CAP 4453 Robot Vision

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Administrative details

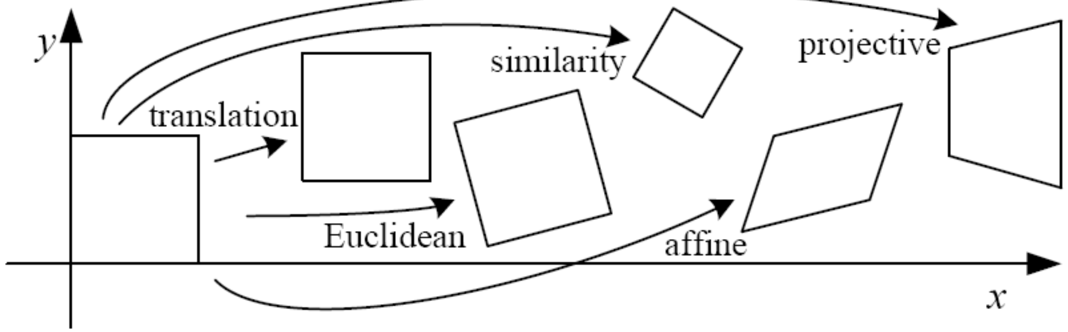
- Issues submitting homework

Short Review from last class

Outline

- Linear algebra
- Image transformations
- 2D transformations.
- Projective geometry 101.
- Transformations in projective geometry.
- Classification of 2D transformations.
- **Determining unknown 2D transformations.**
- Determining unknown image warps.

2D image transformations



Name	Matrix	# D.O.F.	Preserves:	Icon
translation	$\begin{bmatrix} \boldsymbol{I} \mid \boldsymbol{t} \end{bmatrix}_{2\times 3}$	2	orientation $+ \cdots$	
rigid (Euclidean)	$\begin{bmatrix} \boldsymbol{R} \mid \boldsymbol{t} \end{bmatrix}_{2\times 3}$	3	lengths $+ \cdots$	
similarity	$\begin{bmatrix} s\boldsymbol{R} \mid \boldsymbol{t} \end{bmatrix}_{2\times 3}$	4	angles $+ \cdots$	
affine	$\begin{bmatrix} \boldsymbol{A} \end{bmatrix}_{2\times 3}$	6	parallelism $+ \cdots$	
projective	$\begin{bmatrix} \tilde{\boldsymbol{H}} \end{bmatrix}_{3\times 3}$	8	straight lines	

These transformations are a nested set of groups

- Closed under composition and inverse is a member

Least squares

$$\mathbf{A}\mathbf{t} = \mathbf{b}$$

- Find **t** that minimizes

$$||\mathbf{A}\mathbf{t} - \mathbf{b}||^2$$

- To solve, form the *normal equations*

$$\mathbf{A}^{\mathrm{T}}\mathbf{A}\mathbf{t} = \mathbf{A}^{\mathrm{T}}\mathbf{b}$$

$$\mathbf{t} = \left(\mathbf{A}^{\mathrm{T}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{b}$$

Translation transformation

- Can also write as a matrix equation

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ & \vdots \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} x'_1 - x_1 \\ y'_1 - y_1 \\ x'_2 - x_2 \\ y'_2 - y_2 \\ \vdots \\ x'_n - x_n \\ y'_n - y_n \end{bmatrix}$$

$$\underset{2n \times 2}{\mathbf{A}} \quad \underset{2 \times 1}{\mathbf{t}} = \underset{2n \times 1}{\mathbf{b}}$$

Affine transformations

- Matrix form

$$\underset{2n \times 6}{\mathbf{A}} \; \underset{6 \times 1}{\mathbf{t}} = \underset{2n \times 1}{\mathbf{b}}$$

$$\begin{bmatrix} x_1 & y_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1 & y_1 & 1 \\ x_2 & y_2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2 & y_2 & 1 \\ & & & \vdots & & \\ x_n & y_n & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_n & y_n & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} x'_1 \\ y'_1 \\ x'_2 \\ y'_2 \\ \vdots \\ x'_n \\ y'_n \end{bmatrix}$$

Determining the homography matrix

Stack together constraints from multiple point correspondences:

$$\mathbf{A}\boldsymbol{h} = \mathbf{0}$$

$$\begin{bmatrix} \begin{bmatrix} -x & -y & -1 & 0 & 0 & 0 & xx' & yx' & x' \\ 0 & 0 & 0 & -x & -y & -1 & xy' & yy' & y' \end{bmatrix} \\ \begin{bmatrix} -x & -y & -1 & 0 & 0 & 0 & xx' & yx' & x' \\ 0 & 0 & 0 & -x & -y & -1 & xy' & yy' & y' \end{bmatrix} \\ \vdots \\ \begin{bmatrix} -x & -y & -1 & 0 & 0 & 0 & xx' & yx' & x' \\ 0 & 0 & 0 & -x & -y & -1 & xy' & yy' & y' \end{bmatrix} \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \\ h_7 \\ h_8 \\ h_9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Homogeneous linear least squares problem

- Solve with SVD

Robot Vision

10b. Linear Algebra SVD

Linear Algebra

- Matrix as a Linear Transformation
- Eigenvalues and eigenvector
  - Intuition
  - How to compute it
- Singular Value Descomposition (SVD)
  - Definition
  - Intuition
  - Direct Solving Ax=0

Matrix as Linear Transformation

$$T(\vec{v}) = A\vec{v}$$

Example

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$

$$T(\vec{v}) = A\vec{v}$$

$$T(\vec{v}) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

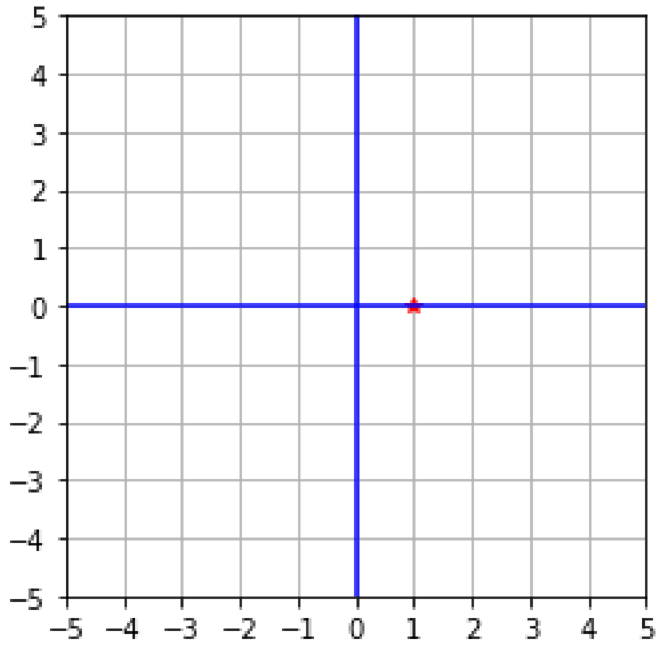
Matrix as Linear Transformation

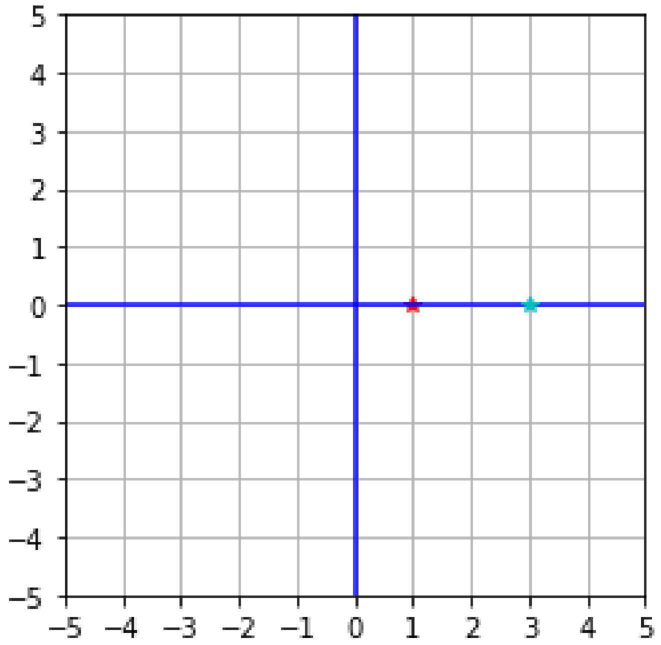
$$T(\vec{v}) = A\vec{v}$$

$$T(\vec{v}) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Case x=1 , y=0

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$



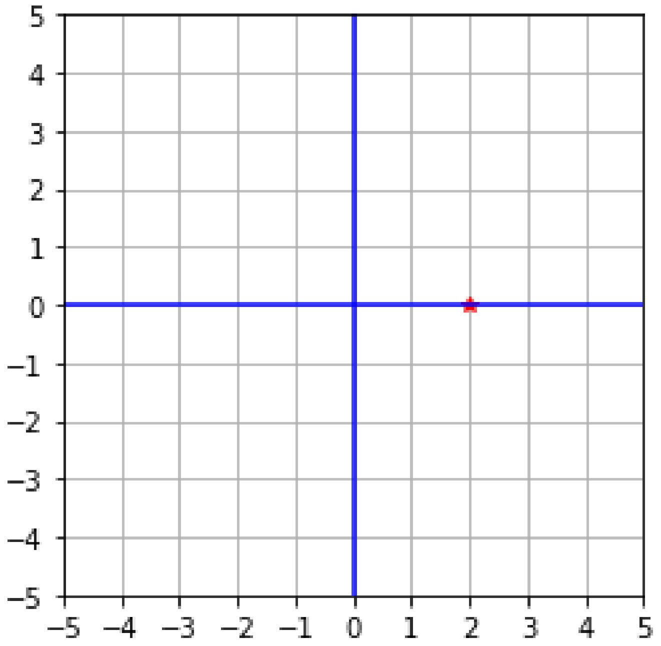


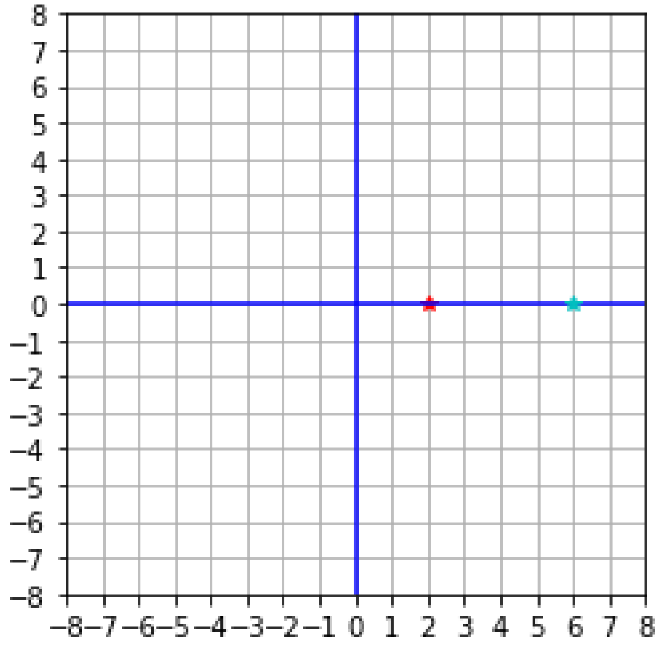
Matrix as Linear Transformation $T(\vec{v}) = A\vec{v}$

