

# CAP 4453

# Robot Vision

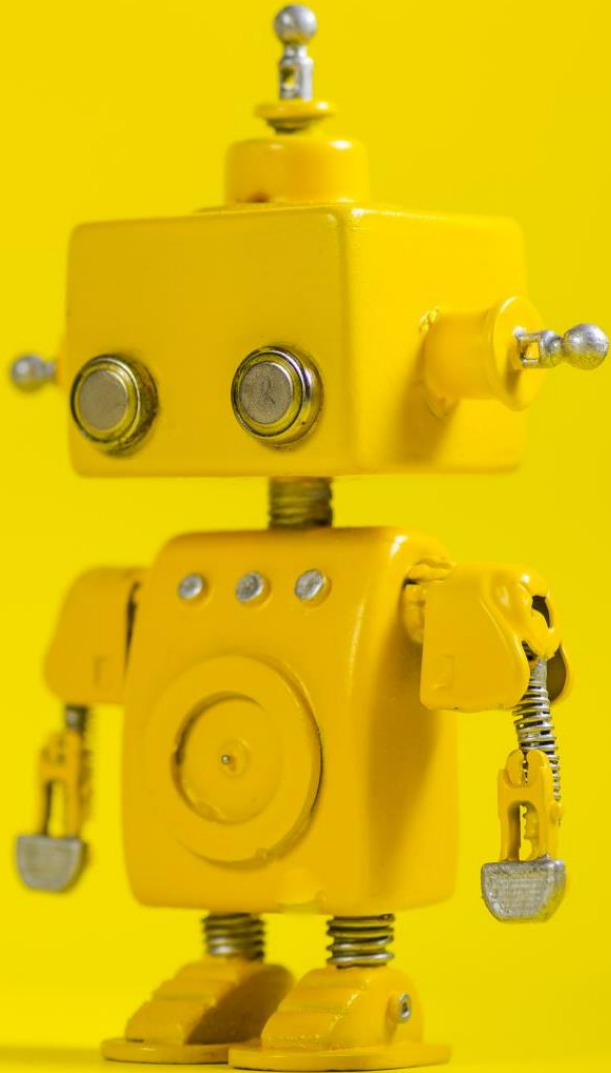
Dr. Gonzalo Vaca-Castaño  
[gonzalo.vacacastano@ucf.edu](mailto:gonzalo.vacacastano@ucf.edu)



# Administrative details

- Issues submitting homework

# Short Review from last class

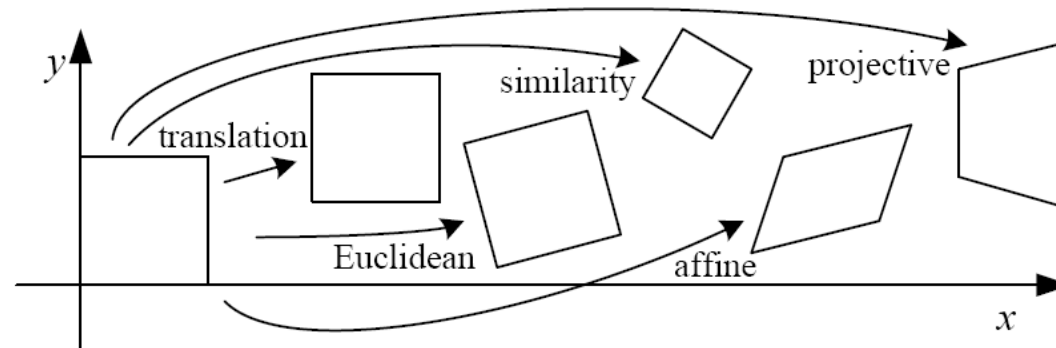









# Outline

- Linear algebra
- Image transformations
- 2D transformations.
- Projective geometry 101.
- Transformations in projective geometry.
- Classification of 2D transformations.
- **Determining unknown 2D transformations.**
- Determining unknown image warps.

# 2D image transformations



Name	Matrix	# D.O.F.	Preserves:	Icon
translation	$\begin{bmatrix} \mathbf{I} & \mathbf{t} \end{bmatrix}_{2 \times 3}$	2	orientation + ...	
rigid (Euclidean)	$\begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix}_{2 \times 3}$	3	lengths + ...	
similarity	$\begin{bmatrix} s\mathbf{R} & \mathbf{t} \end{bmatrix}_{2 \times 3}$	4	angles + ...	
affine	$\begin{bmatrix} \mathbf{A} \end{bmatrix}_{2 \times 3}$	6	parallelism + ...	
projective	$\begin{bmatrix} \tilde{\mathbf{H}} \end{bmatrix}_{3 \times 3}$	8	straight lines	

These transformations are a nested set of groups

- Closed under composition and inverse is a member



# Least squares

$$\mathbf{A}\mathbf{t} = \mathbf{b}$$

- Find  $\mathbf{t}$  that minimizes

$$\|\mathbf{A}\mathbf{t} - \mathbf{b}\|^2$$

- To solve, form the *normal equations*

$$\mathbf{A}^T \mathbf{A} \mathbf{t} = \mathbf{A}^T \mathbf{b}$$

$$\mathbf{t} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{b}$$



# Translation transformation

- Can also write as a matrix equation

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ \vdots & \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} x'_1 - x_1 \\ y'_1 - y_1 \\ x'_2 - x_2 \\ y'_2 - y_2 \\ \vdots \\ x'_n - x_n \\ y'_n - y_n \end{bmatrix}$$

$$\mathbf{A}$$

$2n \times 2$

$$\mathbf{t}$$

$2 \times 1$

=

$$\mathbf{b}$$

$2n \times 1$



# Affine transformations

- Matrix form

$$\begin{bmatrix} x_1 & y_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1 & y_1 & 1 \\ x_2 & y_2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2 & y_2 & 1 \\ \vdots & & & & & \\ x_n & y_n & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_n & y_n & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} x'_1 \\ y'_1 \\ x'_2 \\ y'_2 \\ \vdots \\ x'_n \\ y'_n \end{bmatrix}$$

**A**  
 $2n \times 6$

**t**  
 $6 \times 1$

**=**

**b**  
 $2n \times 1$





# Determining the homography matrix

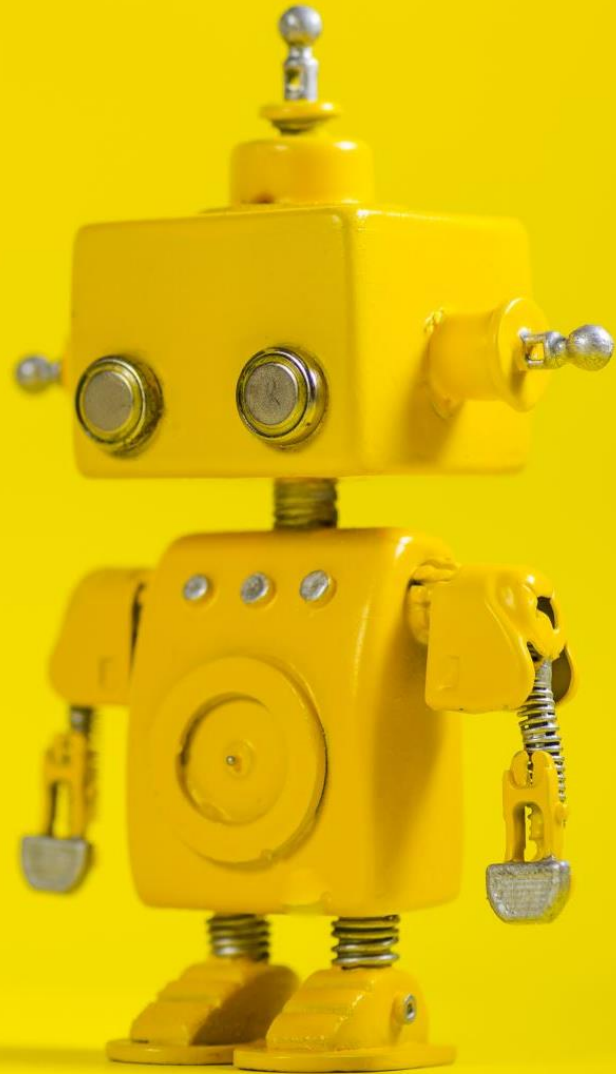
Stack together constraints from multiple point correspondences:

$$\mathbf{A}\mathbf{h} = \mathbf{0}$$

$$\begin{bmatrix} -x & -y & -1 & 0 & 0 & 0 & xx' & yx' & x' \\ 0 & 0 & 0 & -x & -y & -1 & xy' & yy' & y' \\ -x & -y & -1 & 0 & 0 & 0 & xx' & yx' & x' \\ 0 & 0 & 0 & -x & -y & -1 & xy' & yy' & y' \\ \vdots & & & & & & & & \\ -x & -y & -1 & 0 & 0 & 0 & xx' & yx' & x' \\ 0 & 0 & 0 & -x & -y & -1 & xy' & yy' & y' \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \\ h_7 \\ h_8 \\ h_9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

*Homogeneous* linear least squares problem

- Solve with SVD



# Robot Vision

## 10b. Linear Algebra SVD



# Linear Algebra

- Matrix as a Linear Transformation
- Eigenvalues and eigenvector
  - Intuition
  - How to compute it
- Singular Value Decomposition (SVD)
  - Definition
  - Intuition
  - Direct Solving  $Ax=0$



# Matrix as Linear Transformation

$$T(\vec{v}) = A\vec{v}$$

Example

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$

$$T(\vec{v}) = A\vec{v}$$

$$T(\vec{v}) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

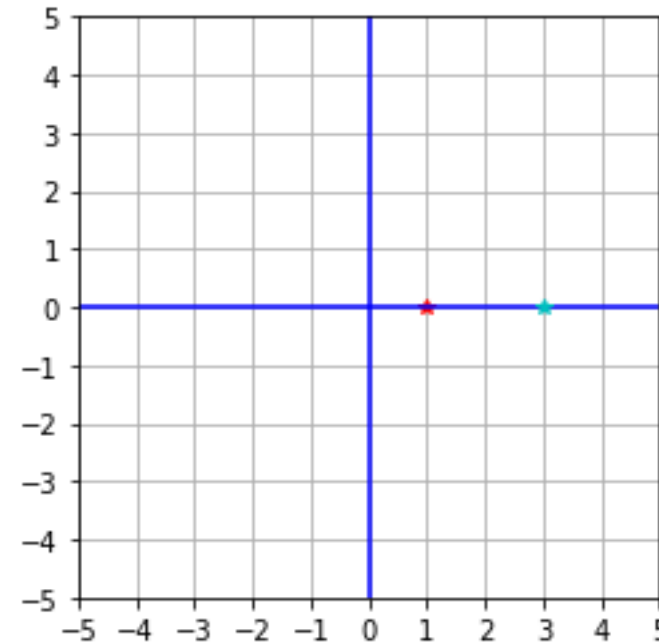
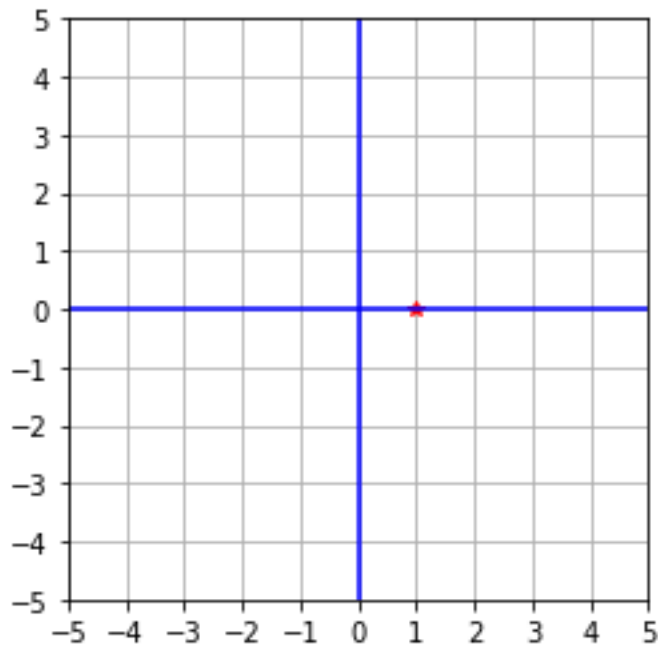
# Matrix as Linear Transformation

