

Euclidean (rigid):
rotation + translation

$$\begin{bmatrix} \cos\theta & -\sin\theta & r_3 \\ \sin\theta & \cos\theta & r_6 \\ 0 & 0 & 1 \end{bmatrix}$$

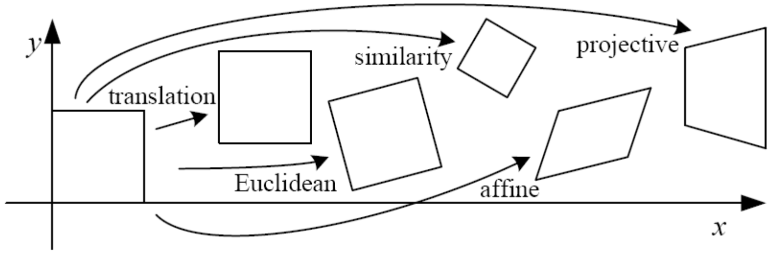


Classification of 2D transformations

Similarity:
uniform scaling + rotation
+ translation

$$\begin{bmatrix} r_1 & r_2 & r_3 \\ r_4 & r_5 & r_6 \\ 0 & 0 & 1 \end{bmatrix}$$

Are there any values that are related?



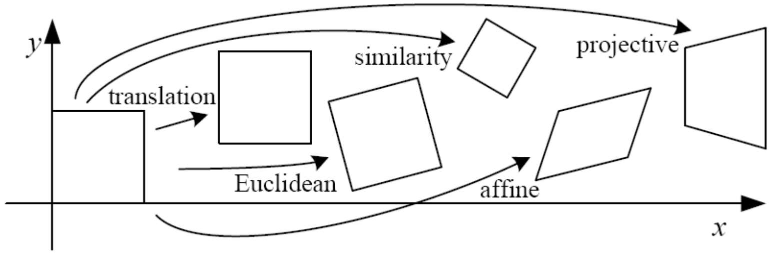
Classification of 2D transformations

multiply these four by scale **s**

Similarity:
uniform scaling + rotation
+ translation

$$\begin{bmatrix} \boxed{\begin{matrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{matrix}} & \begin{matrix} r_3 \\ r_6 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{bmatrix}$$

How many degrees of freedom?

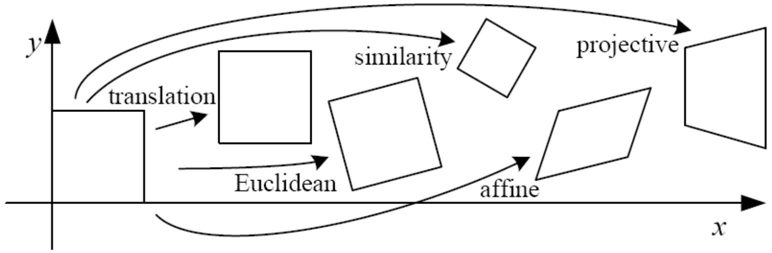


Classification of 2D transformations

what will happen to the image if this increases?

Similarity: uniform scaling + rotation + translation

$$\begin{bmatrix} r_1 & r_2 & r_3 \\ r_4 & r_5 & r_6 \\ 0 & 0 & 1 \end{bmatrix}$$



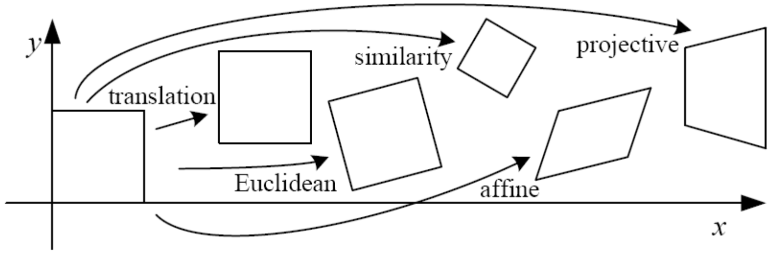
Classification of 2D transformations

Affine transform:
uniform scaling + shearing
+ rotation + translation

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & 1 \end{bmatrix}$$

any transformation represented by a 3x3 matrix with last row [0 0 1] we call an *affine* transformation

Are there any values that are related?