$$T(\vec{v}) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Case x=2 , y=0

$$T\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$





Matrix as Linear Transformation

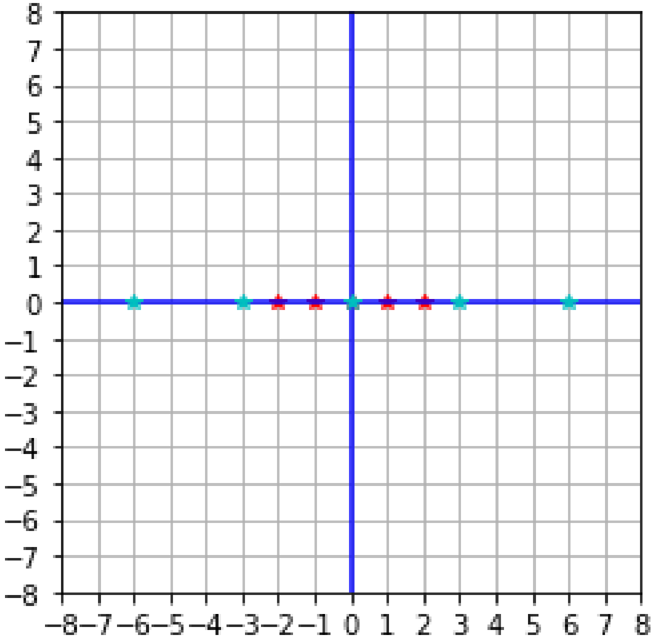
$T(\vec{v}) = A\vec{v}$

X direction

$$T(\vec{v}) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$T\left(\begin{bmatrix} x \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} x$$

Case x=-2,-1,0,1,2 , y=0

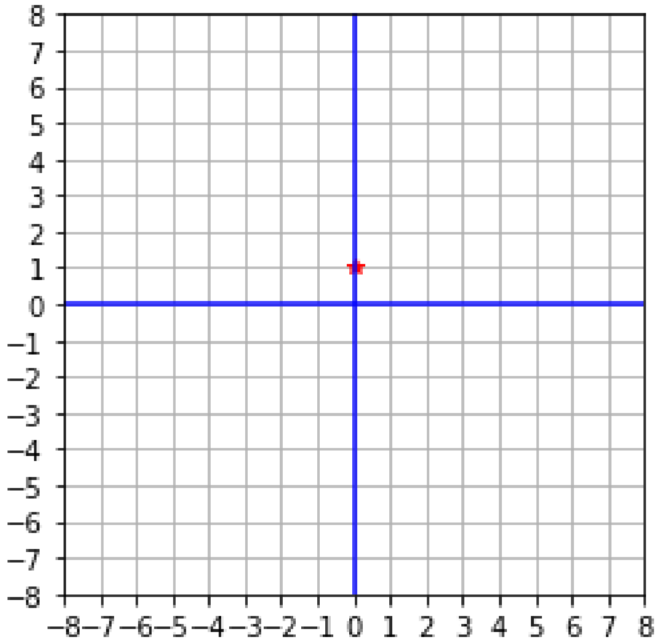


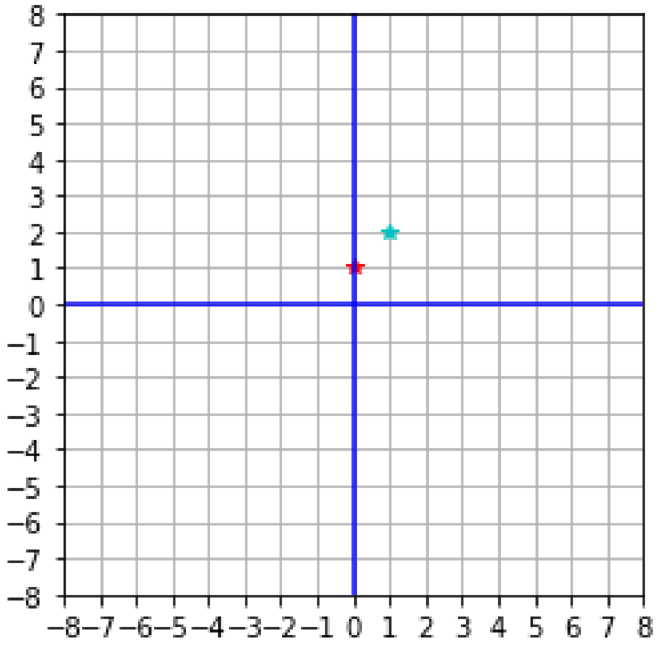
Matrix as Linear Transformation $T(\vec{v}) = A\vec{v}$

$$T(\vec{v}) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Case x=0 , y=1



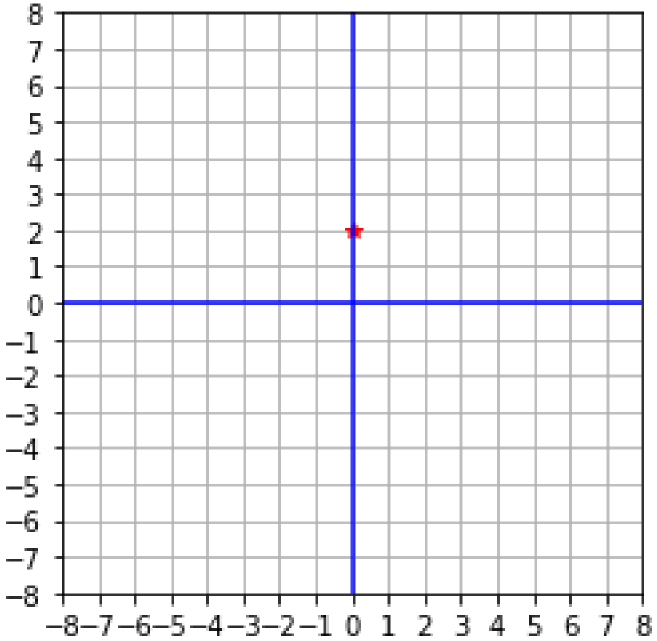


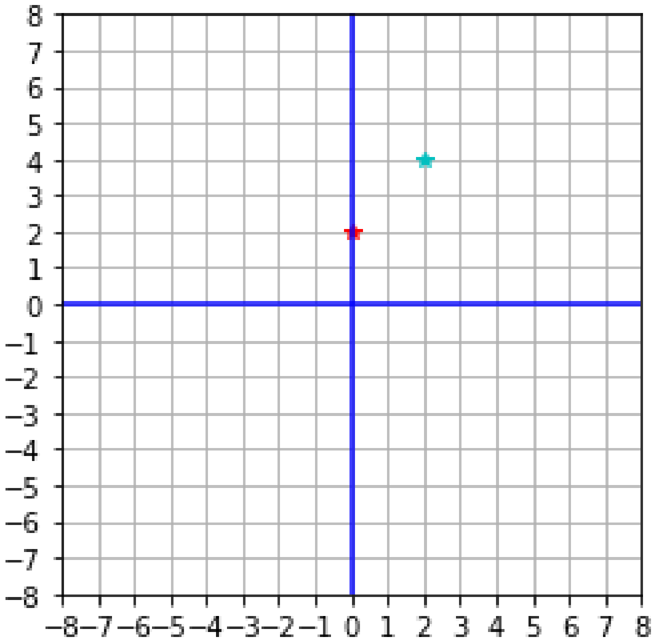
Matrix as Linear Transformation $T(\vec{v}) = A\vec{v}$

$$T(\vec{v}) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Case x=0 , y=2





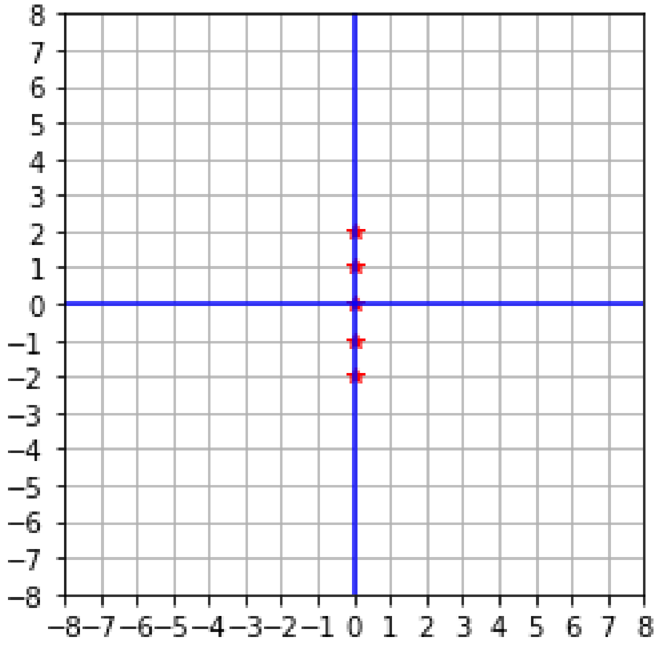
Matrix as Linear Transformation $\quad T(\vec{v}) = A\vec{v}$

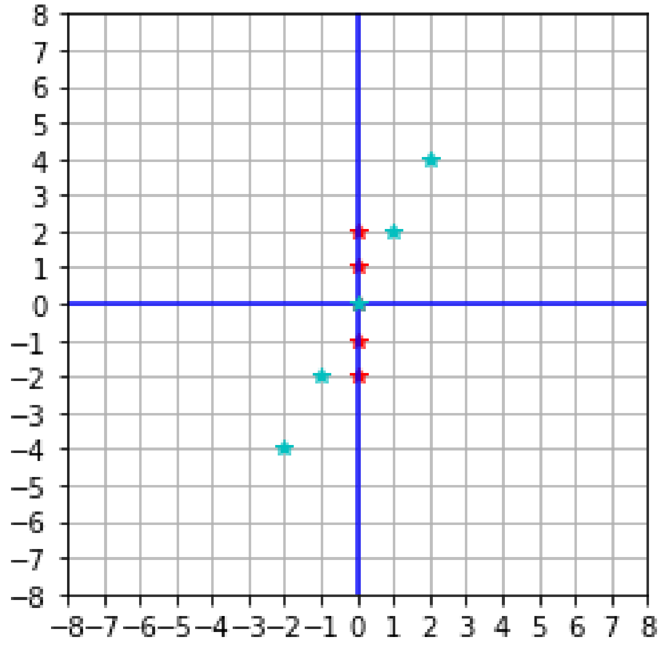
Y direction

$$T(\vec{v}) = \begin{bmatrix} 3 & \boxed{1} \\ 0 & \boxed{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 3 & \boxed{1} \\ 0 & \boxed{2} \end{bmatrix} \begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} y$$