$$T(\vec{v}) = A\vec{v}$$

$$T(\vec{v}) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

Case  $x=1, y=0$

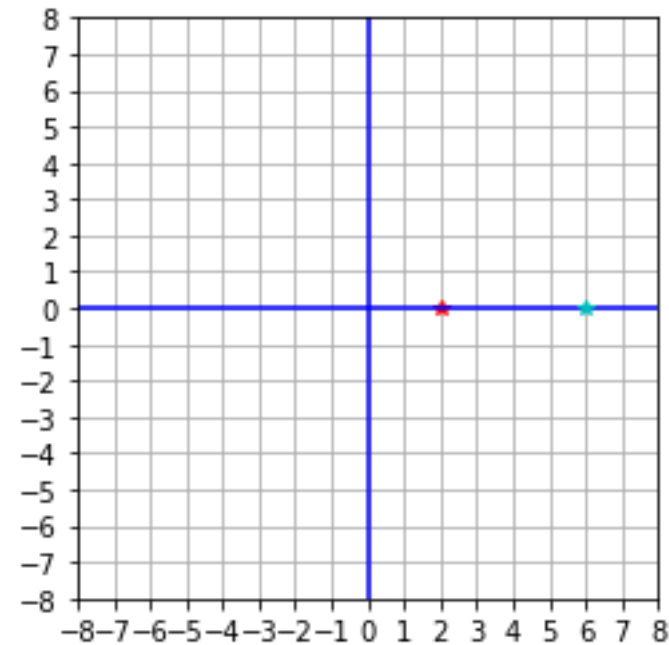
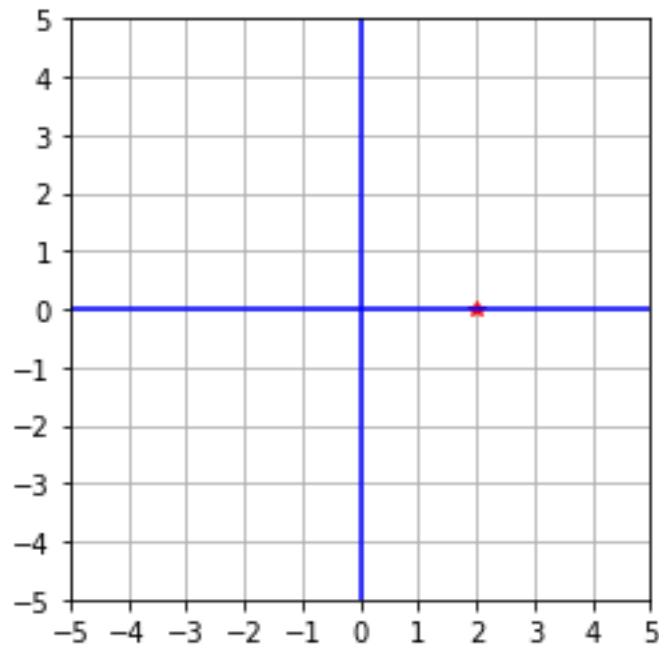


# Matrix as Linear Transformation $T(\vec{v}) = A\vec{v}$

$$T(\vec{v}) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$T\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

Case  $x=2, y=0$



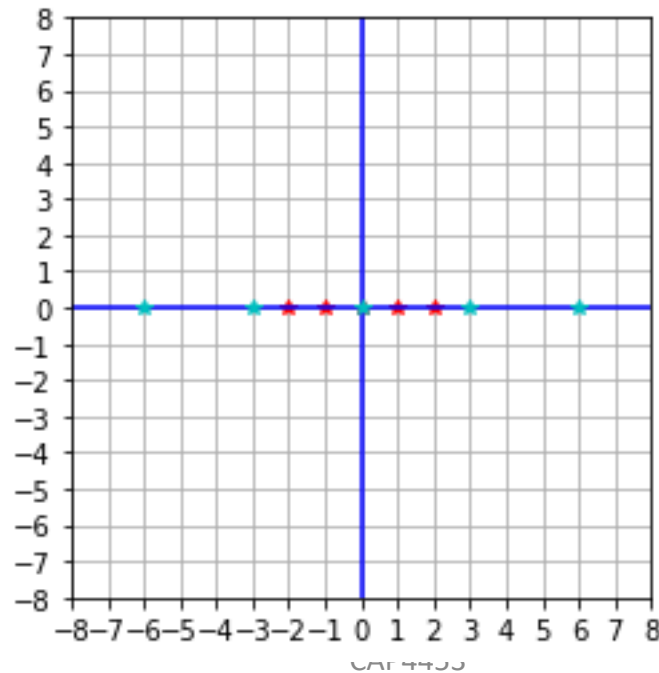
# Matrix as Linear Transformation $T(\vec{v}) = A\vec{v}$

$$T(\vec{v}) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

X direction

$$T\left(\begin{bmatrix} x \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} x$$

Case  $x=-2,-1,0,1,2, y=0$



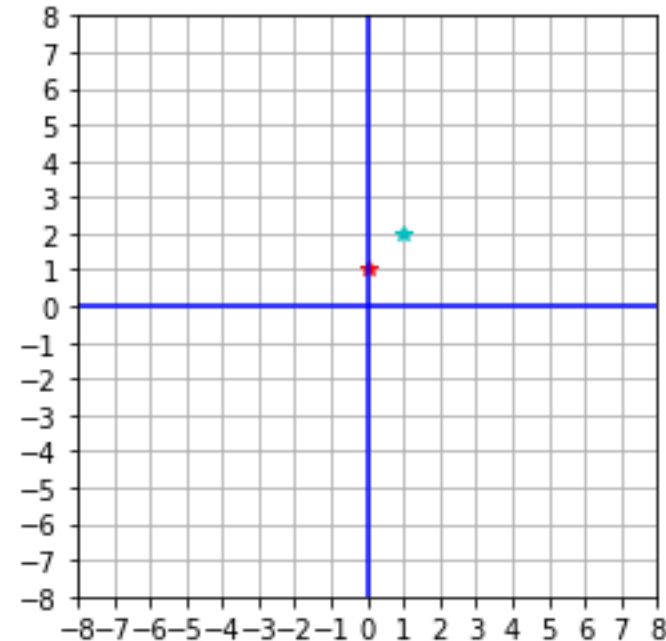
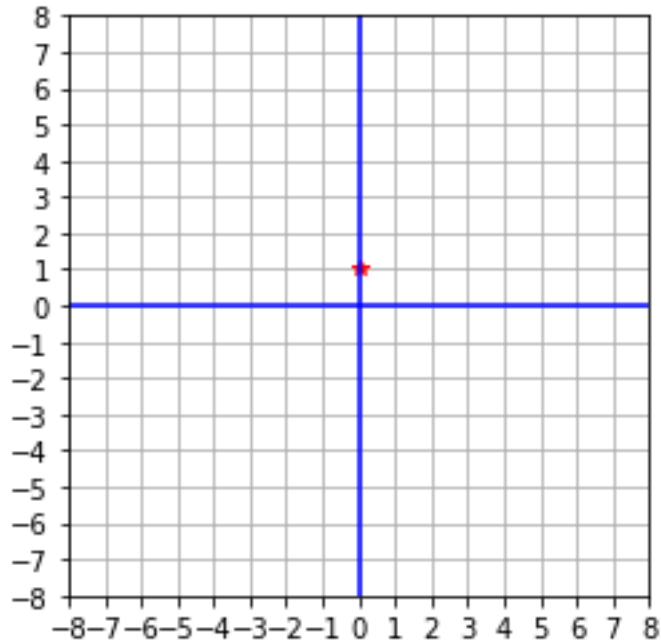
# Matrix as Linear Transformation

$$T(\vec{v}) = A\vec{v}$$

$$T(\vec{v}) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Case  $x=0, y=1$





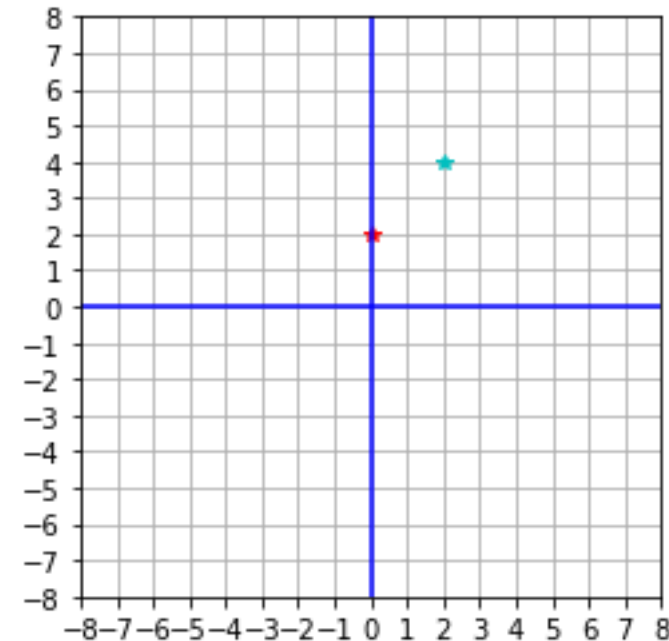
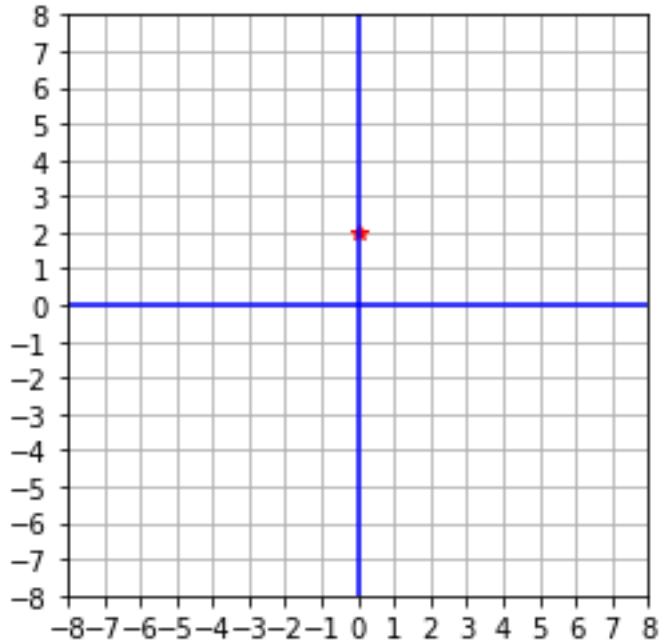
# Matrix as Linear Transformation

$$T(\vec{v}) = A\vec{v}$$

$$T(\vec{v}) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Case  $x=0, y=2$