Classification of 2D transformations

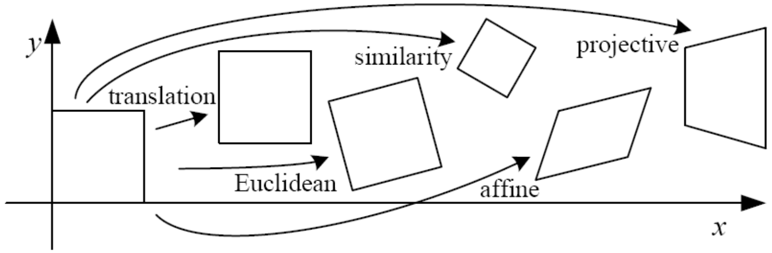
Affine transform:
uniform scaling + shearing
+ rotation + translation

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & 1 \end{bmatrix}$$

Are there any values that are related?

similarity shear

$$\begin{bmatrix} sr_1 & sr_2 \\ sr_3 & sr_4 \end{bmatrix} \begin{bmatrix} 1 & h_1 \\ h_2 & 1 \end{bmatrix} = \begin{bmatrix} sr_1 + h_2 sr_2 & sr_2 + h_1 sr_1 \\ sr_3 + h_2 sr_4 & sr_4 + h_1 sr_3 \end{bmatrix}$$



Classification of 2D transformations

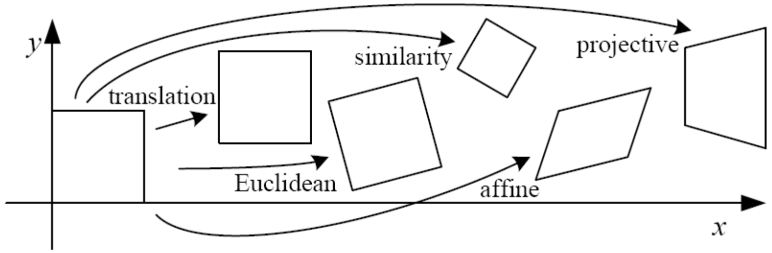
Affine transform:
uniform scaling + shearing
+ rotation + translation

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & 1 \end{bmatrix}$$

How many degrees of freedom?

similarity shear

$$\begin{bmatrix} sr_1 & sr_2 \\ sr_3 & sr_4 \end{bmatrix} \begin{bmatrix} 1 & h_1 \\ h_2 & 1 \end{bmatrix} = \begin{bmatrix} sr_1 + h_2 sr_2 & sr_2 + h_1 sr_1 \\ sr_3 + h_2 sr_4 & sr_4 + h_1 sr_3 \end{bmatrix}$$



Affine transformations

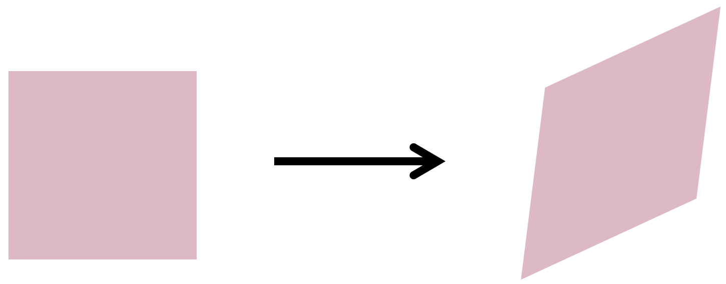
Affine transformations are combinations of

- arbitrary (4-DOF) linear transformations; and
- translations

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

Properties of affine transformations:

- origin does not necessarily map to origin
- lines map to lines
- parallel lines map to parallel lines
- ratios are preserved
- compositions of affine transforms are also affine transforms



Does the last coordinate w ever change?

Affine transformations

Affine transformations are combinations of

- arbitrary (4-DOF) linear transformations; and
- translations

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

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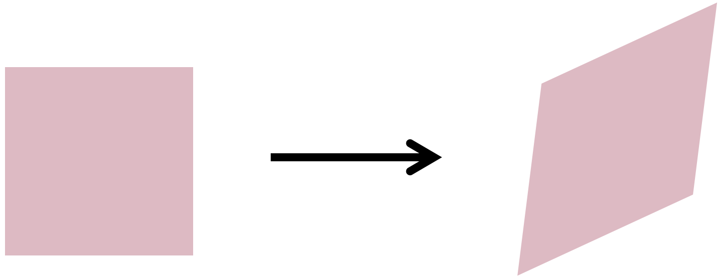