Case x=0 , y=-2,-1,0,1,2

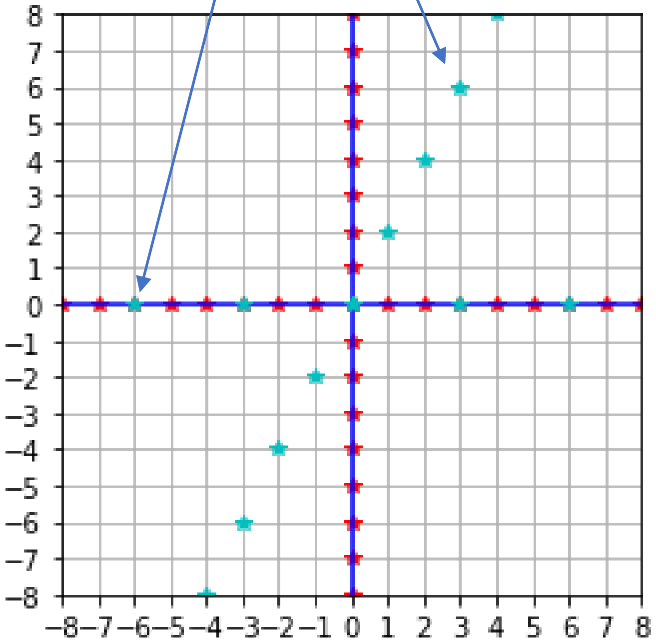




Matrix as Linear Transformation

$$T(\vec{v}) = A\vec{v}$$

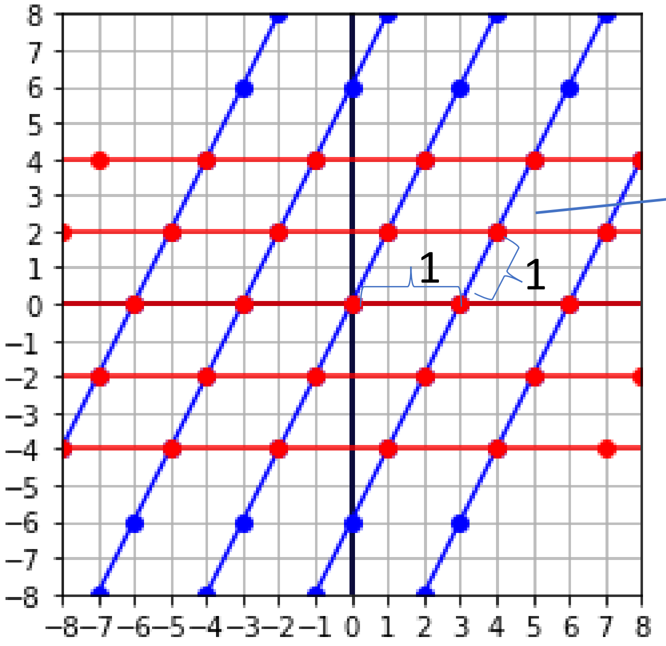
$$T(\vec{v}) = \begin{bmatrix} \boxed{\begin{matrix} 3 \\ 0 \end{matrix}} & \boxed{\begin{matrix} 1 \\ 2 \end{matrix}} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



Matrix as Linear Transformation $T(\vec{v}) = A\vec{v}$

$$T(\vec{v}) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

X=1 y=1

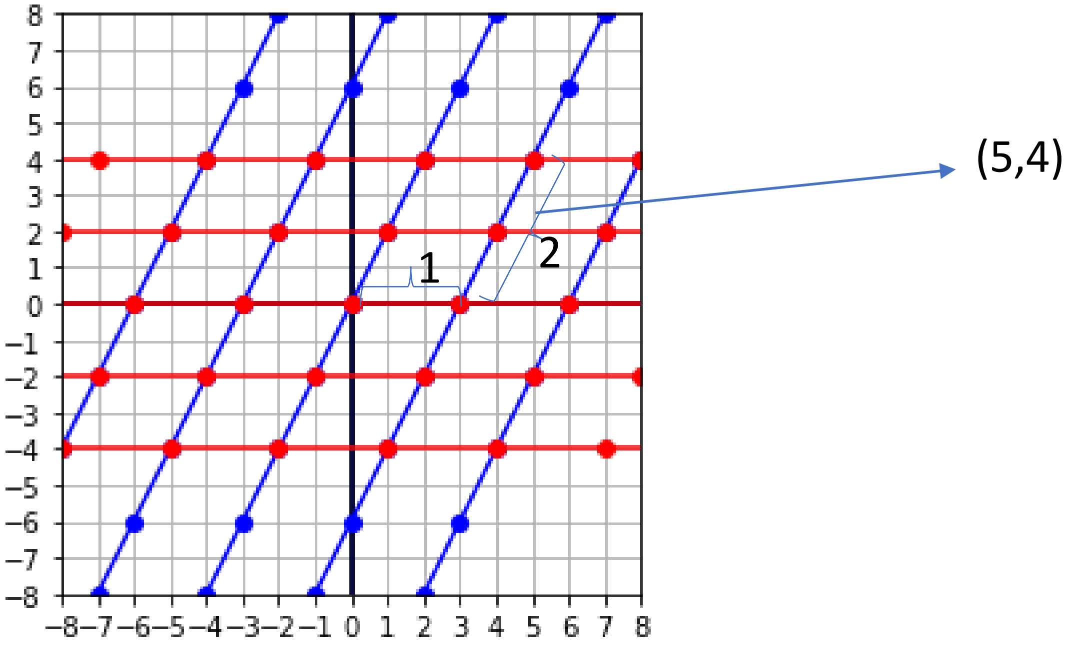


(4,2)

Matrix as Linear Transformation $T(\vec{v}) = A\vec{v}$

$$T(\vec{v}) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

X=1 y=2



Eigenvalues and eigenvector

- An eigenvector is a vector whose direction remains unchanged when a linear transformation is applied to it.

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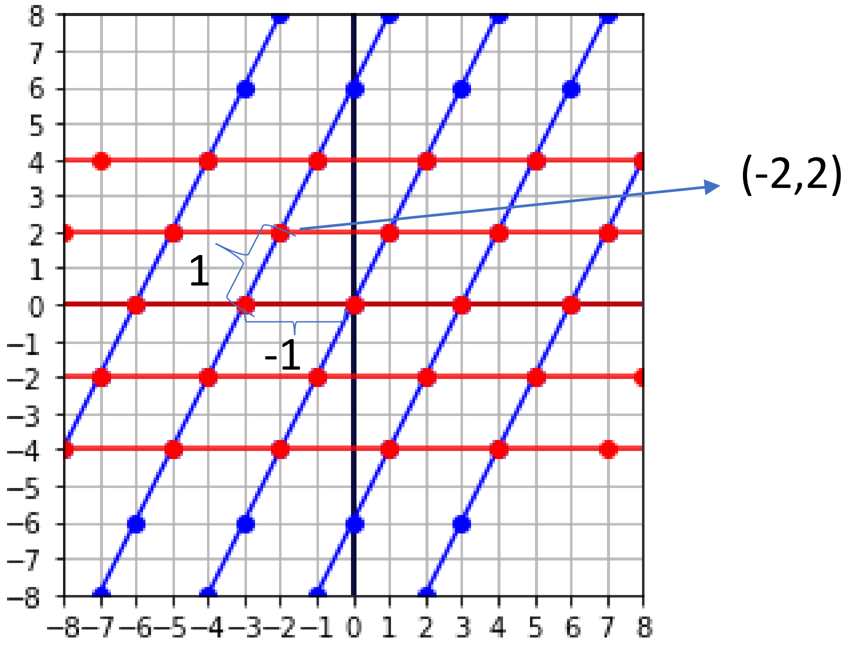
$$T(\vec{v}) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

x=-1 y=1



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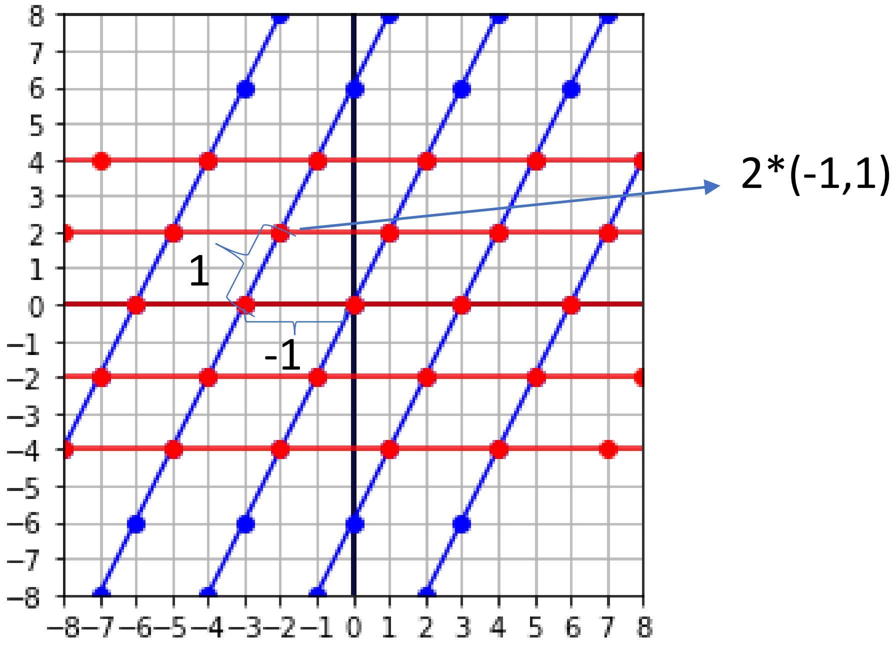

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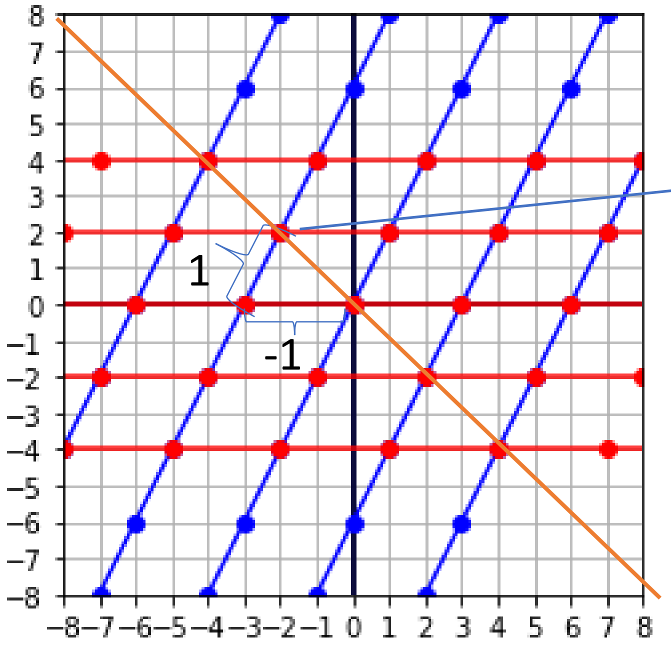
x=-1 y=1

Before Transformation



2*(-1,1)