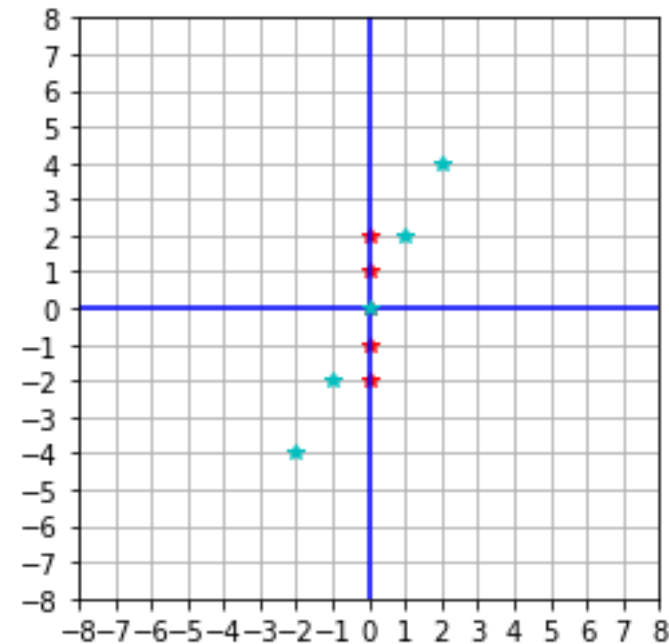
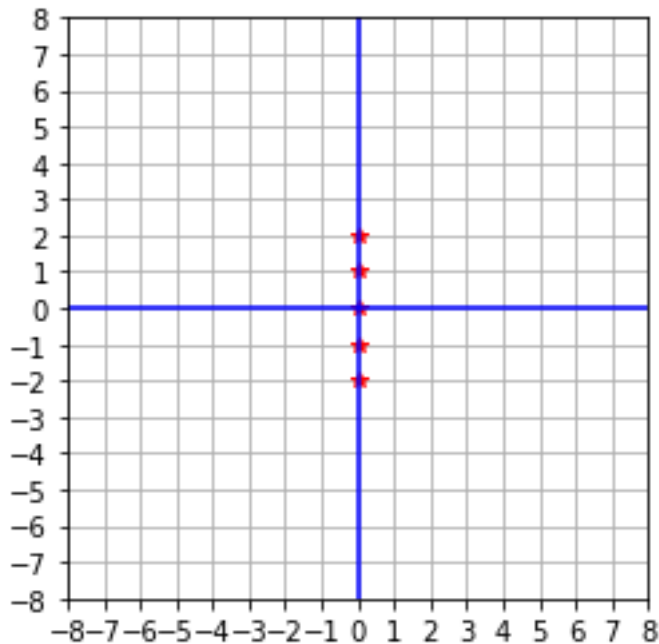
# Matrix as Linear Transformation $T(\vec{v}) = A\vec{v}$

$$T(\vec{v}) = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Y direction

Case  $x=0, y=-2,-1,0,1,2$

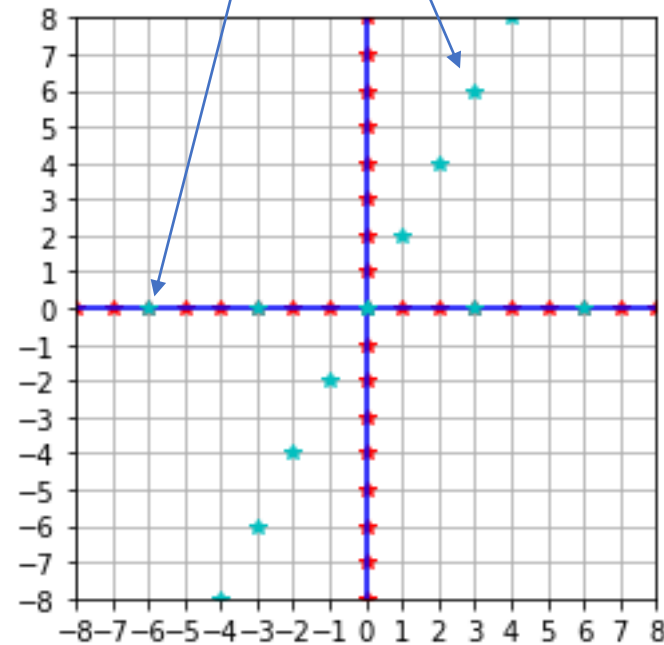
$$T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} y$$



# Matrix as Linear Transformation

$$T(\vec{v}) = A\vec{v}$$

$$T(\vec{v}) = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

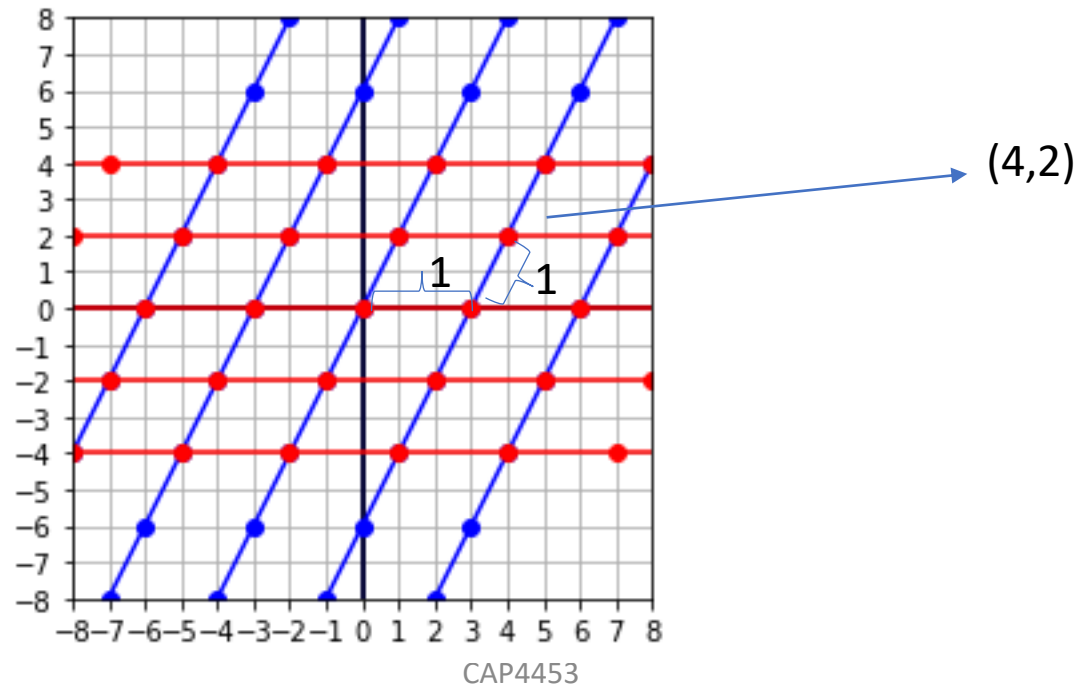


# Matrix as Linear Transformation

$$T(\vec{v}) = A\vec{v}$$

$$T(\vec{v}) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

X=1 y=1

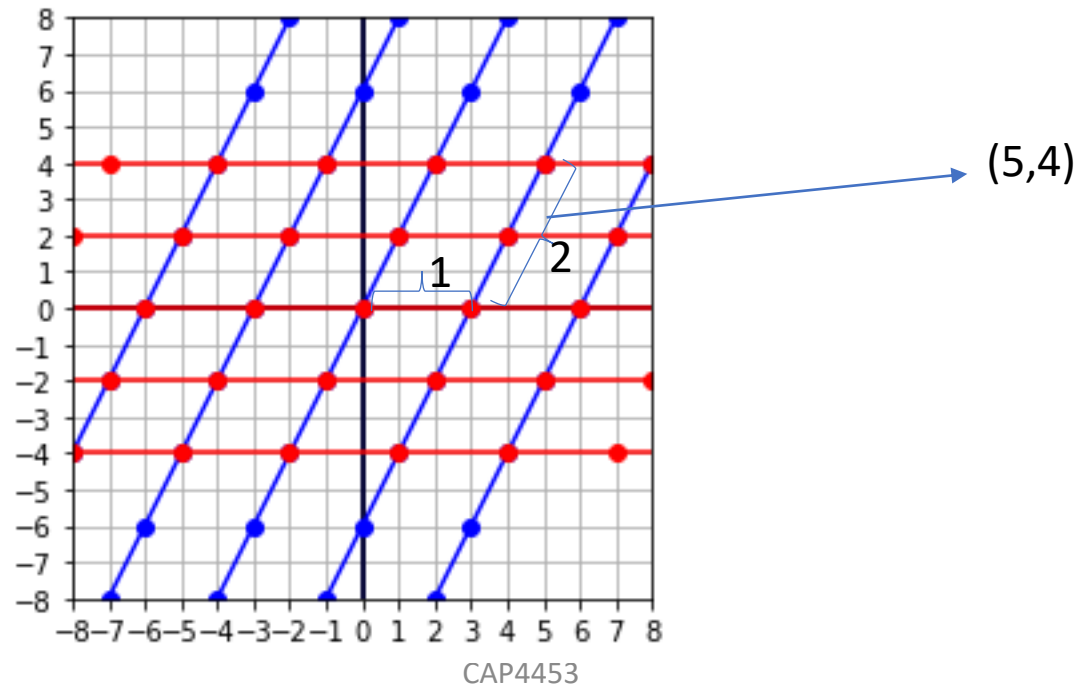


# Matrix as Linear Transformation

$$T(\vec{v}) = A\vec{v}$$

$$T(\vec{v}) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

x=1 y=2





# Eigenvalues and eigenvector

- An eigenvector is a vector whose direction remains unchanged when a linear transformation is applied to it.

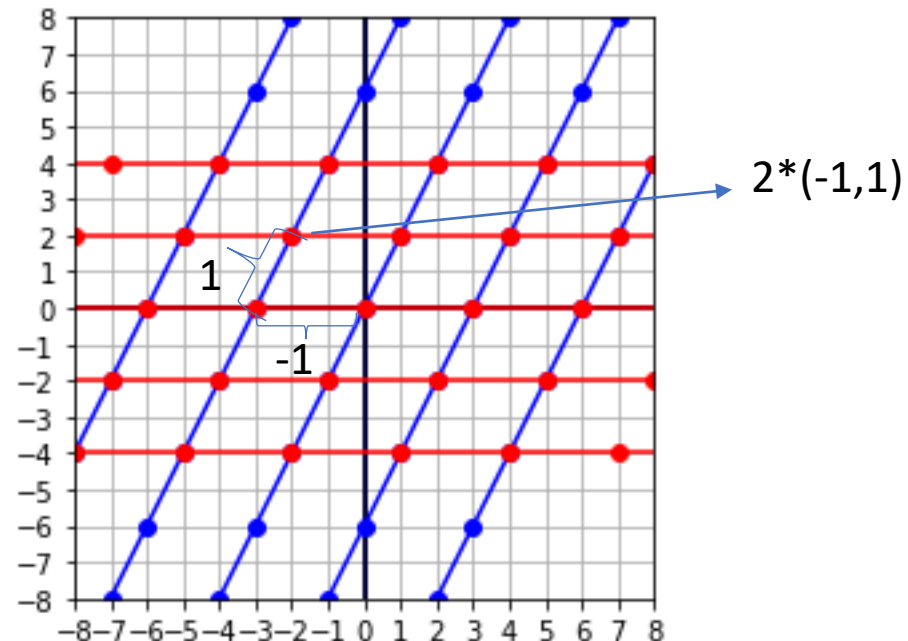


# Eigenvalues and eigenvector

- An eigenvector is a vector whose direction remains unchanged when a linear transformation is applied to it.