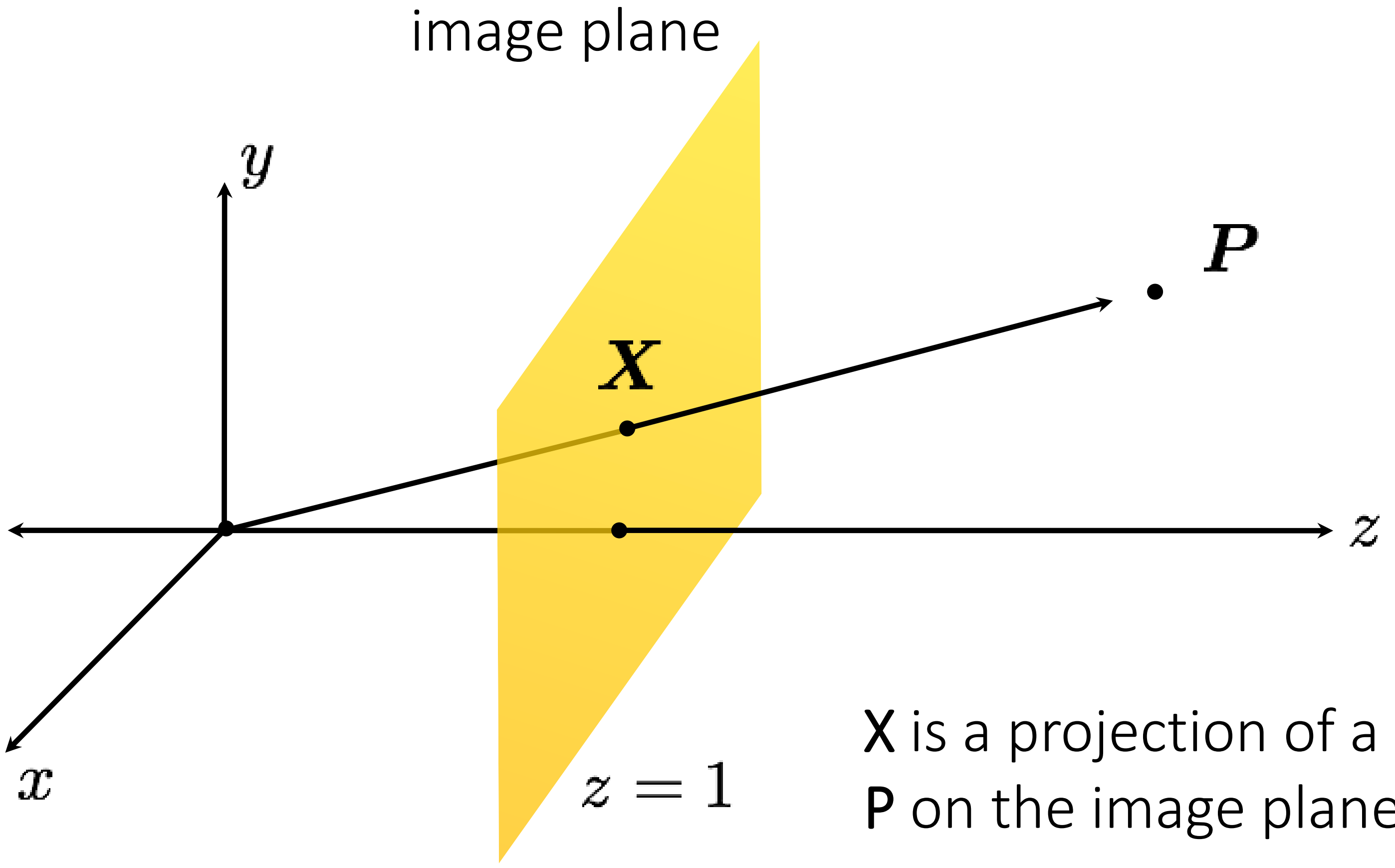
Nope! But what does that mean?

How to interpret affine transformations here?

image point in pixel coordinates $\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix}$

$\Updownarrow$

image point in heterogeneous coordinates $\boldsymbol{X} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$



X is a projection of a point P on the image plane

Where do we go from here?

$$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix}$$

what happens when we mess with this row?

affine transformation

Projective Transformations aka Homographies aka Planar Perspective Maps

$$\mathbf{H} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & 1 \end{bmatrix}$$

Called a *homography*
(or *planar perspective map*)

Homographies

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} \frac{ax+by+c}{gx+hy+1} \\ \frac{dx+ey+f}{gx+hy+1} \\ 1 \end{bmatrix}$$

Alternate formulation for homographies

$$\begin{bmatrix} x'_i \\ y'_i \\ 1 \end{bmatrix} \cong \begin{bmatrix} h_{00} & h_{01} & h_{02} \\ h_{10} & h_{11} & h_{12} \\ h_{20} & h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix}$$

where the length of the vector [h_{00} h_{01} … h_{22}] is 1

Projective transformations (aka homographies)