After Transformation

Eigenvalues and eigenvector

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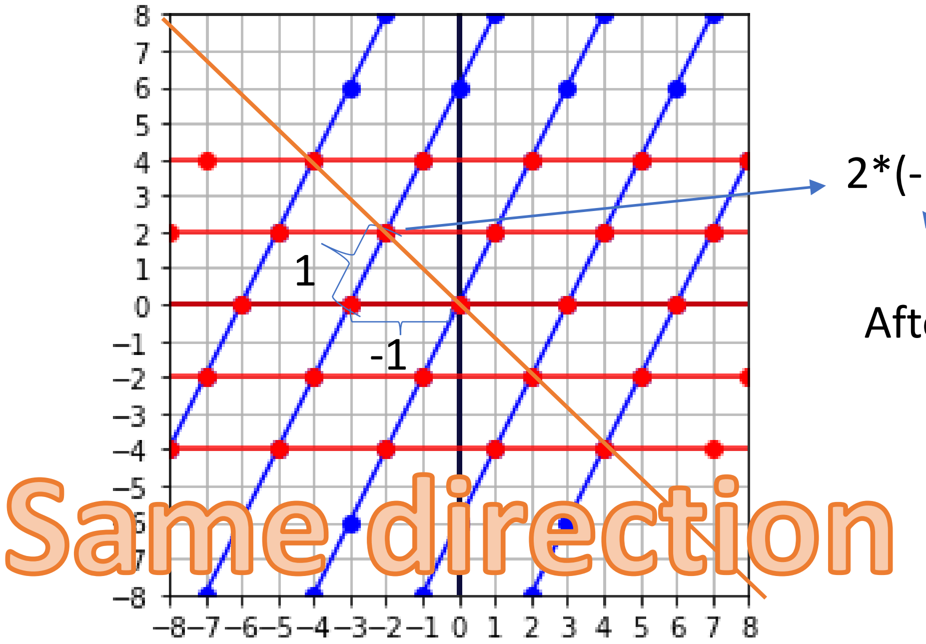
$$T(\vec{v}) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

x=-1 y=1

Before Transformation



2*(-1,1)

After Transformation

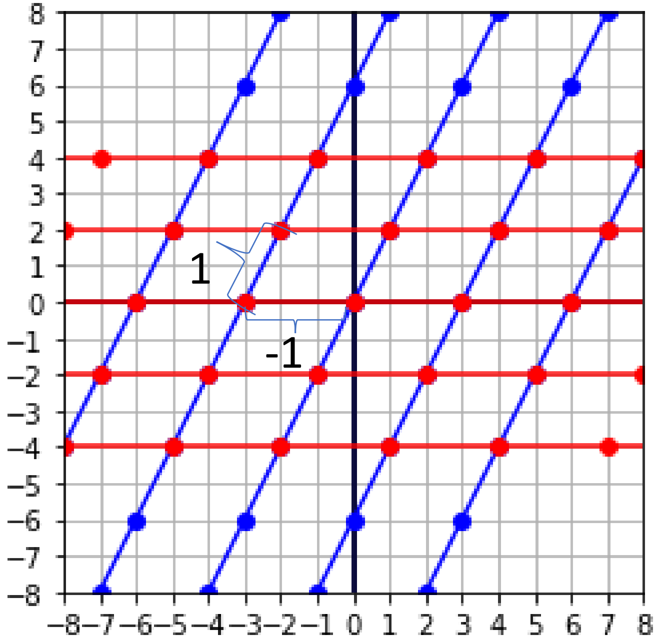
Eigenvalues and eigenvector

- An eigenvector is a vector whose direction remains unchanged when a linear transformation is applied to it.

$$T(\vec{v}) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Eigenvector



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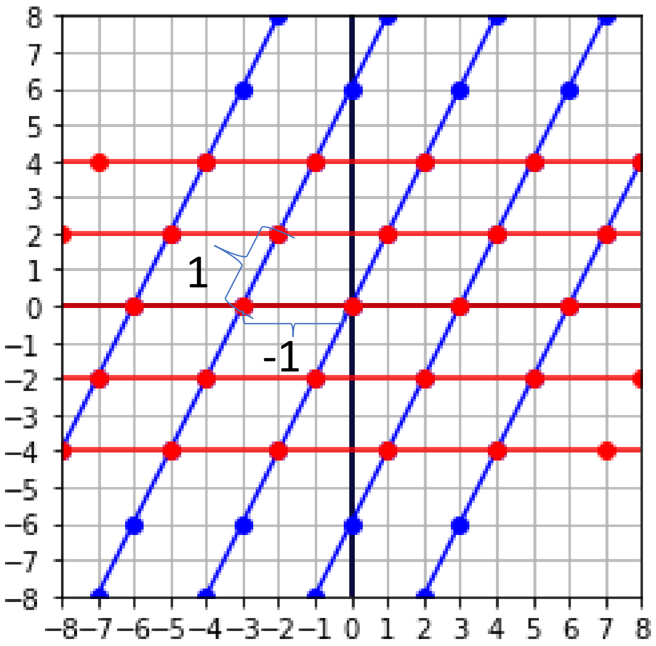
Eigenvalue

Eigenvector



Eigenvalues and eigenvector

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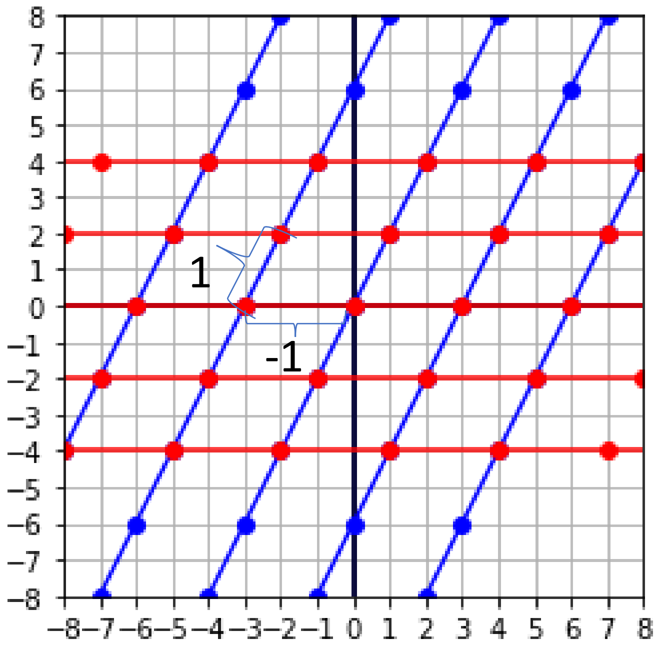
Mathematical definition

$$\overrightarrow{Av} = \lambda \vec{v}$$

Eigenvector

Eigenvalue



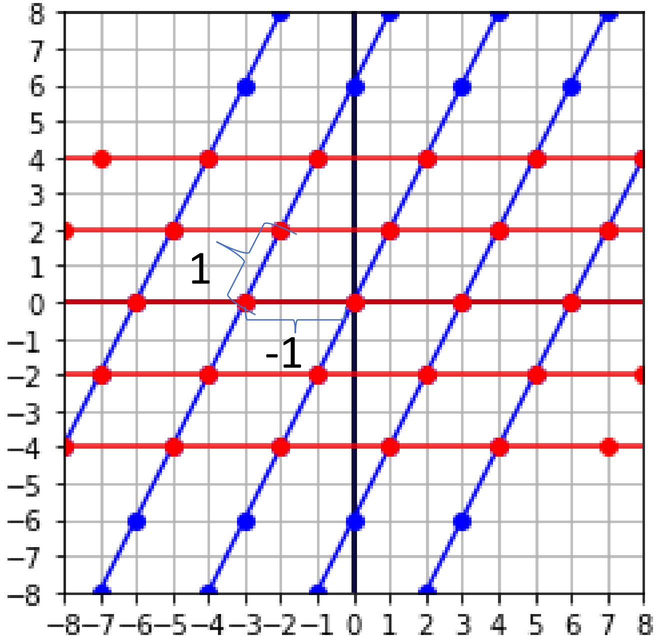
Eigenvalues and eigenvector

$A\vec{v} = \lambda\vec{v}$

- An eigenvector is a vector whose direction remains unchanged when a linear transformation is applied to it.

$$T(\vec{v}) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Is there any other eigenvector?



Eigenvalues and eigenvector

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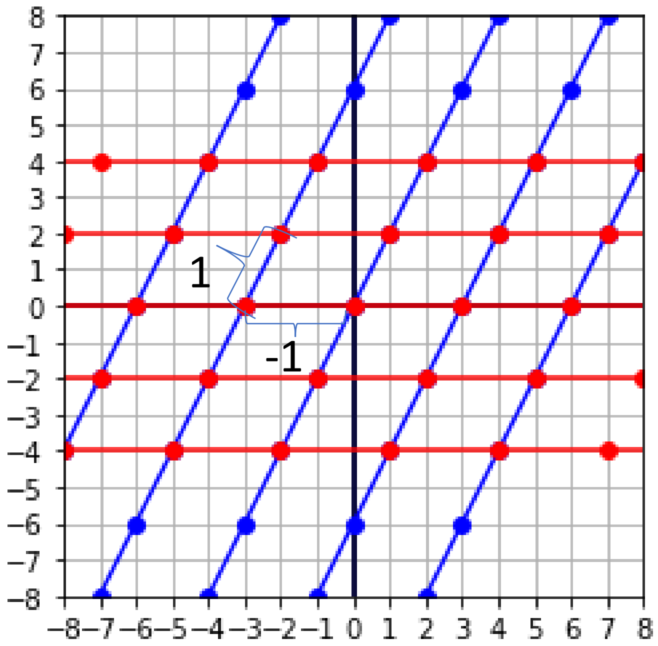
$$T(\vec{v}) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Is there any other eigenvector?
  - Try with $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$



Eigenvalues and eigenvector

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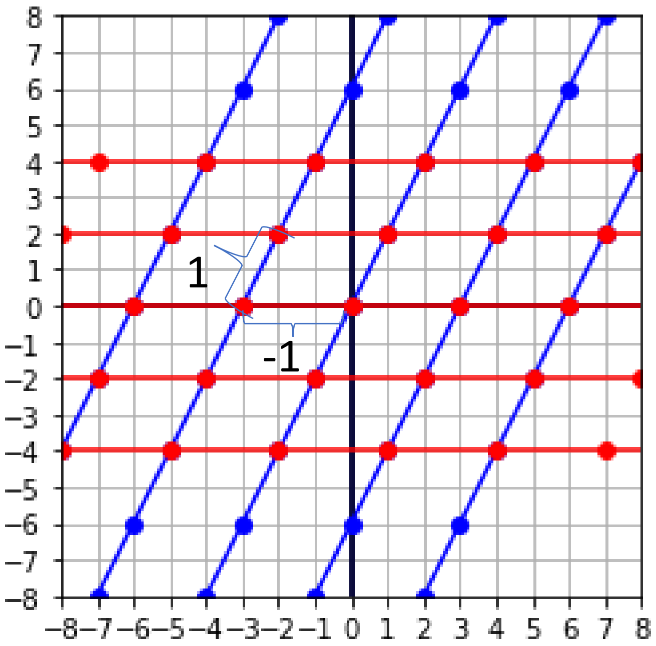
- Is there any other eigenvector?

$$T\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \end{bmatrix} = 3\begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Eigenvalue (stretching)