$$T(\vec{v}) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$x=-1$   $y=1$

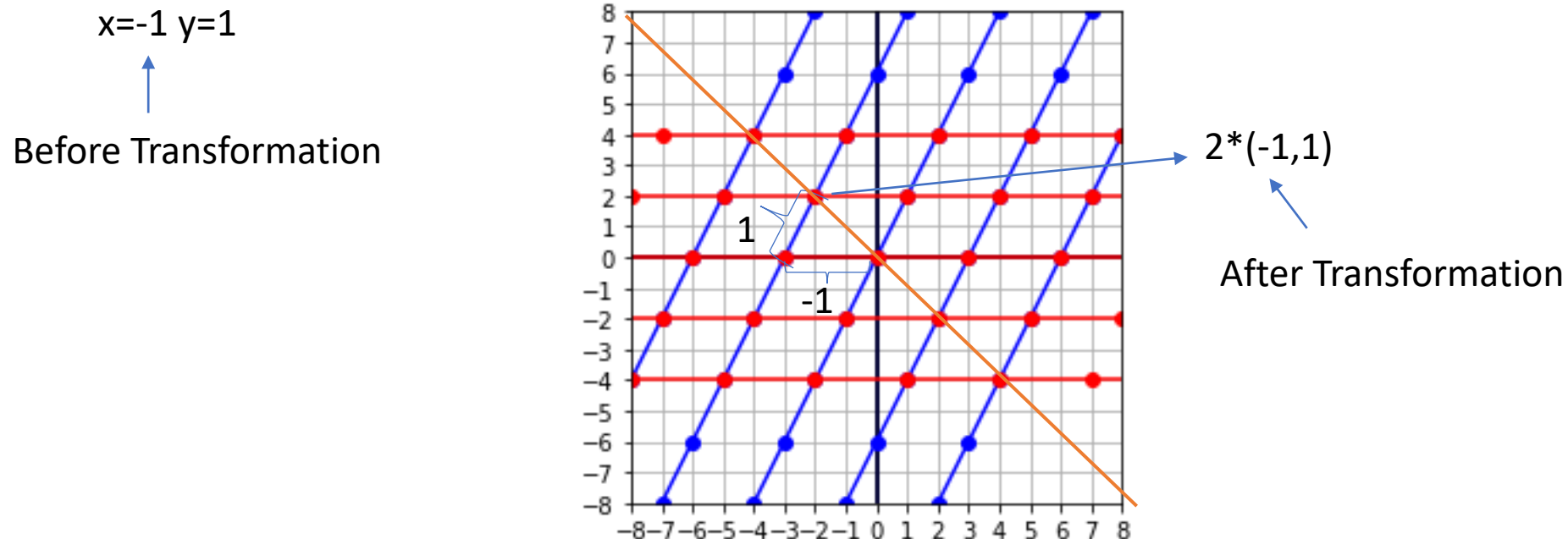




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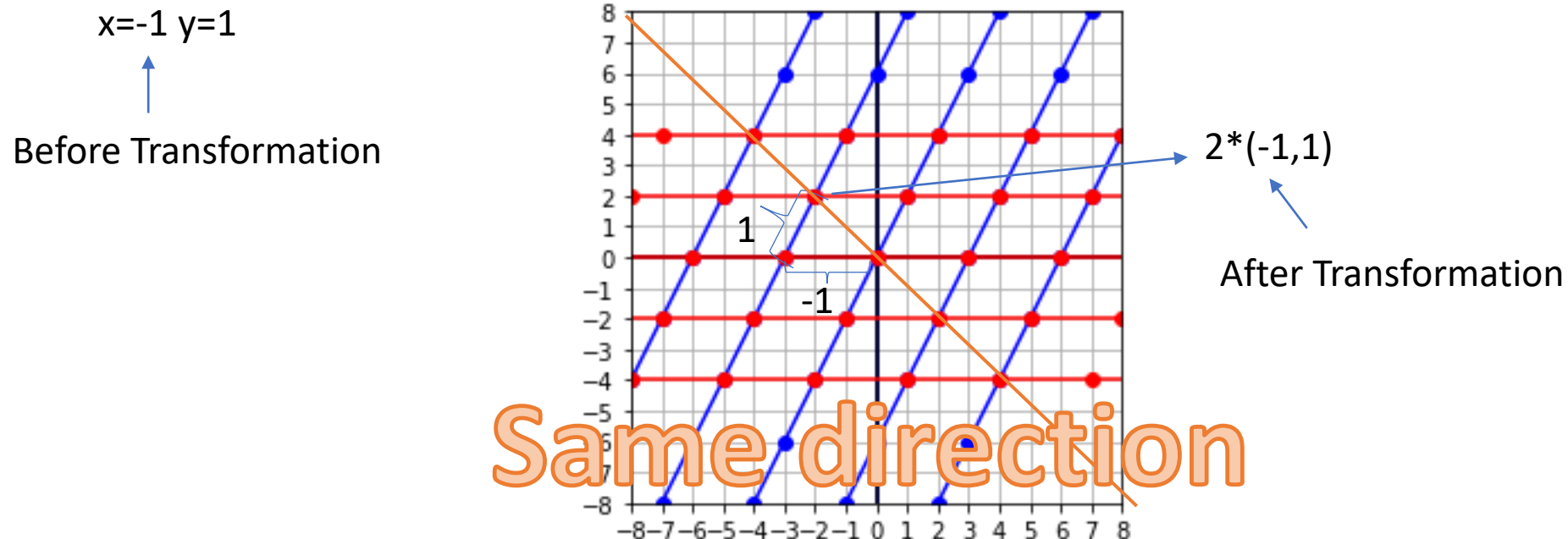
$$T(\vec{v}) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



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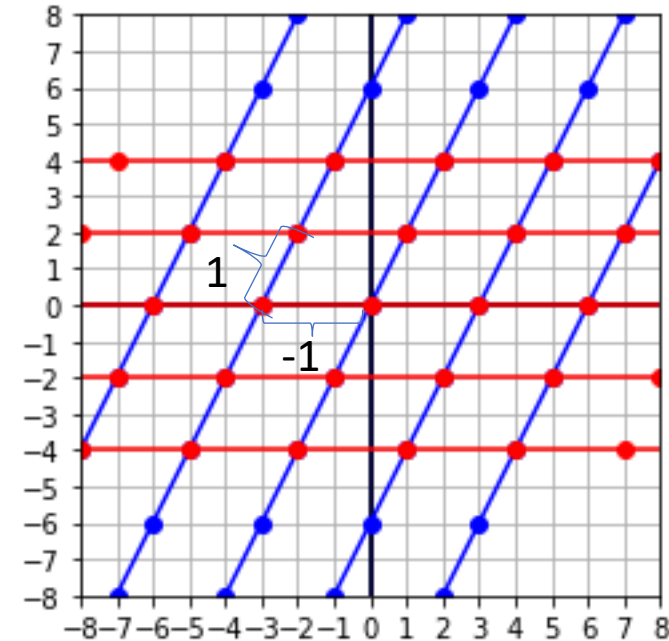
# Eigenvalues and eigenvector

- An eigenvector is a vector whose direction remains unchanged when a linear transformation is applied to it.

$$T(\vec{v}) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Eigenvector



# Eigenvalues and eigenvector

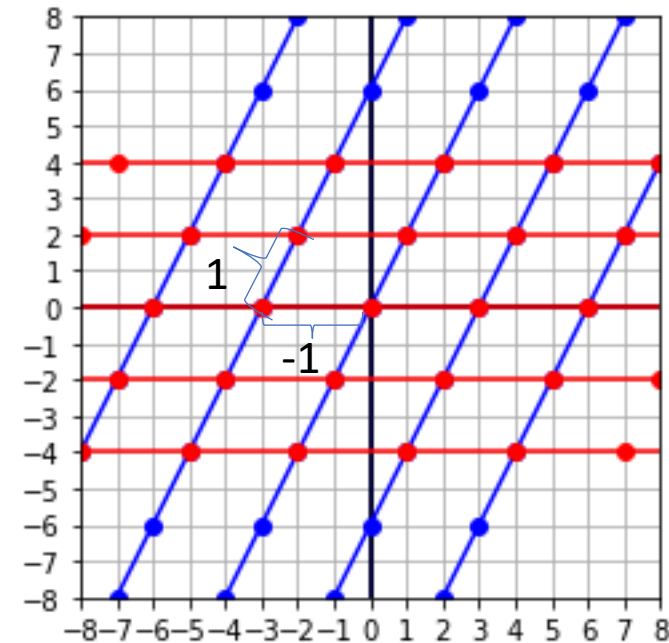
- An eigenvector is a vector whose direction remains unchanged when a linear transformation is applied to it.

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Eigenvalue

Eigenvector



# Eigenvalues and eigenvector

- An eigenvector is a vector whose direction remains unchanged when a linear transformation is applied to it.

$$T(\vec{v}) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

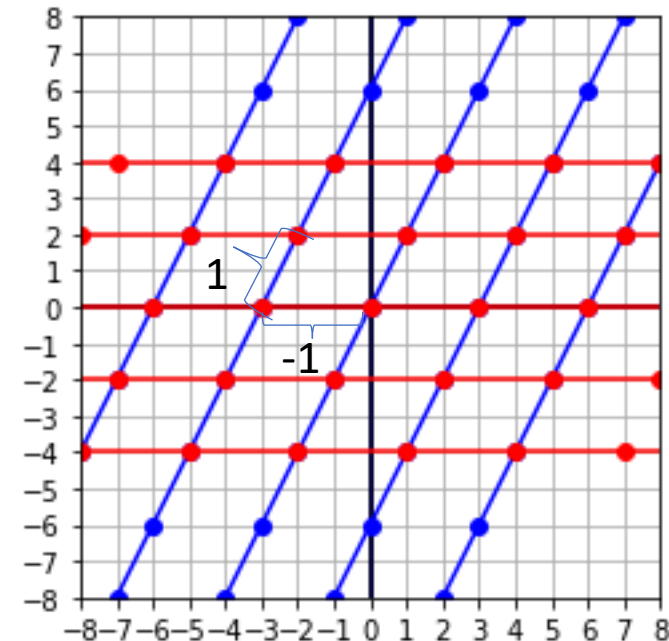
$$T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Mathematical definition

$$\overrightarrow{Av} = \lambda \vec{v}$$

↙
↘

Eigenvalue
Eigenvector



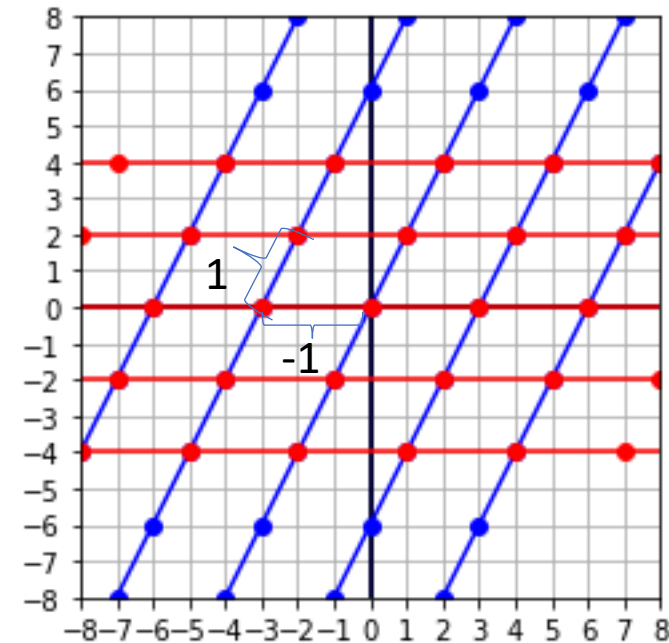
# Eigenvalues and eigenvector

$$A\vec{v} = \lambda\vec{v}$$

- An eigenvector is a vector whose direction remains unchanged when a linear transformation is applied to it.

$$T(\vec{v}) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Is there any other eigenvector?



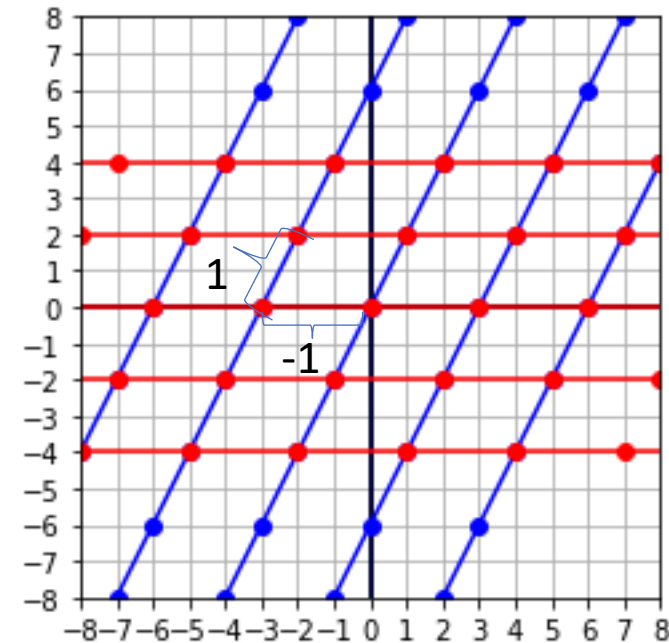
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$$T(\vec{v}) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Is there any other eigenvector?
  - Try with  $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$



# Eigenvalues and eigenvector

$$A\vec{v} = \lambda\vec{v}$$

- An eigenvector is a vector whose direction remains unchanged when a linear transformation is applied to it.

$$T(\vec{v}) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

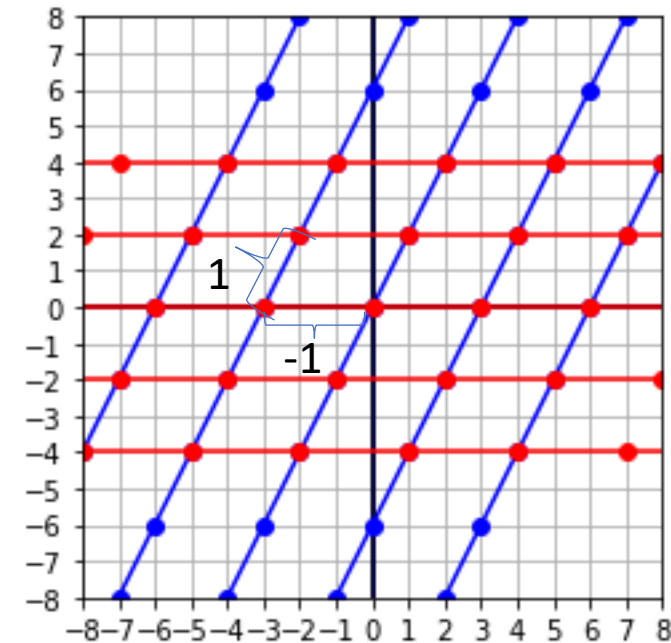
- Is there any other eigenvector?