Projective transformations are combinations of

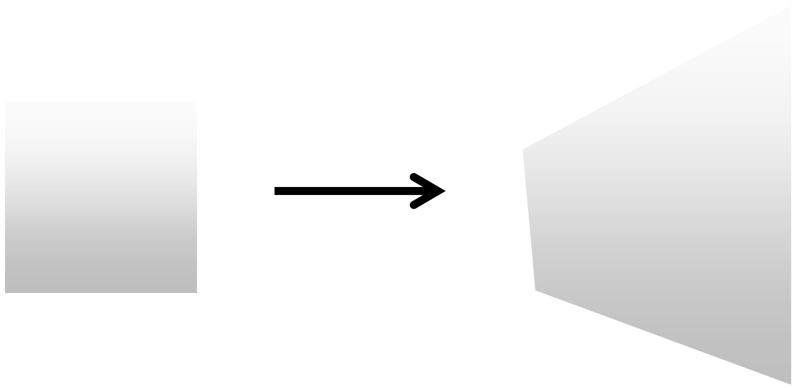
- affine transformations; and
- projective wraps

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

How many degrees of freedom?

Properties of projective transformations:

- origin does not necessarily map to origin
- lines map to lines
- parallel lines do not necessarily map to parallel lines
- ratios are not necessarily preserved
- compositions of projective transforms are also projective transforms



Projective transformations (aka homographies)

Projective transformations are combinations of

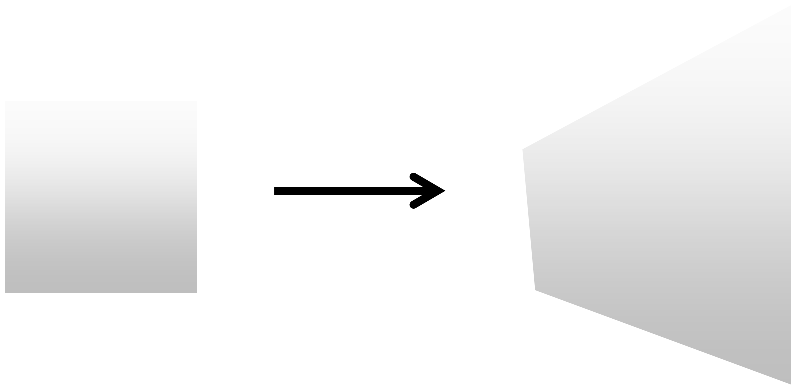
- affine transformations; and
- projective wraps

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

8 DOF: vectors (and therefore matrices) are defined up to scale)

Properties of projective transformations:

- origin does not necessarily map to origin
- lines map to lines
- parallel lines do not necessarily map to parallel lines
- ratios are not necessarily preserved
- compositions of projective transforms are also projective transforms