Eigenvector (direction)



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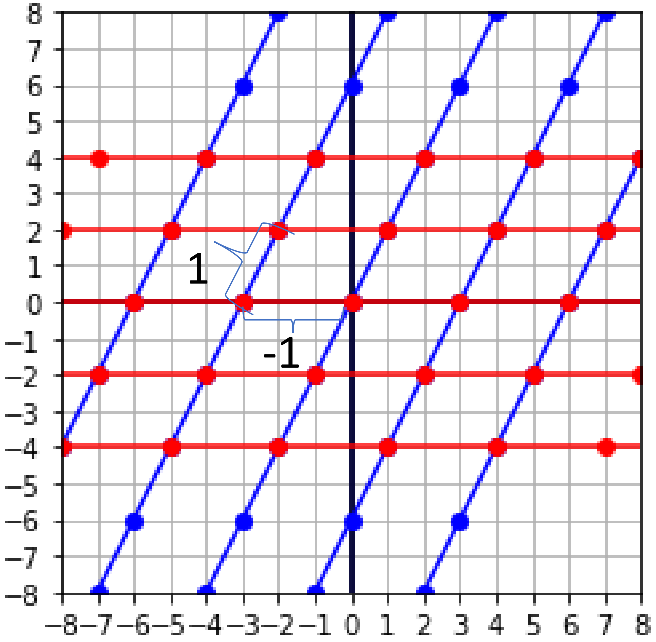

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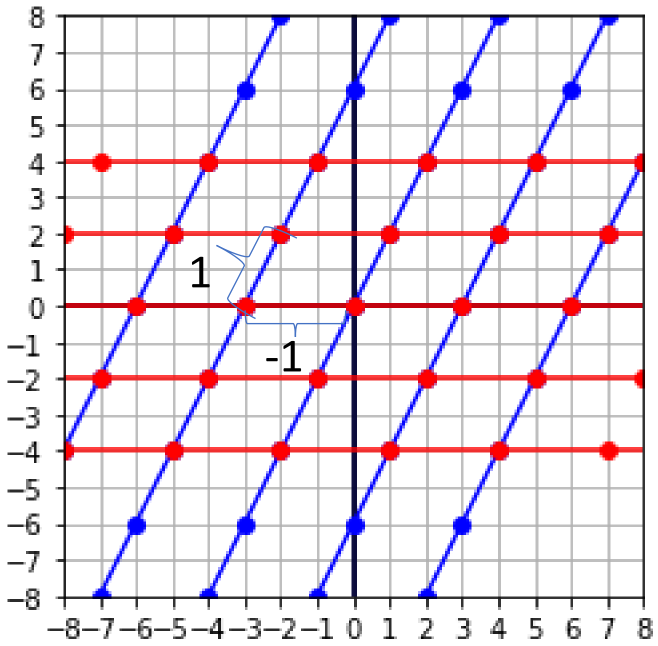
- Is there any other eigenvector?
- NO.



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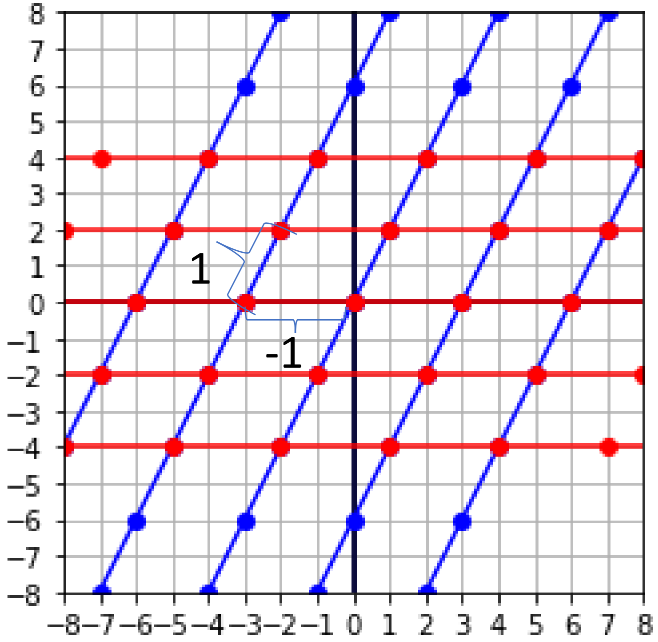
- Is there any other eigenvector?
- NO.
- An $A_{m,m}$ matrix has at most m eigenvectors



Eigenvalues and eigenvector

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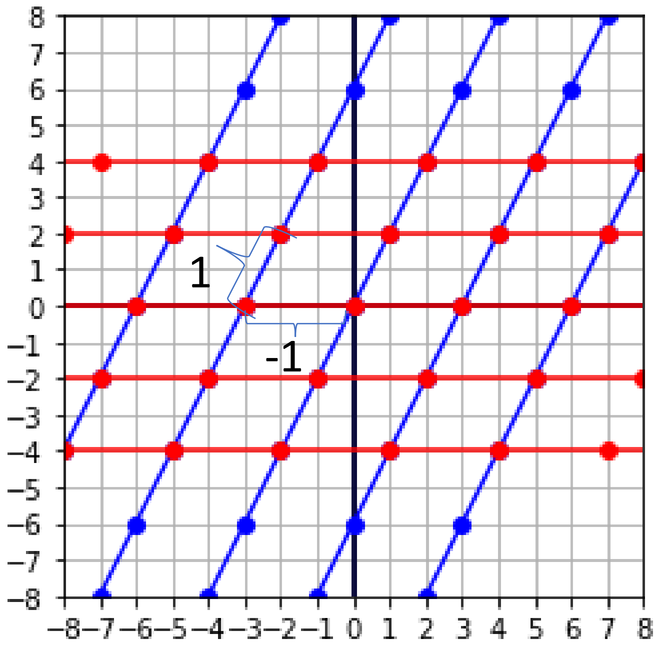
- An eigenvector is a vector whose direction remains unchanged when a linear transformation is applied to it.

$$T(\vec{v}) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Is there any other eigenvector?
- NO.
- An $A_{m,m}$ matrix has at most m eigenvectors

In this example m=2 → maximum 2 eigenvectors



Computing Eigenvalues & Eigenvectors

$$A\vec{v} = \lambda\vec{v}$$

Identity

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$

$$A\vec{v} = \lambda I\vec{v}$$

Computing Eigenvalues & Eigenvectors

$$A\vec{v} = \lambda\vec{v}$$

$$A\vec{v} = \overbrace{\lambda I}^{\begin{matrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{matrix}}\vec{v}$$

$$A\vec{v} - \lambda I\vec{v} = 0$$

Computing Eigenvalues & Eigenvectors

$$A\vec{v} = \lambda\vec{v}$$

$$A\vec{v} = \overbrace{\lambda I}^{\begin{matrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{matrix}}\vec{v}$$

$$A\vec{v} - \lambda I\vec{v} = 0$$

$$(A - \lambda I)\vec{v} = 0$$

Computing Eigenvalues & Eigenvectors

$$A\vec{v} = \lambda\vec{v}$$

$$\begin{matrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{matrix}$$

$$A\vec{v} = \underbrace{\lambda I}\vec{v}$$

$$A\vec{v} - \lambda I\vec{v} = 0$$

$$(\underbrace{A - \lambda I})\vec{v} = 0$$

If $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$

Computing Eigenvalues & Eigenvectors

$$A\vec{v} = \lambda\vec{v}$$

$$\begin{matrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{matrix}$$

$$A\vec{v} = \lambda I\vec{v}$$

$$A\vec{v} - \lambda I\vec{v} = 0$$

If $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$

$$(A - \lambda I)\vec{v} = 0$$

$$\begin{bmatrix} 3-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix}$$

Computing Eigenvalues & Eigenvectors

$$A\vec{v} = \lambda\vec{v}$$

$$A\vec{v} = \lambda I\vec{v}$$

$$A\vec{v} - \lambda I\vec{v} = 0$$

$$(A - \lambda I)\vec{v} = 0$$

If $\vec{v}$ is not null

Computing Eigenvalues & Eigenvectors

$$A\vec{v} = \lambda\vec{v}$$

$$A\vec{v} = \lambda I\vec{v}$$

$$A\vec{v} - \lambda I\vec{v} = 0$$

$$\underbrace{(A - \lambda I)}_{\text{Must be not invertible}}\vec{v} = 0$$

If $\vec{v}$ is not null

Computing Eigenvalues & Eigenvectors

$$A\vec{v} = \lambda\vec{v}$$

$$A\vec{v} = \lambda I\vec{v}$$

$$A\vec{v} - \lambda I\vec{v} = 0$$

$$(A - \lambda I)\vec{v} = 0$$

If $\vec{v}$ is not null

Must be not invertible

Determinant =zero

Computing Eigenvalues & Eigenvectors

$$A\vec{v} - \lambda I \vec{v} = 0$$

$$\underbrace{(A - \lambda I)}\vec{v} = 0$$

If $\vec{v}$ is not null

Must be not invertible

Determinant =zero

$$det(A - \lambda I) = 0$$

Computing Eigenvalues & Eigenvectors

$$A\vec{v} = \lambda\vec{v}$$

$$(A - \lambda I)\vec{v} = 0$$

$$det(A - \lambda I) = 0$$

$$T(\vec{v}) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathrm{A} = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$

$$(A - \lambda I) = \begin{bmatrix} 3 - \lambda & 1 \\ 0 & 2 - \lambda \end{bmatrix}$$

Computing Eigenvalues & Eigenvectors

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$$det(A - \lambda I) = det \begin{bmatrix} 3 - \lambda & 1 \\ 0 & 2 - \lambda \end{bmatrix} = 0$$

$$(3 - \lambda)(2 - \lambda) - 0 * 1 = 0$$

Computing Eigenvalues & Eigenvectors

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$$det(A - \lambda I) = det \begin{bmatrix} 3 - \lambda & 1 \\ 0 & 2 - \lambda \end{bmatrix} = 0$$

$$(3 - \lambda)(2 - \lambda) = 0$$

$\lambda = 3$ $\qquad$ $\lambda = 2$

Computing Eigenvector for λ=2

$$A\vec{v} = \lambda\vec{v}$$

$$(A - \lambda I)\vec{v} = 0$$

$$det(A - \lambda I) = 0$$

$$T(\vec{v}) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$$

Matrix multiplication

$$3x + 2y = 2x$$

$$0x + 2y = 2y$$

Computing Eigenvector for λ=2

$$A\vec{v} = \lambda\vec{v}$$
$$(A - \lambda I)\vec{v} = 0$$
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Matrix multiplication

$$3x + y = 2x$$
$$0x + 2y = 2y$$

$$3x + y = 2x$$
$$3x - 2x = -y$$
$$x = -y$$

Computing Eigenvector for λ=2

$$A\vec{v} = \lambda\vec{v}$$
$$(A - \lambda I)\vec{v} = 0$$
$$det(A - \lambda I) = 0$$

$$T(\vec{v}) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix}$$

Matrix multiplication

$$3x + y = 2x$$
$$0x + 2y = 2y$$

$$3x + y = 2x$$
$$3x - 2x = -y$$
$$x = -y$$

If x=-1 then y=1

for λ=2, $\vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Computing Eigenvector for λ=3

$$A\vec{v} = \lambda\vec{v}$$
$$(A - \lambda I)\vec{v} = 0$$
$$det(A - \lambda I) = 0$$

$$T(\vec{v}) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 3\begin{bmatrix} x \\ y \end{bmatrix}$$