$$T\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Eigenvalue  
(stretching)

Eigenvector  
(direction)

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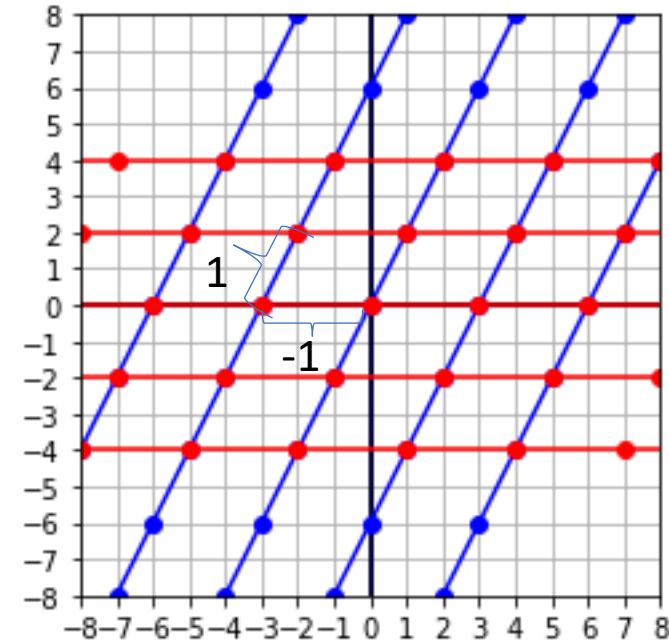
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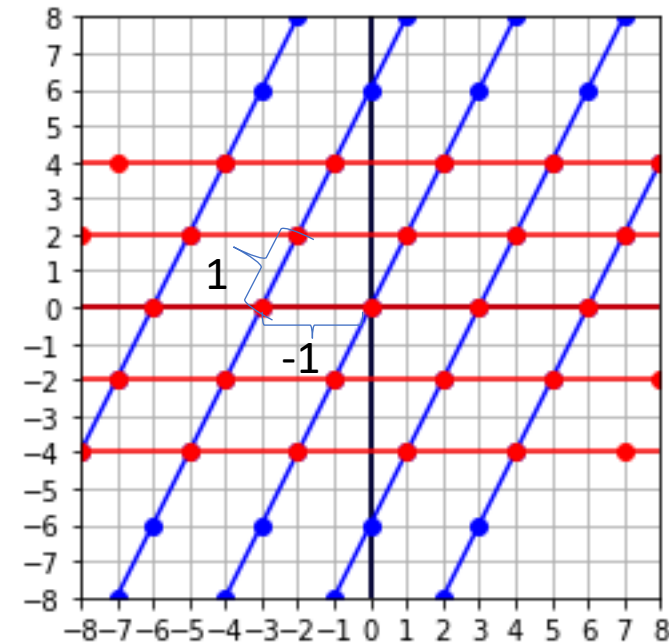
# Eigenvalues and eigenvector

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- An eigenvector is a vector whose direction remains unchanged when a linear transformation is applied to it.

$$T(\vec{v}) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Is there any other eigenvector?
- NO.



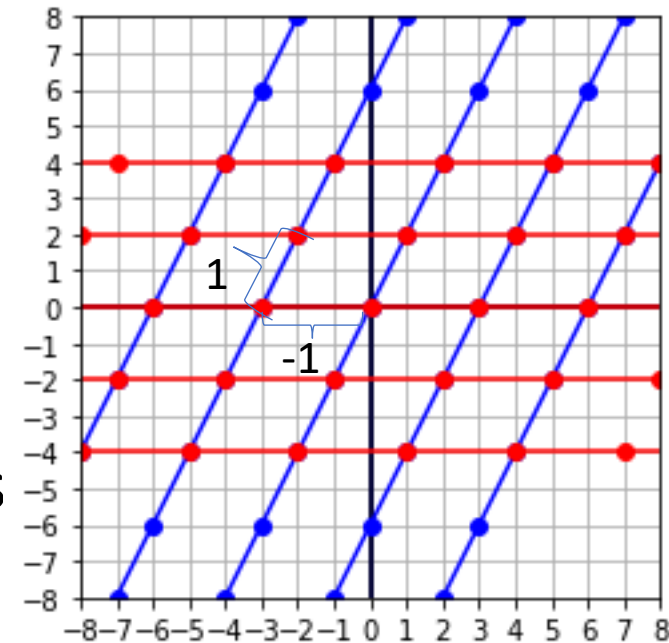
# Eigenvalues and eigenvector

$$A\vec{v} = \lambda\vec{v}$$

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$$T(\vec{v}) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Is there any other eigenvector?
- NO.
- An  $A_{m,m}$  matrix has at most  $m$  eigenvectors



# Eigenvalues and eigenvector

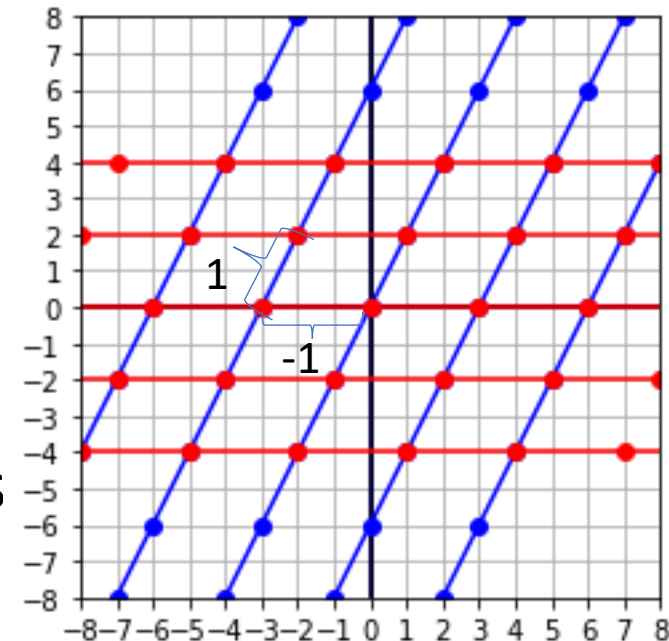
$$A\vec{v} = \lambda\vec{v}$$

- An eigenvector is a vector whose direction remains unchanged when a linear transformation is applied to it.

$$T(\vec{v}) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Is there any other eigenvector?
- NO.
- An  $A_{m,m}$  matrix has at most  $m$  eigenvectors

In this example  $m=2 \rightarrow$  maximum 2 eigenvectors





# Computing Eigenvalues & Eigenvectors

$$A\vec{v} = \lambda\vec{v}$$

Identity

$$A\vec{v} = \lambda I\vec{v}$$

1	0	0
0	1	0
0	0	1



# Computing Eigenvalues & Eigenvectors

$$A\vec{v} = \lambda\vec{v} \quad \begin{matrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{matrix}$$
$$A\vec{v} = \underbrace{\lambda I}_{\leftarrow} \vec{v}$$

$$A\vec{v} - \lambda I\vec{v} = 0$$



# Computing Eigenvalues & Eigenvectors

$$A\vec{v} = \lambda\vec{v} \quad \begin{matrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{matrix}$$
$$A\vec{v} = \underbrace{\lambda I}_{\leftarrow} \vec{v}$$

$$A\vec{v} - \lambda I\vec{v} = 0$$

$$(A - \lambda I)\vec{v} = 0$$



# Computing Eigenvalues & Eigenvectors

$$A\vec{v} = \lambda\vec{v}$$
$$A\vec{v} = \underbrace{\lambda I}_{\leftarrow} \vec{v}$$
$$\begin{matrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{matrix}$$

$$A\vec{v} - \lambda I\vec{v} = 0$$

if  $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$

$$\underbrace{(A - \lambda I)}_{\leftarrow} \vec{v} = 0$$





# Computing Eigenvalues & Eigenvectors

$$A\vec{v} = \lambda\vec{v} \quad \begin{matrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{matrix}$$
$$A\vec{v} = \underbrace{\lambda I}_{\leftarrow} \vec{v}$$

$$A\vec{v} - \lambda I\vec{v} = 0 \quad \text{if } A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$
$$\underbrace{(A - \lambda I)}_{\leftarrow} \vec{v} = 0 \quad \begin{bmatrix} 3 - \lambda & 1 \\ 0 & 2 - \lambda \end{bmatrix}$$



# Computing Eigenvalues & Eigenvectors

$$A\vec{v} = \lambda\vec{v}$$

$$A\vec{v} = \lambda I\vec{v}$$

$$A\vec{v} - \lambda I\vec{v} = 0$$

$$(A - \lambda I)\vec{v} = 0$$

↑  
If  $\vec{v}$  is not null



# Computing Eigenvalues & Eigenvectors

$$A\vec{v} = \lambda\vec{v}$$

$$A\vec{v} = \lambda I\vec{v}$$

$$A\vec{v} - \lambda I\vec{v} = 0$$

$$(A - \lambda I)\vec{v} = 0$$

If  $\vec{v}$  is not null

Must be not invertible



# Computing Eigenvalues & Eigenvectors

$$A\vec{v} = \lambda\vec{v}$$

$$A\vec{v} = \lambda I\vec{v}$$

$$A\vec{v} - \lambda I\vec{v} = 0$$

$$(A - \lambda I)\vec{v} = 0$$

If  $\vec{v}$  is not null

Must be not invertible

Determinant = zero



# Computing Eigenvalues & Eigenvectors

$$A\vec{v} - \lambda I\vec{v} = 0$$

$$(A - \lambda I)\vec{v} = 0$$

If  $\vec{v}$  is not null

Must be not invertible

Determinant = zero

$$\det(A - \lambda I) = 0$$



# Computing Eigenvalues & Eigenvectors

$$T(\vec{v}) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$

$$(A - \lambda I) = \begin{bmatrix} 3 - \lambda & 1 \\ 0 & 2 - \lambda \end{bmatrix}$$

$$A\vec{v} = \lambda\vec{v}$$

$$(A - \lambda I)\vec{v} = 0$$

$$\det(A - \lambda I) = 0$$



# Computing Eigenvalues & Eigenvectors

$$T(\vec{v}) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$

$$\det(A - \lambda I) = \det \begin{bmatrix} 3 - \lambda & 1 \\ 0 & 2 - \lambda \end{bmatrix} = 0$$

$$A\vec{v} = \lambda\vec{v}$$

$$(A - \lambda I)\vec{v} = 0$$

$$\det(A - \lambda I) = 0$$



# Computing Eigenvalues & Eigenvectors

$$T(\vec{v}) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$

$$\det(A - \lambda I) = \det \begin{bmatrix} 3 - \lambda & 1 \\ 0 & 2 - \lambda \end{bmatrix} = 0$$

$$(3 - \lambda)(2 - \lambda) - 0 * 1 = 0$$

$$A\vec{v} = \lambda\vec{v}$$

$$(A - \lambda I)\vec{v} = 0$$

$$\det(A - \lambda I) = 0$$





# Computing Eigenvalues & Eigenvectors

$$T(\vec{v}) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$

$$\det(A - \lambda I) = \det \begin{bmatrix} 3 - \lambda & 1 \\ 0 & 2 - \lambda \end{bmatrix} = 0$$

$$(3 - \lambda)(2 - \lambda) = 0$$

$\lambda = 3$        $\lambda = 2$

$$A\vec{v} = \lambda\vec{v}$$

$$(A - \lambda I)\vec{v} = 0$$

$$\det(A - \lambda I) = 0$$



# Computing Eigenvector for $\lambda=2$

$$T(\vec{v}) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix}$$

Matrix multiplication

$$3x + 2y = 2x$$

$$0x + 2y = 2y$$

$$A\vec{v} = \lambda\vec{v}$$

$$(A - \lambda I)\vec{v} = 0$$

$$\det(A - \lambda I) = 0$$



# Computing Eigenvector for $\lambda=2$

$$T(\vec{v}) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix}$$

Matrix multiplication

$$3x + y = 2x$$

$$0x + 2y = 2y$$

$$3x + y = 2x$$

$$3x - 2x = -y$$

$$x = -y$$

$$A\vec{v} = \lambda\vec{v}$$

$$(A - \lambda I)\vec{v} = 0$$

$$\det(A - \lambda I) = 0$$



# Computing Eigenvector for $\lambda=2$

$$T(\vec{v}) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix}$$

Matrix multiplication

$$3x + y = 2x$$

$$0x + 2y = 2y$$

If  $x=-1$  then  $y=1$

$$\text{for } \lambda=2, \vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$A\vec{v} = \lambda\vec{v}$$

$$(A - \lambda I)\vec{v} = 0$$

$$\det(A - \lambda I) = 0$$

$$3x + y = 2x$$

$$3x - 2x = -y$$

$$x = -y$$



# Computing Eigenvector for $\lambda=3$

$$T(\vec{v}) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 3 \begin{bmatrix} x \\ y \end{bmatrix}$$