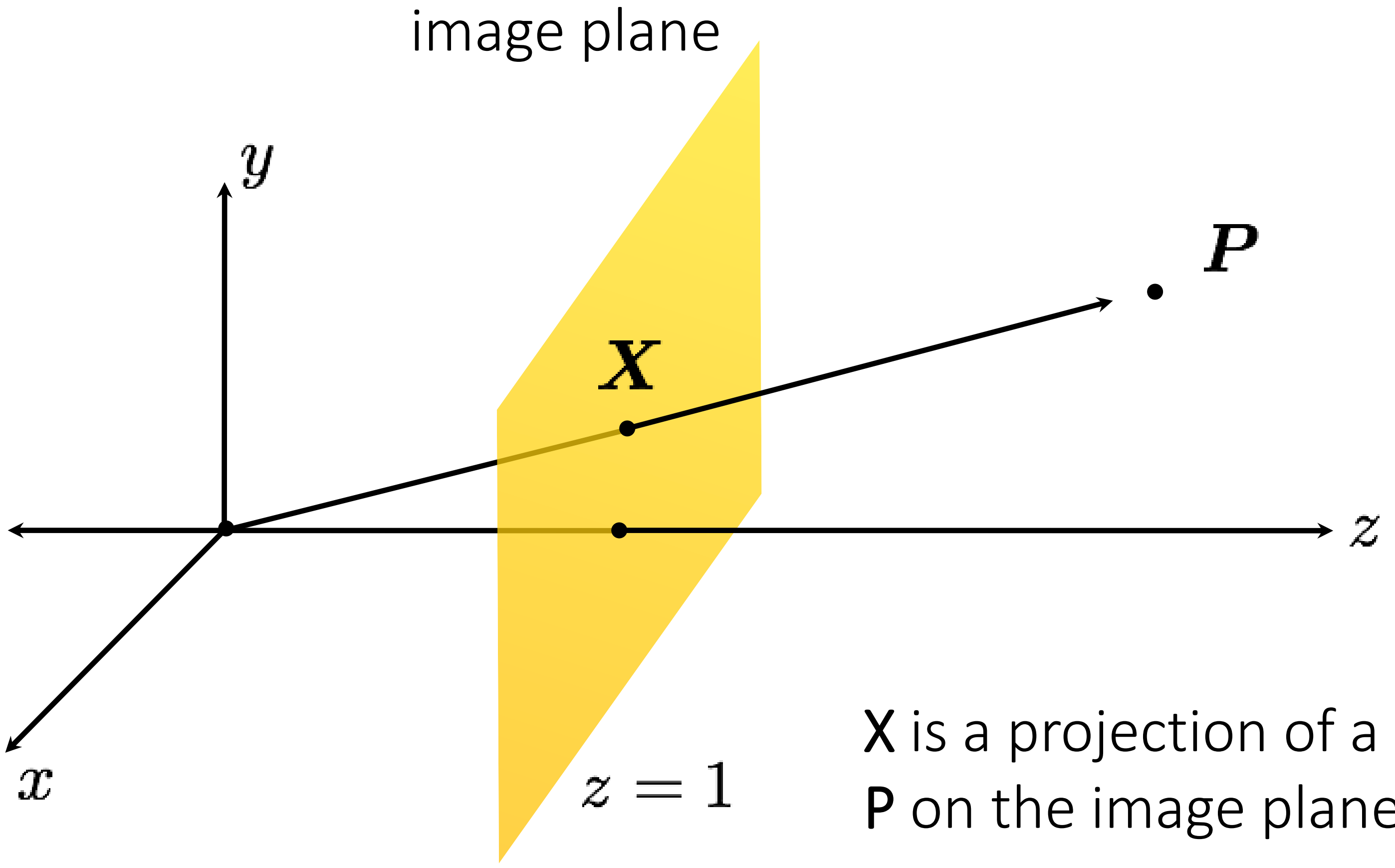


How to interpret projective transformations here?

image point in pixel coordinates $\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix}$