Matrix multiplication

$$3x + y = 3x$$
$$0x + 2y = 3y$$

$$3x + y = 3x$$
$$3x - 3x = -y$$
$$0 = y$$

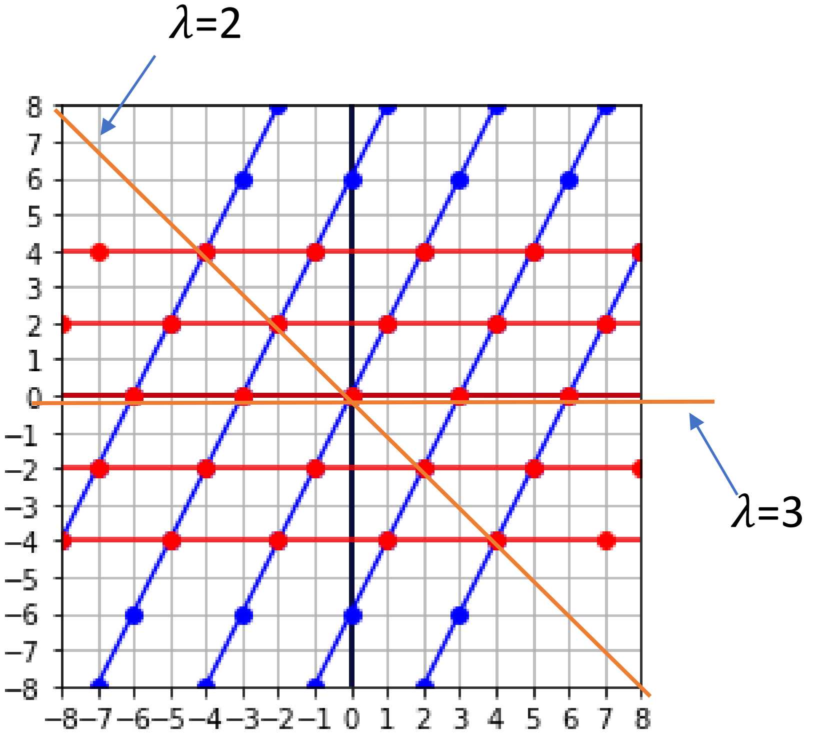
y=0

for λ=3, $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Computing Eigenvectors

$$T(\vec{v}) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 3 \begin{bmatrix} x \\ y \end{bmatrix}$$

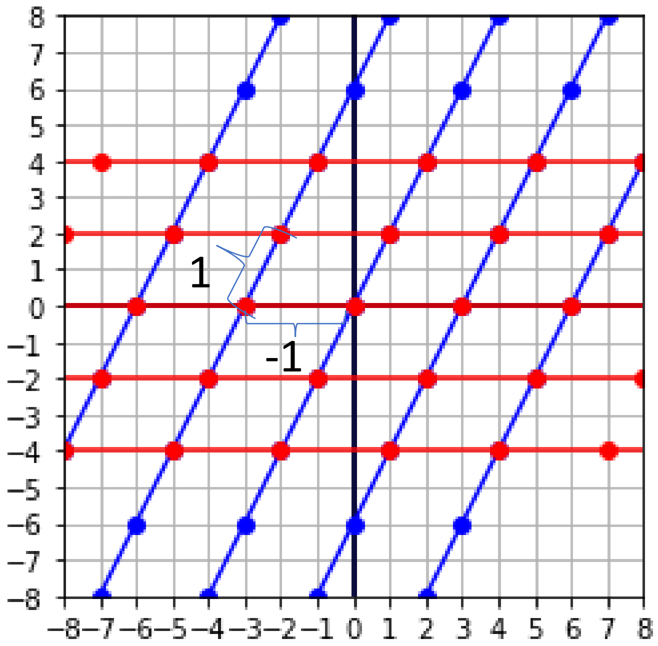
$$A\vec{v} = \lambda\vec{v}$$
$$(A - \lambda I)\vec{v} = 0$$
$$det(A - \lambda I) = 0$$



Eigenvalues and eigenvector

- An eigenvector is a vector whose direction remains unchanged when a linear transformation is applied to it.

$$A\vec{v} = \lambda\vec{v}$$



- Does the definition make sense for a non-square matrix $A_{m,n}$?

Eigenvalues and eigenvector

- An eigenvector is a vector whose direction remains unchanged when a linear transformation is applied to it.

$$A\vec{v} = \lambda\vec{v}$$

- Does the definition make sense for a non-square matrix $A_{m,n}$?
  - NO
  - Transformation changes dimension of vector $\vec{v}$.

Linear Algebra

- Matrix as a Linear Transformation
- Eigenvalues and eigenvector
  - Intuition
  - How to compute it
- **Singular Value Descomposition (SVD)**
  - Definition, derivation
  - Intuition
  - Direct Solving Ax=0

Singular Value Decomposition

ortho-normal diagonal ortho-normal

$$\underset{n \times m}{\mathbf{A}} = \underbrace{\mathbf{U}}_{n \times n}\,\underbrace{\boldsymbol{\Sigma}}_{n \times m}\,\underbrace{\mathbf{V}}_{m \times m}{}^{\top}$$

unit norm constraint

Singular Value Decomposition

$$\underset{n \times m}{\mathbf{A}} = \underset{n \times n}{\overset{\text{ortho-normal}}{\mathbf{U}}}\,\underset{n \times m}{\overset{\text{diagonal}}{\mathbf{\Sigma}}}\,\underset{m \times m}{\overset{\text{ortho-normal}}{\mathbf{V}}}^{\top}$$

$$A\overrightarrow{v_1} = \sigma_1 \overrightarrow{u_1}$$

$$A\overrightarrow{v_2} = \sigma_2 \overrightarrow{u_2}$$

.

.

.

$$A\overrightarrow{v_m} = \sigma_m \overrightarrow{u_n}$$

$\overrightarrow{v_i}$ is ortho-normal

Singular Value Decomposition

$$\underset{n \times m}{\mathbf{A}} = \overset{\text{ortho-normal}}{\underset{n \times n}{\mathbf{U}}} \overset{\text{diagonal}}{\underset{n \times m}{\mathbf{\Sigma}}} \overset{\text{ortho-normal}}{\underset{m \times m}{\mathbf{V}}}{}^{\top}$$

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$\overrightarrow{v_i}$ is ortho-normal

$$\overrightarrow{v_i} \cdot \overrightarrow{v_i} = 1$$

$$\overrightarrow{v_i} \cdot \overrightarrow{v_j} = 0$$

Singular Value Decomposition

$$\underset{n \times m}{\mathbf{A}} = \overset{\text{ortho-normal}}{\underset{n \times n}{\mathbf{U}}}\,\overset{\text{diagonal}}{\underset{n \times m}{\mathbf{\Sigma}}}\,\overset{\text{ortho-normal}}{\underset{m \times m}{\mathbf{V}^{\top}}}$$

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dimension of $\overrightarrow{v_i}$ is m x 1

Singular Value Decomposition

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$$\vdots$$
$$A\overrightarrow{v_m} = \sigma_m\overrightarrow{u_m}$$

$\overrightarrow{v_i}$ is ortho-normal

dimension of $\overrightarrow{v_i}$ is m x 1

$\overrightarrow{u_i}$ is a unit vector

dimension of $\overrightarrow{u_i}$ is n x 1

σ_i is magnitude of vector

Singular Value Decomposition

$$\underset{n \times m}{\mathbf{A}} = \underset{n \times n}{\overset{\text{ortho-normal}}{\mathbf{U}}} \underset{n \times m}{\overset{\text{diagonal}}{\mathbf{\Sigma}}} \underset{m \times m}{\overset{\text{ortho-normal}}{\mathbf{V}^\top}}$$

Any vector $\vec{x} \in R^m$

$$\vec{x} = (\vec{x} \cdot \overrightarrow{v_1})\overrightarrow{v_1} + (\vec{x} \cdot \overrightarrow{v_2})\overrightarrow{v_2} + \cdots + (\vec{x} \cdot \overrightarrow{v_m})\overrightarrow{v_m}$$

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Singular Value Decomposition

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Singular Value Decomposition

$$\underset{n \times m}{\mathbf{A}} = \underset{\substack{\text{ortho-normal} \\ n \times n}}{\mathbf{U}} \underset{\substack{\text{diagonal} \\ n \times m}}{\mathbf{\Sigma}} \underset{\substack{\text{ortho-normal} \\ m \times m}}{\mathbf{V}^{\top}}$$

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$$A\overrightarrow{v_1} = \sigma_1\overrightarrow{u_1}$$

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$$\vdots$$

$$A\overrightarrow{v_m} = \sigma_m\overrightarrow{u_m}$$

$$\vec{x} \cdot \overrightarrow{v_i} = \overrightarrow{v_i}^T\vec{x}$$

Singular Value Decomposition

ortho-normal diagonal ortho-normal

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top$$

n x m n x n n x m m x m

Any vector $\vec{x} \in R^m$

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$$A\vec{x} = (\vec{u_1}\sigma_1\vec{v_1}^T + \vec{u_2}\sigma_2\vec{v_2}^T + \cdots + \vec{u_m}\sigma_m\vec{v_m}^T)\vec{x}$$

$$A = (\vec{u_1}\sigma_1\vec{v_1}^T + \vec{u_2}\sigma_2\vec{v_2}^T + \cdots + \vec{u_m}\sigma_m\vec{v_m}^T)$$

$$A\vec{v_1} = \sigma_1\vec{u_1}$$

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$$\vec{x}\cdot\vec{v_i} = \vec{v_i}^T\vec{x}$$

n>m

Singular Value Decomposition $\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\top}$

ortho-normal, diagonal, ortho-normal

$\mathbf{A}$: n x m, $\mathbf{U}$: n x n, $\boldsymbol{\Sigma}$: n x m, $\mathbf{V}^{\top}$: m x m

$$A = (\overrightarrow{u_1}\sigma_1\overrightarrow{v_1}^T + \overrightarrow{u_2}\sigma_2\overrightarrow{v_2}^T + \cdots + \overrightarrow{u_m}\sigma_m\overrightarrow{v_m}^T)$$

$$A = U\Sigma V^T$$

$$\mathrm{U} = [\overrightarrow{u_1}|\overrightarrow{u_2}|\ldots|\overrightarrow{u_m}|\ldots]_{n\times n}$$

dimension of $\overrightarrow{u_i}$ is n x 1

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \ddots\ 0 \\ 0 & 0 & \sigma_m \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{n\times m}$$

$$\mathrm{V} = [\overrightarrow{v_1}|\overrightarrow{v_2}|\ldots|\overrightarrow{v_m}]_{m\times m}$$

dimension of $\overrightarrow{v_i}$ is m x 1

$$\mathrm{V}^T = \begin{bmatrix} \overrightarrow{v_1}^T \\ \overrightarrow{v_2}^T \\ \vdots \\ \overrightarrow{v_m}^T \end{bmatrix}_{m\times m}$$