Matrix multiplication

$$3x + y = 3x$$

$$0x + 2y = 3y$$

$$y=0$$

$$\text{for } \lambda=3, \vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A\vec{v} = \lambda\vec{v}$$

$$(A - \lambda I)\vec{v} = 0$$

$$\det(A - \lambda I) = 0$$

$$3x + y = 3x$$

$$3x - 3x = -y$$

$$0 = y$$

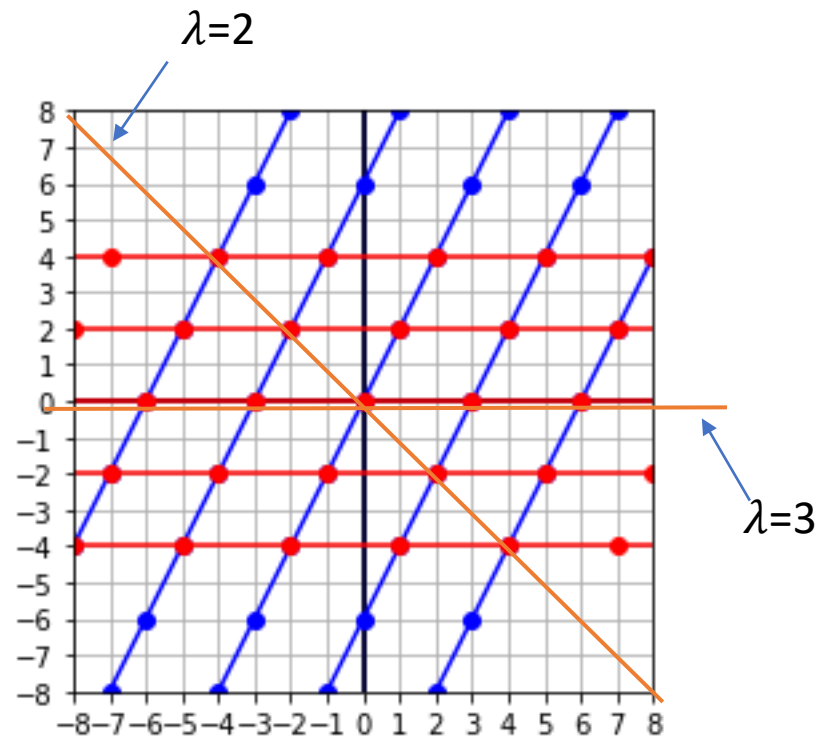
# Computing Eigenvectors

$$T(\vec{v}) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 3 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$A\vec{v} = \lambda\vec{v}$$

$$(A - \lambda I)\vec{v} = 0$$

$$\det(A - \lambda I) = 0$$

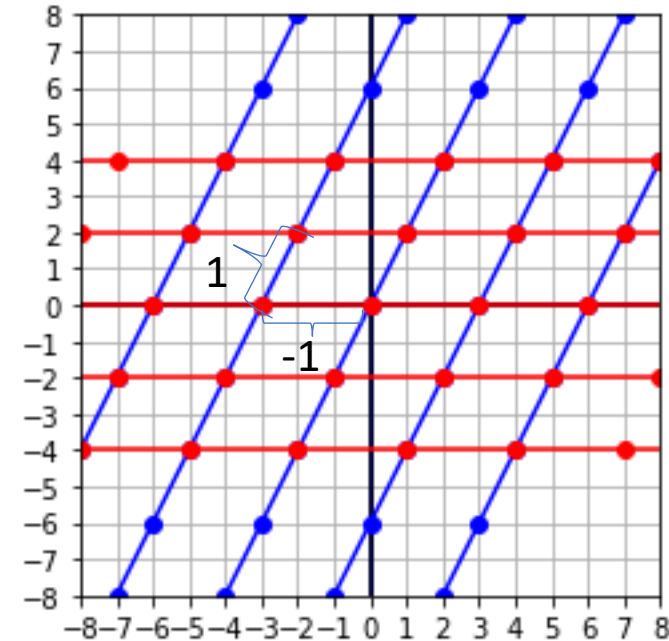


# Eigenvalues and eigenvector

- An eigenvector is a vector whose direction remains unchanged when a linear transformation is applied to it.

$$A\vec{v} = \lambda\vec{v}$$

- Does the definition make sense for a non-square matrix  $A_{m,n}$ ?





# Eigenvalues and eigenvector

- An eigenvector is a vector whose direction remains unchanged when a linear transformation is applied to it.

$$A\vec{v} = \lambda\vec{v}$$

- Does the definition make sense for a non-square matrix  $A_{m,n}$ ?
  - NO
  - Transformation changes dimension of vector  $\vec{v}$ .





# Linear Algebra

- Matrix as a Linear Transformation
- Eigenvalues and eigenvector
  - Intuition
  - How to compute it
- **Singular Value Decomposition (SVD)**
  - Definition, derivation
  - Intuition
  - Direct Solving  $Ax=0$




# Singular Value Decomposition

$$\mathbf{A} = \mathbf{U} \mathbf{\Sigma} \mathbf{V}^T$$

ortho-normal    diagonal    ortho-normal

$n \times m$              $n \times n$      $n \times m$      $m \times m$

unit norm constraint





# Singular Value Decomposition

$$\mathbf{A} = \mathbf{U} \mathbf{\Sigma} \mathbf{V}^T$$

ortho-normal    diagonal    ortho-normal

n x m            n x n    n x m    m x m

$$A\vec{v}_1 = \sigma_1 \vec{u}_1$$

$$A\vec{v}_2 = \sigma_2 \vec{u}_2$$

⋮

$$A\vec{v}_m = \sigma_m \vec{u}_n$$

$\vec{v}_i$  is ortho-normal



# Singular Value Decomposition

$$\mathbf{A} = \mathbf{U} \mathbf{\Sigma} \mathbf{V}^T$$

ortho-normal    diagonal    ortho-normal

n x m            n x n    n x m    m x m

$$A\vec{v}_1 = \sigma_1 \vec{u}_1$$

$$A\vec{v}_2 = \sigma_2 \vec{u}_2$$

⋮  
⋮  
⋮

$$A\vec{v}_m = \sigma_m \vec{u}_n$$

$\vec{v}_i$  is ortho-normal

$$\vec{v}_i \cdot \vec{v}_i = 1$$

$$\vec{v}_i \cdot \vec{v}_j = 0$$



# Singular Value Decomposition

$$\mathbf{A} = \mathbf{U} \mathbf{\Sigma} \mathbf{V}^T$$

ortho-normal    diagonal    ortho-normal

n x m            n x n    n x m    m x m

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⋮  
⋮  
⋮

$$A\vec{v}_m = \sigma_m \vec{u}_n$$

$\vec{v}_i$  is ortho-normal

$$\vec{v}_i \cdot \vec{v}_i = 1$$

$$\vec{v}_i \cdot \vec{v}_j = 0$$

dimension of  $\vec{v}_i$  is  $m \times 1$



# Singular Value Decomposition

$$\mathbf{A} = \mathbf{U} \mathbf{\Sigma} \mathbf{V}^T$$

ortho-normal    diagonal    ortho-normal

n x m            n x n    n x m    m x m

$$A\vec{v}_1 = \sigma_1 \vec{u}_1$$

$$A\vec{v}_2 = \sigma_2 \vec{u}_2$$

⋮

⋮

⋮

$$A\vec{v}_m = \sigma_m \vec{u}_m$$

$\vec{v}_i$  is ortho-normal

dimension of  $\vec{v}_i$  is  $m \times 1$

$\vec{u}_i$  is a unit vector

dimension of  $\vec{u}_i$  is  $n \times 1$

$\sigma_i$  is magnitude of vector



# Singular Value Decomposition

$$\begin{matrix}
 \text{ortho-normal} & & \text{diagonal} & & \text{ortho-normal} \\
 \mathbf{A} = \mathbf{U} \mathbf{\Sigma} \mathbf{V}^T \\
 n \times m & & n \times n & n \times m & m \times m
 \end{matrix}$$

Any vector  $\vec{x} \in R^m$

$$\vec{x} = (\vec{x} \cdot \vec{v}_1) \vec{v}_1 + (\vec{x} \cdot \vec{v}_2) \vec{v}_2 + \dots + (\vec{x} \cdot \vec{v}_m) \vec{v}_m$$

$$A \vec{v}_1 = \sigma_1 \vec{u}_1$$

$$A \vec{v}_2 = \sigma_2 \vec{u}_2$$

⋮

$$A \vec{v}_m = \sigma_m \vec{u}_m$$



# Singular Value Decomposition

$$\begin{matrix}
 \text{ortho-normal} & & \text{diagonal} & & \text{ortho-normal} \\
 \mathbf{A} = \mathbf{U} \mathbf{\Sigma} \mathbf{V}^T \\
 n \times m & & n \times n & & n \times m & & m \times m
 \end{matrix}$$

Any vector  $\vec{x} \in R^m$

$$\vec{x} = (\vec{x} \cdot \vec{v}_1)\vec{v}_1 + (\vec{x} \cdot \vec{v}_2)\vec{v}_2 + \dots + (\vec{x} \cdot \vec{v}_m)\vec{v}_m$$

$$A\vec{v}_1 = \sigma_1\vec{u}_1$$

$$A\vec{v}_2 = \sigma_2\vec{u}_2$$

A transformation  $A\vec{x}$  from  $R^m$  to  $R^n$  is:

$$A\vec{x} = (\vec{x} \cdot \vec{v}_1)A\vec{v}_1 + (\vec{x} \cdot \vec{v}_2)A\vec{v}_2 + \dots + (\vec{x} \cdot \vec{v}_m)A\vec{v}_m$$

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$$A\vec{v}_m = \sigma_m\vec{u}_m$$





# Singular Value Decomposition

$$\underset{n \times m}{\mathbf{A}} = \underset{n \times n}{\mathbf{U}} \underset{n \times m}{\mathbf{\Sigma}} \underset{m \times m}{\mathbf{V}}^T$$

ortho-normal
diagonal
ortho-normal

Any vector  $\vec{x} \in R^m$

$$\vec{x} = (\vec{x} \cdot \vec{v}_1)\vec{v}_1 + (\vec{x} \cdot \vec{v}_2)\vec{v}_2 + \dots + (\vec{x} \cdot \vec{v}_m)\vec{v}_m$$

$$A\vec{v}_1 = \sigma_1\vec{u}_1$$

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$$A\vec{x} = (\vec{x} \cdot \vec{v}_1)\sigma_1\vec{u}_1 + (\vec{x} \cdot \vec{v}_2)\sigma_2\vec{u}_2 + \dots + (\vec{x} \cdot \vec{v}_m)\sigma_m\vec{u}_m$$

$$A\vec{v}_m = \sigma_m\vec{u}_m$$



# Singular Value Decomposition

$$\underset{n \times m}{\mathbf{A}} = \underset{n \times n}{\mathbf{U}} \underset{n \times m}{\mathbf{\Sigma}} \underset{m \times m}{\mathbf{V}}^T$$

ortho-normal
diagonal
ortho-normal

Any vector  $\vec{x} \in R^m$

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$$A\vec{x} = (\vec{x} \cdot \vec{v}_1)\sigma_1\vec{u}_1 + (\vec{x} \cdot \vec{v}_2)\sigma_2\vec{u}_2 + \dots + (\vec{x} \cdot \vec{v}_m)\sigma_m\vec{u}_m$$

$$A\vec{v}_m = \sigma_m\vec{u}_m$$

$$A\vec{x} = \vec{u}_1\sigma_1(\vec{x} \cdot \vec{v}_1) + \vec{u}_2\sigma_2(\vec{x} \cdot \vec{v}_2) + \dots + \vec{u}_m\sigma_m(\vec{x} \cdot \vec{v}_m)$$



ortho-normal    diagonal    ortho-normal

# Singular Value Decomposition

$$\mathbf{A} = \mathbf{U} \mathbf{\Sigma} \mathbf{V}^T$$

$n \times m$                        $n \times n$      $n \times m$                        $m \times m$