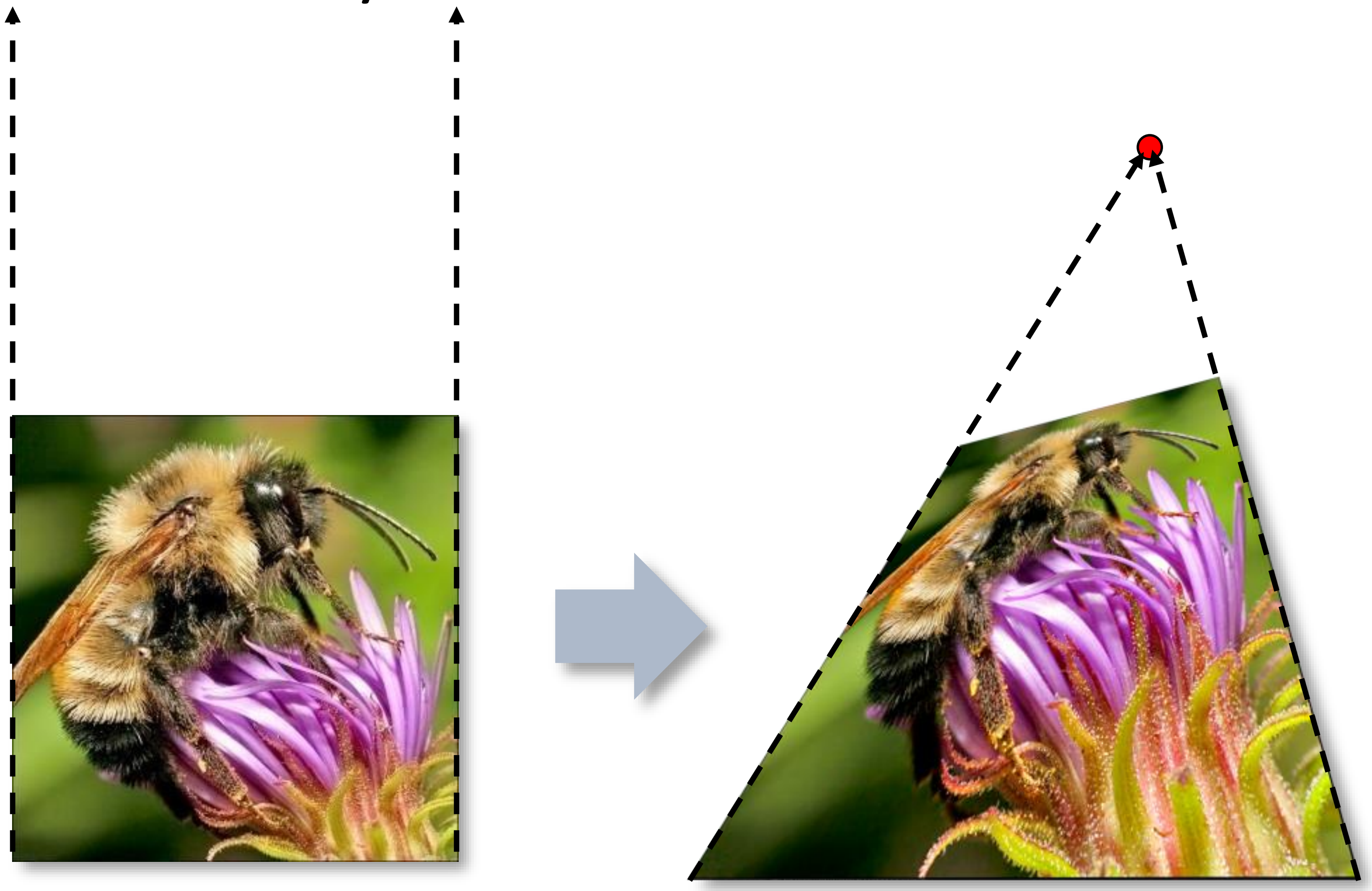
$$\Updownarrow$$

image point in heterogeneous coordinates $\boldsymbol{X} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$



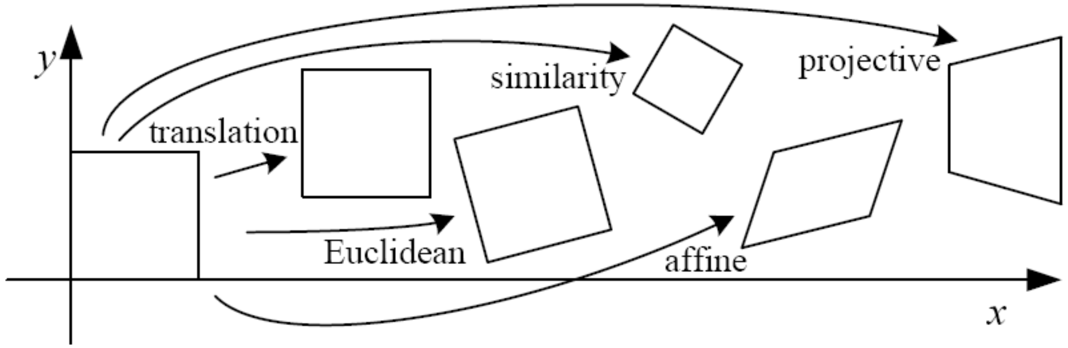
X is a projection of a point P on the image plane

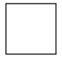
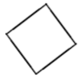
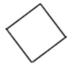
Points at infinity



Is this an affine transformation?

2D image transformations



Name	Matrix	# D.O.F.	Preserves:	Icon
translation	$\begin{bmatrix} \boldsymbol{I} \mid \boldsymbol{t} \end{bmatrix}_{2\times3}$	2	orientation $+\cdots$	
rigid (Euclidean)	$\begin{bmatrix} \boldsymbol{R} \mid \boldsymbol{t} \end{bmatrix}_{2\times3}$	3	lengths $+\cdots$	
similarity	$\begin{bmatrix} s\boldsymbol{R} \mid \boldsymbol{t} \end{bmatrix}_{2\times3}$	4	angles $+\cdots$	
affine	$\begin{bmatrix} \boldsymbol{A} \end{bmatrix}_{2\times3}$	6	parallelism $+\cdots$	
projective	$\begin{bmatrix} \tilde{\boldsymbol{H}} \end{bmatrix}_{3\times3}$	8	straight lines	

These transformations are a nested set of groups

- Closed under composition and inverse is a member

When can we use homographies?