n>m

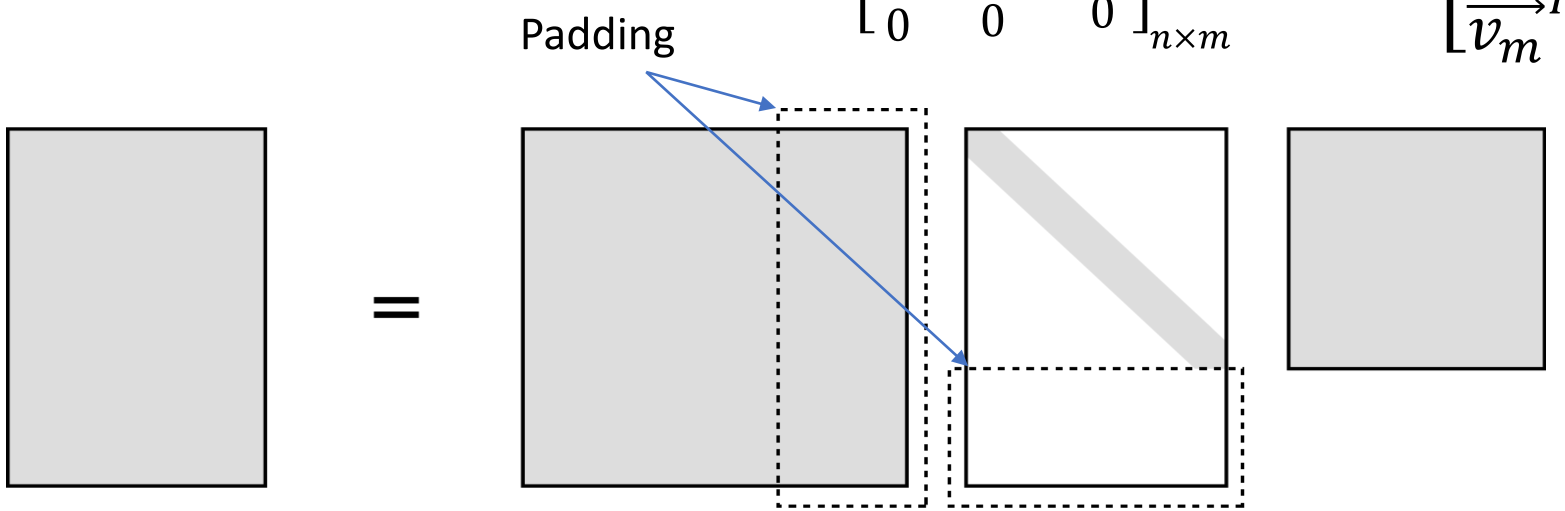
Singular Value Decomposition $\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^\top$

$$\underset{n\times m}{\mathbf{A}} = \underset{\substack{\text{ortho-normal} \\ n\times n}}{\mathbf{U}}\ \underset{\substack{\text{diagonal} \\ n\times m}}{\boldsymbol{\Sigma}}\ \underset{\substack{\text{ortho-normal} \\ m\times m}}{\mathbf{V}^\top}$$

$$A = (\overrightarrow{u_1}\sigma_1\overrightarrow{v_1}^T + \overrightarrow{u_2}\sigma_2\overrightarrow{v_2}^T + \cdots + \overrightarrow{u_m}\sigma_m\overrightarrow{v_m}^T)$$

$$A = U\Sigma V^T$$

$$\mathrm{U} = [\overrightarrow{u_1}|\overrightarrow{u_2}| \dots |\overrightarrow{u_m} \dots]_{n\times n} \qquad \Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \ddots\ \sigma_m \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{n\times m} \qquad \mathrm{V}^T = \begin{bmatrix} \overrightarrow{v_1}^T \\ \overrightarrow{v_2}^T \\ \vdots \\ \overrightarrow{v_m}^T \end{bmatrix}_{m\times m}$$



A U Σ V*

n>m

Singular Value Decomposition $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top$

ortho-normal diagonal ortho-normal

n x m n x n n x m m x m

$$A = (\overrightarrow{u_1}\sigma_1\overrightarrow{v_1}^T + \overrightarrow{u_2}\sigma_2\overrightarrow{v_2}^T + \cdots + \overrightarrow{u_m}\sigma_m\overrightarrow{v_m}^T)$$

$$A = U\Sigma V^T$$

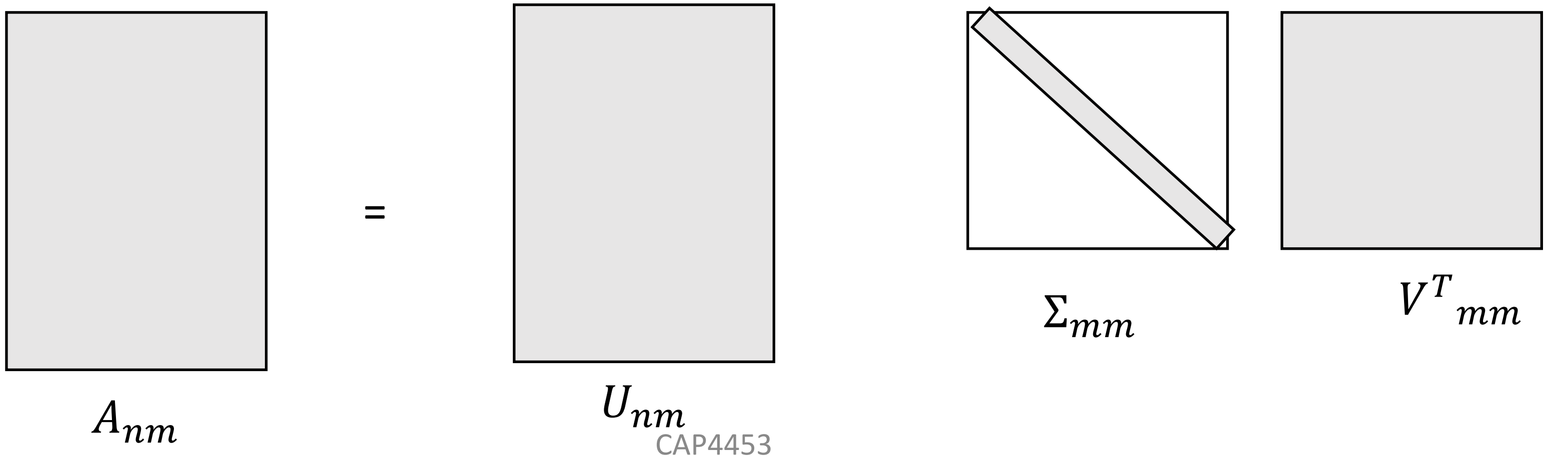
$$\mathrm{U} = [\overrightarrow{u_1}|\overrightarrow{u_2}|\ldots|\overrightarrow{u_m}]_{n\times m}$$

dimension of $\overrightarrow{u_i}$ is n×1

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 \ddots & 0 \\ 0 & 0 & \sigma m \end{bmatrix}_{m\times m}$$

dimension of $\overrightarrow{v_i}$ is m×1

$$\mathrm{V}^T = \begin{bmatrix} \overrightarrow{v_1}^T \\ \overrightarrow{v_2}^T \\ \vdots \\ \overrightarrow{v_m}^T \end{bmatrix}_{m\times m}$$

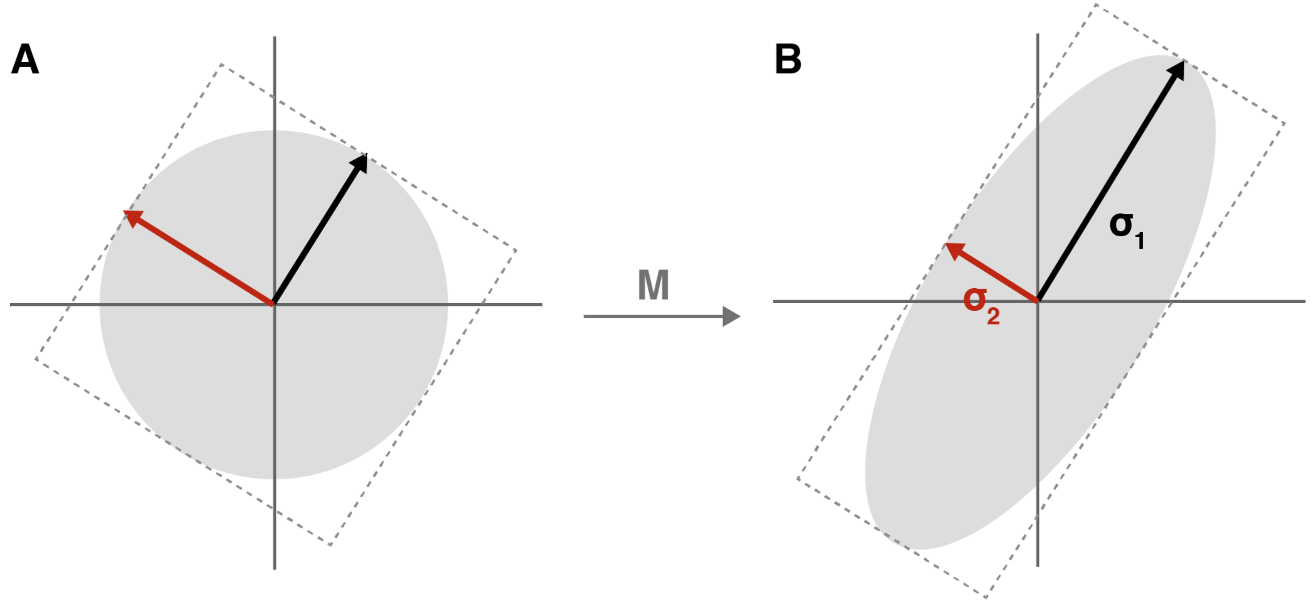


Linear Algebra

- Matrix as a Linear Transformation
- Eigenvalues and eigenvector
  - Intuition
  - How to compute it
- **Singular Value Descomposition (SVD)**
  - Definition, derivation
  - **Intuition**
  - Direct Solving Ax=0

Pseudo inverse intuition

- Since the SVD is a decomposition of a given matrix into 2 Unitary matrices and a diagonal matrix, all matrices could be described as a rotation, scaling and another rotation.



(A) An oriented circle; if it helps, imagine that circle inscribed in our original square. (B) Our circle transformed into an ellipse. The length of the major and minor axes of the ellipse have values $\sigma 1$ and $\sigma 2$ respectively, called the *singular values*.