Any vector  $\vec{x} \in R^m$

$$\vec{x} = (\vec{x} \cdot \vec{v}_1) \vec{v}_1 + (\vec{x} \cdot \vec{v}_2) \vec{v}_2 + \dots + (\vec{x} \cdot \vec{v}_m) \vec{v}_m$$

$$A\vec{v}_1 = \sigma_1 \vec{u}_1$$

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A transformation  $A\vec{x}$  from  $R^m$  to  $R^n$  is:

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$$A\vec{v}_m = \sigma_m \vec{u}_m$$

$$A\vec{x} = \vec{u}_1 \sigma_1 (\vec{x} \cdot \vec{v}_1) + \vec{u}_2 \sigma_2 (\vec{x} \cdot \vec{v}_2) + \dots + \vec{u}_m \sigma_m (\vec{x} \cdot \vec{v}_m)$$

$$A\vec{x} = \vec{u}_1 \sigma_1 \vec{v}_1^T \vec{x} + \vec{u}_2 \sigma_2 \vec{v}_2^T \vec{x} + \dots + \vec{u}_m \sigma_m \vec{v}_m^T \vec{x}$$

$$\vec{x} \cdot \vec{v}_i = \vec{v}_i^T \vec{x}$$



# Singular Value Decomposition

$$\underset{n \times m}{\mathbf{A}} = \underset{n \times n}{\mathbf{U}} \underset{n \times m}{\mathbf{\Sigma}} \underset{m \times m}{\mathbf{V}}^T$$

ortho-normal
diagonal
ortho-normal

Any vector  $\vec{x} \in R^m$

$$\vec{x} = (\vec{x} \cdot \vec{v}_1)\vec{v}_1 + (\vec{x} \cdot \vec{v}_2)\vec{v}_2 + \dots + (\vec{x} \cdot \vec{v}_m)\vec{v}_m$$

$$A\vec{v}_1 = \sigma_1\vec{u}_1$$

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A transformation  $A\vec{x}$  from  $R^m$  to  $R^n$  is:

$$A\vec{x} = (\vec{x} \cdot \vec{v}_1)A\vec{v}_1 + (\vec{x} \cdot \vec{v}_2)A\vec{v}_2 + \dots + (\vec{x} \cdot \vec{v}_m)A\vec{v}_m$$

$\cdot$   
 $\cdot$   
 $\cdot$

$$A\vec{x} = (\vec{x} \cdot \vec{v}_1)\sigma_1\vec{u}_1 + (\vec{x} \cdot \vec{v}_2)\sigma_2\vec{u}_2 + \dots + (\vec{x} \cdot \vec{v}_m)\sigma_m\vec{u}_m$$

$$A\vec{v}_m = \sigma_m\vec{u}_m$$

$$A\vec{x} = \vec{u}_1\sigma_1(\vec{x} \cdot \vec{v}_1) + \vec{u}_2\sigma_2(\vec{x} \cdot \vec{v}_2) + \dots + \vec{u}_m\sigma_m(\vec{x} \cdot \vec{v}_m)$$

$$A\vec{x} = \vec{u}_1\sigma_1\vec{v}_1^T\vec{x} + \vec{u}_2\sigma_2\vec{v}_2^T\vec{x} + \dots + \vec{u}_m\sigma_m\vec{v}_m^T\vec{x}$$

$$\vec{x} \cdot \vec{v}_i = \vec{v}_i^T\vec{x}$$

$$A\vec{x} = (\vec{u}_1\sigma_1\vec{v}_1^T + \vec{u}_2\sigma_2\vec{v}_2^T + \dots + \vec{u}_m\sigma_m\vec{v}_m^T)\vec{x}$$



# Singular Value Decomposition

$$\mathbf{A} = \mathbf{U} \mathbf{\Sigma} \mathbf{V}^T$$

n x m
n x n
n x m
m x m

Any vector  $\vec{x} \in R^m$

$$\vec{x} = (\vec{x} \cdot \vec{v}_1) \vec{v}_1 + (\vec{x} \cdot \vec{v}_2) \vec{v}_2 + \dots + (\vec{x} \cdot \vec{v}_m) \vec{v}_m$$

A transformation  $A\vec{x}$  from  $R^m$  to  $R^n$  is:

$$A\vec{x} = (\vec{x} \cdot \vec{v}_1) A\vec{v}_1 + (\vec{x} \cdot \vec{v}_2) A\vec{v}_2 + \dots + (\vec{x} \cdot \vec{v}_m) A\vec{v}_m$$

$$A\vec{x} = (\vec{x} \cdot \vec{v}_1) \sigma_1 \vec{u}_1 + (\vec{x} \cdot \vec{v}_2) \sigma_2 \vec{u}_2 + \dots + (\vec{x} \cdot \vec{v}_m) \sigma_m \vec{u}_m$$

$$A\vec{x} = \vec{u}_1 \sigma_1 (\vec{x} \cdot \vec{v}_1) + \vec{u}_2 \sigma_2 (\vec{x} \cdot \vec{v}_2) + \dots + \vec{u}_m \sigma_m (\vec{x} \cdot \vec{v}_m)$$

$$A\vec{x} = \vec{u}_1 \sigma_1 \vec{v}_1^T \vec{x} + \vec{u}_2 \sigma_2 \vec{v}_2^T \vec{x} + \dots + \vec{u}_m \sigma_m \vec{v}_m^T \vec{x}$$

$$A\vec{x} = (\vec{u}_1 \sigma_1 \vec{v}_1^T + \vec{u}_2 \sigma_2 \vec{v}_2^T + \dots + \vec{u}_m \sigma_m \vec{v}_m^T) \vec{x}$$

$$A = (\vec{u}_1 \sigma_1 \vec{v}_1^T + \vec{u}_2 \sigma_2 \vec{v}_2^T + \dots + \vec{u}_m \sigma_m \vec{v}_m^T)$$

$$A\vec{v}_1 = \sigma_1 \vec{u}_1$$

$$A\vec{v}_2 = \sigma_2 \vec{u}_2$$

⋮

$$A\vec{v}_m = \sigma_m \vec{u}_m$$

$$\vec{x} \cdot \vec{v}_i = \vec{v}_i^T \vec{x}$$



n>m

ortho-normal    diagonal    ortho-normal

# Singular Value Decomposition $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$

$n \times m$                        $n \times n$      $n \times m$                        $m \times m$

$$A = (\vec{u}_1\sigma_1\vec{v}_1^T + \vec{u}_2\sigma_2\vec{v}_2^T + \dots + \vec{u}_m\sigma_m\vec{v}_m^T)$$

$$A = U\Sigma V^T$$

$\mathbf{U}$   
 $= [\vec{u}_1 | \vec{u}_2 | \dots | \vec{u}_m | \dots]_{n \times n}$   
dimension of  $\vec{u}_i$  is  $n \times 1$

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \ddots \\ 0 & 0 & \sigma_m \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{n \times m}$$

$\mathbf{V} = [\vec{v}_1 | \vec{v}_2 | \dots | \vec{v}_m]_{m \times m}$   
dimension of  $\vec{v}_i$  is  $m \times 1$

$$V^T = \begin{bmatrix} \vec{v}_1^T \\ \vec{v}_2^T \\ \vdots \\ \vec{v}_m^T \end{bmatrix}_{m \times m}$$



n>m

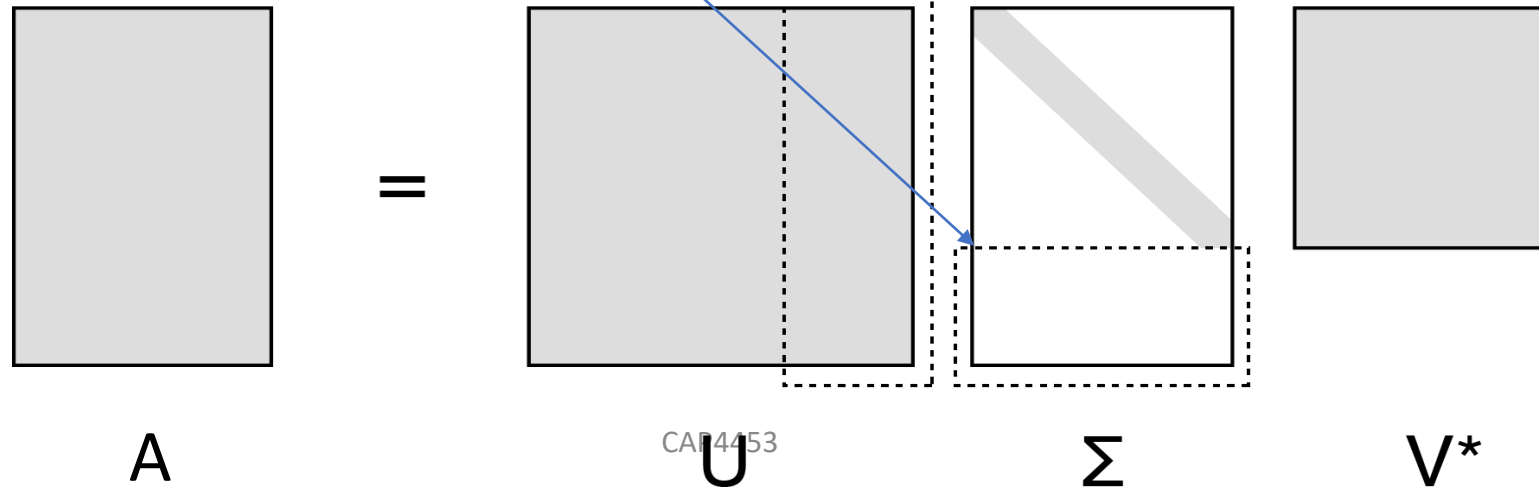
ortho-normal    diagonal    ortho-normal

# Singular Value Decomposition $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$

$$A = (\vec{u}_1\sigma_1\vec{v}_1^T + \vec{u}_2\sigma_2\vec{v}_2^T + \dots + \vec{u}_m\sigma_m\vec{v}_m^T)$$

$$A = U\Sigma V^T$$

$$U = [\vec{u}_1 | \vec{u}_2 | \dots | \vec{u}_m | \dots ]_{n \times n} \quad \Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \ddots \\ 0 & 0 & \sigma_m & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{n \times m} \quad V^T = \begin{bmatrix} \vec{v}_1^T \\ \vec{v}_2^T \\ \vdots \\ \vec{v}_m^T \end{bmatrix}_{m \times m}$$





$n > m$

# Singular Value Decomposition $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$

ortho-normal    diagonal    ortho-normal

$n \times m$      $n \times n$      $n \times m$      $m \times m$

$$A = (\vec{u}_1 \sigma_1 \vec{v}_1^T + \vec{u}_2 \sigma_2 \vec{v}_2^T + \dots + \vec{u}_m \sigma_m \vec{v}_m^T)$$

$$A = U \Sigma V^T$$

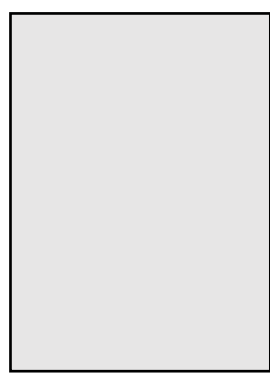
dimension of  $\vec{v}_i$  is  $m \times 1$

$$U = [\vec{u}_1 | \vec{u}_2 | \dots | \vec{u}_m]_{n \times m}$$

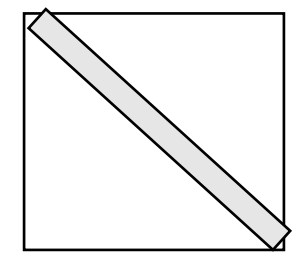
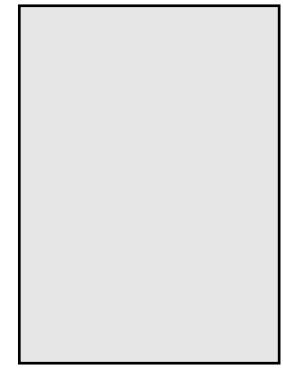
dimension of  $\vec{u}_i$  is  $n \times 1$