We can use homographies when…

1. … the scene is planar; or

2. … the scene is very far or has small (relative) depth variation
→ scene is approximately planar

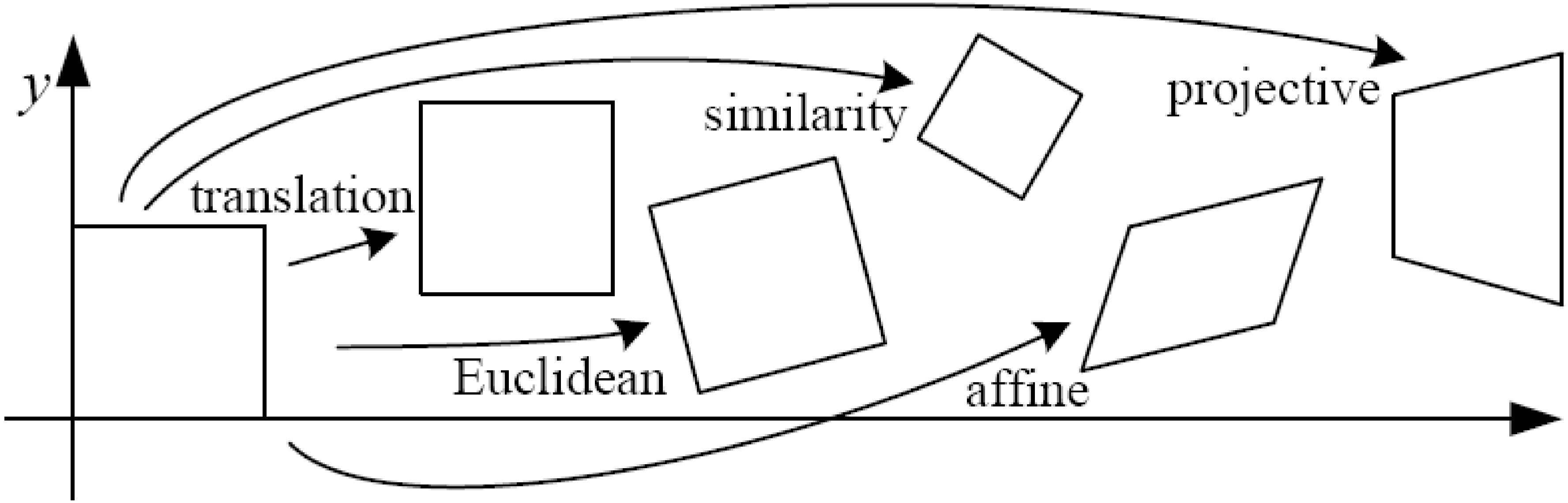
We can use homographies when…

3. … the scene is captured under camera rotation only (no translation or pose change)

More on why this is the case in a later lecture.

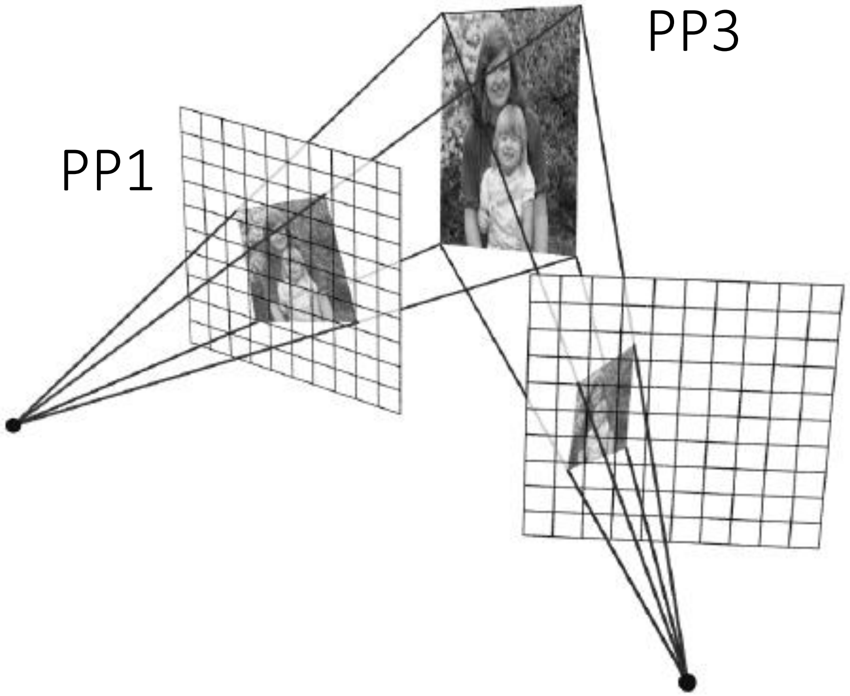
Computing with homographies

Classification of 2D transformations



Which kind transformation is needed to warp projective plane 1 into projective plane 2?

- A projective transformation (a.k.a. a homography).