Interesting properties of SVD

- The diagonal values of Σ are the square root of eigenvalues of $A^T A$

$$A = U \Sigma V^{-1} \qquad \Sigma = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \cdot & \\ & & & \sigma_N \end{bmatrix}$$

U, V = orthogonal matrix

$$\sigma_i = \sqrt{\lambda_i}$$

σ = singular value

λ = eigenvalue of $A^t A$

Interesting properties of SVD

- The diagonal values of Σ are the square root of eigenvalues of $A^T A$
- Eigenvectors of $A^T A$ corresponds to V
- SVD consists of matrices U,Σ,V which are always real
  - this is unlike eigenvectors and eigenvalues of A which may be complex even if A is real
  - The singular values are always non-negative, even though the eigenvalues may be negative
- While writing the SVD, the following convention is assumed, and the left and right singular vectors are also arranged accordingly:

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{m-1} \geq \sigma_m$$

Interesting properties of SVD

- The rank of a rectangular matrix A is equal to the number of non-zero singular values. Note that rank(A) = rank(Σ).
- SVD always exist
- It is used to compute pseudoinverse

The Pseudo Inverse of a matrix $A = U\Sigma V^H$, denoted $A^\dagger$ is given by

$$A^\dagger = V\Sigma^\dagger U^H$$

Where $\Sigma^\dagger$ is obtained by transposing Σ and inverting all non zero entries.

Computing SVD

- Compute SVD for $A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}$

  - Calculate the eigenvalues of AA^T

$$AA^T = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$$

  - $\det(AA^T - \lambda I) = 0$

$$(17 - \lambda)(17 - \lambda) + 64 = 0$$

$$\lambda^2 - 34\lambda + 225 = 0$$

$$= (\lambda - 25)(\lambda - 9)$$

$$\sigma_i = \sqrt{\lambda_i} \qquad \sigma_1 = 5\,;\, \sigma_2 = 3 \qquad \Sigma = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}$$

Compute SVD

- Eigenvector of $A^T A$

$$A^T A = \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix}$$

$\lambda = 25$

$$AA^T - 25 \cdot I = \begin{bmatrix} -12 & 12 & 2 \\ 12 & -12 & -2 \\ 2 & -2 & -17 \end{bmatrix}$$

$$\begin{aligned} -12x + 12y + 2z &= 0 \\ 12x - 12y - 2z &= 0 \\ 2x - \quad 2y - 17z &= 0 \end{aligned}$$

$-12x + 12y + 2z = 0$

$6(2x - \quad 2y - 17z) = 0$

$$\begin{aligned} -12x \;\; + 12y + 2z &= 0 \\ 12x - \quad 12y - 102z &= 0 \\ \hline -100z &= 0 \end{aligned}$$

$\boldsymbol{z = 0}$

$2x - \quad 2y - 17z = 0$

$2x - \quad 2y = 0$

$\boldsymbol{x = y}$

$$v_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$

Compute SVD

- Eigenvector of $A^T A$

$$A^T A = \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix}$$

$\lambda = 9$

$$A^T A - 9I = \begin{bmatrix} 4 & 12 & 2 \\ 12 & 4 & -2 \\ 2 & -2 & -1 \end{bmatrix}$$

$$\begin{aligned} 4x + 12y + 2z &= 0 \\ 12x + 4y - 2z &= 0 \\ 2x - \quad 2y - 1z &= 0 \end{aligned}$$

$$4x + 12y + 2z = 0$$

$$12x + 9y - 2z = 0$$

$$16x \;\; + 16y \;\; = 0$$

$$\boldsymbol{x = -y}$$

$$4x + 12y + 2z = 0$$

-2$(2x - \;\; 2y - 1z) = 0$

$$16y = -4z$$

$$\boldsymbol{4y = -z}$$

$$v_2 = \begin{bmatrix} -y \\ y \\ -4y \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -4 \end{bmatrix} y =$$

$$v_2 = \begin{pmatrix} 1/\sqrt{18} \\ -1/\sqrt{18} \\ 4/\sqrt{18} \end{pmatrix}.$$

Compute SVD

- Eigenvector of $A^T A$

$$A^T A = \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix}$$

$\lambda = 0$

$$A^T A - 0I = \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix}$$

$$\begin{aligned} 13x + 12y + 2z &= 0 \\ 12x + 13y - 2z &= 0 \\ 2x - \quad 2y + 8z &= 0 \end{aligned}$$

$$13x + 12y + 2z = 0$$

$$12x + 13y - 2z = 0$$

$$25x \;\; + 25y \;\; = 0$$

$$\boldsymbol{x = -y}$$

$$12x + 13y - 2z = 0$$

$$-6(2x - \quad 2y + 8z) = 0$$

$$25y = 50z$$

$$\boldsymbol{y = 2z}$$

$$v_3 = \begin{bmatrix} -y \\ y \\ y/2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0.5 \end{bmatrix} y =$$

$$v_3 = \begin{pmatrix} 2/3 \\ -2/3 \\ -1/3 \end{pmatrix}.$$

Compute SVD

- So far:

$$\sigma_1 = 5\,;\sigma_2 = 3 \qquad v_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1/\sqrt{18} \\ -1/\sqrt{18} \\ 4/\sqrt{18} \end{pmatrix}. \quad v_3 = \begin{pmatrix} 2/3 \\ -2/3 \\ -1/3 \end{pmatrix}.$$

$$A = U\Sigma V^T = U \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{18} & -1/\sqrt{18} & 4/\sqrt{18} \\ 2/3 & -2/3 & -1/3 \end{pmatrix}.$$

- Using $A\vec{v_i} = \sigma_i \vec{u_i}$ $\qquad \dfrac{A\vec{v_i}}{\sigma_i} = \vec{u_i}$

$$U = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

$$\frac{A\vec{v_1}}{\sigma_1} = \frac{\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}}{5} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\frac{A\vec{v_2}}{\sigma_2} = \frac{\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}\begin{bmatrix} \frac{1}{3\sqrt{2}} \\ \frac{-1}{3\sqrt{2}} \\ \frac{4}{3\sqrt{2}} \end{bmatrix}}{3} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

Compute SVD

- In total

$$A = U\Sigma V^T = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{18} & -1/\sqrt{18} & 4/\sqrt{18} \\ 2/3 & -2/3 & -1/3 \end{pmatrix}.$$

```
import numpy as np
A=np.array([[3,2,2 ],[2,3,-2]])
u, s, vh = np.linalg.svd(A, full_matrices=True)
```

```
In [115]: u
Out[115]:
array([[-0.70710678, -0.70710678],
       [-0.70710678,  0.70710678]])

In [116]: s
Out[116]: array([5., 3.])

In [117]: vh
Out[117]:
array([[-7.07106781e-01, -7.07106781e-01, -6.47932334e-17],
       [-2.35702260e-01,  2.35702260e-01, -9.42809042e-01],
       [-6.66666667e-01,  6.66666667e-01,  3.33333333e-01]])
```

Linear Algebra

- Matrix as a Linear Transformation
- Eigenvalues and eigenvector
  - Intuition
  - How to compute it
- **Singular Value Descomposition (SVD)**
  - Definition, derivation
  - Intuition
  - **Direct Solving Ax=0**

Derivation using Least squares

$$Ah = 0$$

The sum squared error can be written as:

$$
\begin{aligned}
f(\mathbf{h}) &= \frac{1}{2}(A\mathbf{h} - \mathbf{0})^T (A\mathbf{h} - \mathbf{0}) \\
f(\mathbf{h}) &= \frac{1}{2}(A\mathbf{h})^T (A\mathbf{h}) \\
f(\mathbf{h}) &= \frac{1}{2}\mathbf{h}^T A^T A\mathbf{h}.
\end{aligned}
$$

Taking the derivative of f with respect to $\mathbf{h}$ and setting the result to zero,

$$
\begin{aligned}
\frac{d}{d\mathbf{h}} f = 0 &= \frac{1}{2}\left(A^T A + (A^T A)^T\right)\mathbf{h} \\
0 &= A^T A\mathbf{h}.
\end{aligned}
$$

h should equal the eigenvector of $B = A^T A$ that has an eigenvalue of zero

$$B\vec{h} = \lambda\vec{h}$$

(or, in the presence of noise the eigenvalue closest to zero)

General form of total least squares

(**Warning:** change of notation. x is a vector of parameters!)

$$E_{\mathrm{TLS}} = \sum_i (\boldsymbol{a}_i \boldsymbol{x})^2$$

$$= \|\mathbf{A}\boldsymbol{x}\|^2$$ (matrix form)

$$\|\boldsymbol{x}\|^2 = 1$$ constraint

minimize $\|\mathbf{A}\boldsymbol{x}\|^2$

subject to $\|\boldsymbol{x}\|^2 = 1$

→

(Rayleigh quotient)

minimize $\dfrac{\|\mathbf{A}\boldsymbol{x}\|^2}{\|\boldsymbol{x}\|^2}$

Solution is the eigenvector corresponding to smallest eigenvalue of

$$\mathbf{A}^\top \mathbf{A}$$

(equivalent)

Solution is the column of **V** corresponding to smallest singular value

$$\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^\top$$

Homogeneous Linear Least Squares problem

$$A\mathbf{x} = \mathbf{0}$$

$$A = U\Sigma V^\top = \sum_{i=1}^{9} \sigma_i \mathbf{u}_i \mathbf{v}_i^\top$$

- If the homography is *exactly determined*, then $\sigma_9 = 0$, and there exists a homography that fits the points exactly.
- If the homography is *overdetermined*, then $\sigma_9 \geq 0$. Here σ_9 represents a "residual" or goodness of fit.
- We will not handle the case of the homography being *underdetermined*.

Solving for H using DLT

Given $\{\boldsymbol{x}_i, \boldsymbol{x}'_i\}$ solve for H such that $\boldsymbol{x}' = \mathbf{H}\boldsymbol{x}$

1. For each correspondence, create 2x9 matrix $\mathbf{A}_i$

2. Concatenate into single 2n x 9 matrix $\mathbf{A}$

3. Compute SVD of $\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^\top$

4. Store singular vector of the smallest singular value $\boldsymbol{h} = \boldsymbol{v}_{\hat{i}}$

5. Reshape to get $\mathbf{H}$

Recap: Two Common Optimization Problems

Problem statement

$$\text{minimize} \quad \|\mathbf{Ax}-\mathbf{b}\|^2$$

$$\text{least squares solution to } \mathbf{Ax}=\mathbf{b}$$

Solution

$$\mathbf{x}=\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{b}$$

```python
import numpy as np
x,resid,rank,s = np.linalg.lstsq(A,b)
```

Problem statement

$$\text{minimize} \quad \mathbf{x}^T\mathbf{A}^T\mathbf{Ax} \quad \text{s.t. } \mathbf{x}^T\mathbf{x}=1$$

$$\text{non-trivial lsq solution to } \mathbf{Ax}=0$$

Solution

$$[\mathbf{v},\lambda]=\mathrm{eig}(\mathbf{A}^T\mathbf{A})$$

$$\lambda_1<\lambda_{2..n}:\mathbf{x}=\mathbf{v}_1$$

References

Basic reading:

- Szeliski textbook, Section 3.6.

Additional reading:

- Hartley and Zisserman, “Multiple View Geometry in Computer Vision,” Cambridge University Press 2004.
  a comprehensive treatment of all aspects of projective geometry relating to computer vision, and also a very useful reference for the second part of the class.
- Richter-Gebert, “Perspectives on projective geometry,” Springer 2011.
  a beautiful, thorough, and very accessible mathematics textbook on projective geometry (available online for free from CMU’s library).

Questions?