Applying a homography

1. Convert to homogeneous coordinates: $p = \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow P = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

What is the size of the homography matrix?

2. Multiply by the homography matrix: $P' = H \cdot P$

3. Convert back to heterogeneous coordinates: $P' = \begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} \Rightarrow p' = \begin{bmatrix} x'/w' \\ y'/w' \end{bmatrix}$

Applying a homography

1. Convert to homogeneous coordinates: $p = \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow P = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

What is the size of the homography matrix? Answer: 3 x 3

2. Multiply by the homography matrix: $P' = H \cdot P$

How many degrees of freedom does the homography matrix have?

3. Convert back to heterogeneous coordinates: $P' = \begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} \Rightarrow p' = \begin{bmatrix} x'/w' \\ y'/w' \end{bmatrix}$

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How do we compute the homography matrix?

Homography

Under homography, we can write the transformation of points in 3D from camera 1 to camera 2 as:

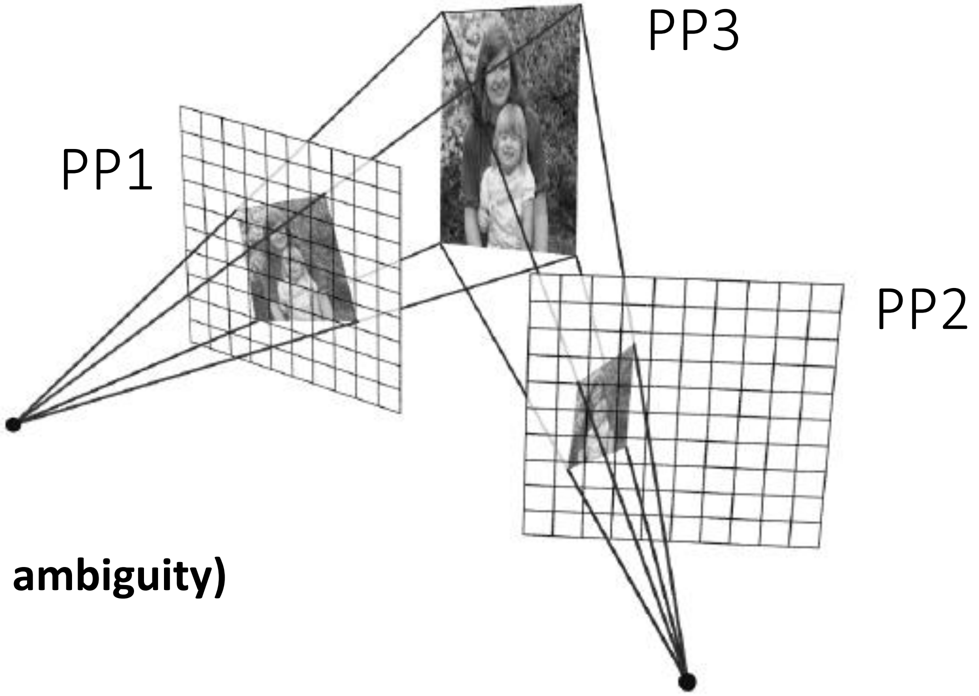
$$\mathbf{X}_2 = H\mathbf{X}_1 \quad \mathbf{X}_1, \mathbf{X}_2 \in \mathbb{R}^3$$

Homogeneous coordinates

In the image planes, using homogeneous coordinates, we have

$$\lambda_1 \mathbf{x}_1 = \mathbf{X}_1, \quad \lambda_2 \mathbf{x}_2 = \mathbf{X}_2, \quad \text{therefore} \quad \lambda_2 \mathbf{x}_2 = H\lambda_1 \mathbf{x}_1$$

Heterogeneous coordinates



This means that x2 is equal to Hx1 up to a scale (due to universal scale ambiguity)

Outline

- Linear algebra
- Image transformations
- 2D transformations.
- Projective geometry 101.
- Transformations in projective geometry.
- Classification of 2D transformations.
- **Determining unknown 2D transformations.**
- Determining unknown image warps.

References

Basic reading:

- Szeliski textbook, Section 3.6.

Additional reading:

- Hartley and Zisserman, “Multiple View Geometry in Computer Vision,” Cambridge University Press 2004.
  a comprehensive treatment of all aspects of projective geometry relating to computer vision, and also a very useful reference for the second part of the class.
- Richter-Gebert, “Perspectives on projective geometry,” Springer 2011.
  a beautiful, thorough, and very accessible mathematics textbook on projective geometry (available online for free from CMU’s library).

Questions?