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \sigma_m \end{bmatrix}_{m \times m}$$

$$V^T = \begin{bmatrix} \vec{v}_1^T \\ \vec{v}_2^T \\ \vdots \\ \vec{v}_m^T \end{bmatrix}_{m \times m}$$



=



$A_{nm}$

$U_{nm}$

$\Sigma_{mm}$

$V^T_{mm}$



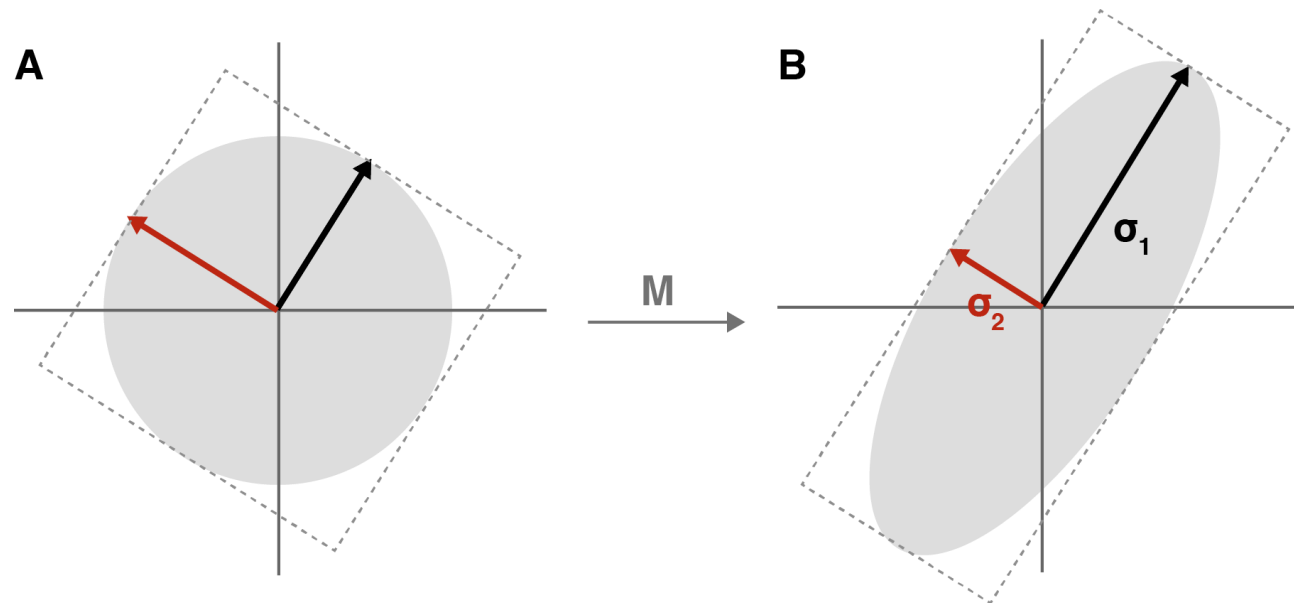


# Linear Algebra

- Matrix as a Linear Transformation
- Eigenvalues and eigenvector
  - Intuition
  - How to compute it
- **Singular Value Decomposition (SVD)**
  - Definition, derivation
  - **Intuition**
  - Direct Solving  $Ax=0$

# Pseudo inverse intuition

- Since the SVD is a decomposition of a given matrix into 2 Unitary matrices and a diagonal matrix, all matrices could be described as a rotation, scaling and another rotation.



(A) An oriented circle; if it helps, imagine that circle inscribed in our original square. (B) Our circle transformed into an ellipse. The length of the major and minor axes of the ellipse have values  $\sigma_1$  and  $\sigma_2$  respectively, called the *singular values*.

# Interesting properties of SVD

- The diagonal values of  $\Sigma$  are the square root of eigenvalues of  $A^T A$

$$A = U \Sigma V^{-1} \quad \Sigma = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \cdot & \\ & & & \sigma_N \end{bmatrix}$$

$U, V =$  orthogonal matrix

$$\sigma_i = \sqrt{\lambda_i} \quad \begin{array}{l} \sigma = \text{singular value} \\ \lambda = \text{eigenvalue of } A^t A \end{array}$$

# Interesting properties of SVD

- The diagonal values of  $\Sigma$  are the square root of eigenvalues of  $A^T A$
- Eigenvectors of  $A^T A$  corresponds to  $V$
- SVD consists of matrices  $U, \Sigma, V$  which are always real
  - this is unlike eigenvectors and eigenvalues of  $A$  which may be complex even if  $A$  is real
  - The singular values are always non-negative, even though the eigenvalues may be negative
- While writing the SVD, the following convention is assumed, and the left and right singular vectors are also arranged accordingly:

$$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_{m-1} \geq \sigma_m$$



# Interesting properties of SVD

- The rank of a rectangular matrix  $A$  is equal to the number of non-zero singular values. Note that  $\text{rank}(A) = \text{rank}(\Sigma)$ .
- SVD always exist
- It is used to compute pseudoinverse

The Pseudo Inverse of a matrix  $A = U\Sigma V^H$ , denoted  $A^\dagger$  is given by

$$A^\dagger = V\Sigma^\dagger U^H$$

Where  $\Sigma^\dagger$  is obtained by transposing  $\Sigma$  and inverting all non zero entries.



# Computing SVD

- Compute SVD for

$$A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}$$

- Calculate the eigenvalues of  $AA^T$

$$AA^T = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$$

- $\det(AA^T - \lambda I) = 0$

$$(17 - \lambda)(17 - \lambda) + 64 = 0$$

$$\lambda^2 - 34\lambda + 225 = 0$$

$$= (\lambda - 25)(\lambda - 9)$$

$$\sigma_i = \sqrt{\lambda_i}$$

$$\sigma_1 = 5; \sigma_2 = 3$$

$$\Sigma = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}$$



# Compute SVD

- Eigenvector of  $A^T A$

$$\lambda = 25$$

$$A^T A = \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix}$$

$$AA^T - 25 \cdot I = \begin{bmatrix} -12 & 12 & 2 \\ 12 & -12 & -2 \\ 2 & -2 & -17 \end{bmatrix}$$

$$\begin{aligned} -12x + 12y + 2z &= 0 \\ 12x - 12y - 2z &= 0 \\ 2x - 2y - 17z &= 0 \end{aligned}$$

$$-12x + 12y + 2z = 0$$

$$-12x + 12y + 2z = 0$$

$$6(2x - 2y - 17z) = 0$$

$$12x - 12y - 102z = 0$$

---


$$-100z = 0$$

$$\mathbf{z} = \mathbf{0}$$

$$2x - 2y - 17z = 0$$

$$2x - 2y = 0$$

$$\mathbf{x} = \mathbf{y}$$

$$v_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$



# Compute SVD

- Eigenvector of  $A^T A$

$$\lambda = 9$$

$$A^T A = \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix}$$

$$A^T A - 9I = \begin{bmatrix} 4 & 12 & 2 \\ 12 & 4 & -2 \\ 2 & -2 & -1 \end{bmatrix} \quad \begin{aligned} 4x + 12y + 2z &= 0 \\ 12x + 4y - 2z &= 0 \\ 2x - 2y - 1z &= 0 \end{aligned}$$

$$4x + 12y + 2z = 0$$

$$12x + 4y - 2z = 0$$

---


$$16x + 16y = 0$$

$$\mathbf{x} = -\mathbf{y}$$

$$v_2 = \begin{bmatrix} -y \\ y \\ -4y \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -4 \end{bmatrix} y = v_2 = \begin{pmatrix} 1/\sqrt{18} \\ -1/\sqrt{18} \\ 4/\sqrt{18} \end{pmatrix}.$$

$$4x + 12y + 2z = 0$$

$$-2(2x - 2y - 1z) = 0$$

---


$$16y = -4z$$

$$\mathbf{4y} = -\mathbf{z}$$





# Compute SVD

- Eigenvector of  $A^T A$

$$\lambda = 0$$

$$A^T A = \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix}$$

$$A^T A - 0I = \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix} \quad \begin{aligned} 13x + 12y + 2z &= 0 \\ 12x + 13y - 2z &= 0 \\ 2x - 2y + 8z &= 0 \end{aligned}$$

$$13x + 12y + 2z = 0$$

$$12x + 13y - 2z = 0$$

---


$$25x + 25y = 0$$

$$\mathbf{x} = -\mathbf{y}$$

$$v_3 = \begin{bmatrix} -y \\ y \\ y/2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0.5 \end{bmatrix} y =$$

$$v_3 = \begin{pmatrix} 2/3 \\ -2/3 \\ -1/3 \end{pmatrix}.$$

$$12x + 13y - 2z = 0$$

$$-6(2x - 2y + 8z) = 0$$

---


$$25y = 50z$$

$$\mathbf{y} = 2\mathbf{z}$$

# Compute SVD

- So far:  $\sigma_1 = 5; \sigma_2 = 3$   $v_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$   $v_2 = \begin{pmatrix} 1/\sqrt{18} \\ -1/\sqrt{18} \\ 4/\sqrt{18} \end{pmatrix}$   $v_3 = \begin{pmatrix} 2/3 \\ -2/3 \\ -1/3 \end{pmatrix}$ .

$$A = U\Sigma V^T = U \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{18} & -1/\sqrt{18} & 4/\sqrt{18} \\ 2/3 & -2/3 & -1/3 \end{pmatrix}.$$

- Using  $A\vec{v}_i = \sigma_i \vec{u}_i$   $\frac{A\vec{v}_i}{\sigma_i} = \vec{u}_i$   $U = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$

$$\frac{A\vec{v}_1}{\sigma_1} = \frac{\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}}{5} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\frac{A\vec{v}_2}{\sigma_2} = \frac{\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} 1/3\sqrt{2} \\ -1/3\sqrt{2} \\ 4/3\sqrt{2} \end{bmatrix}}{3} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$



# Compute SVD

- In total

$$A = U\Sigma V^T = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{18} & -1/\sqrt{18} & 4/\sqrt{18} \\ 2/3 & -2/3 & -1/3 \end{pmatrix}.$$

```
import numpy as np
A=np.array([[3,2,2],[2,3,-2]])
u, s, vh = np.linalg.svd(A, full_matrices=True)
```

```
In [115]: u
Out[115]: array([[ -0.70710678, -0.70710678],
                [ -0.70710678,  0.70710678]])

In [116]: s
Out[116]: array([5., 3.])

In [117]: vh
Out[117]: array([[ -7.07106781e-01, -7.07106781e-01, -6.47932334e-17],
                 [ -2.35702260e-01,  2.35702260e-01, -9.42809042e-01],
                 [ -6.66666667e-01,  6.66666667e-01,  3.33333333e-01]])
```



# Linear Algebra

- Matrix as a Linear Transformation
- Eigenvalues and eigenvector
  - Intuition
  - How to compute it
- **Singular Value Decomposition (SVD)**
  - Definition, derivation
  - Intuition
  - **Direct Solving  $Ax=0$**

# Derivation using Least squares

$$A\mathbf{h} = \mathbf{0}$$

The sum squared error can be written as:

$$f(\mathbf{h}) = \frac{1}{2} (A\mathbf{h} - \mathbf{0})^T (A\mathbf{h} - \mathbf{0})$$

$$f(\mathbf{h}) = \frac{1}{2} (A\mathbf{h})^T (A\mathbf{h})$$

$$f(\mathbf{h}) = \frac{1}{2} \mathbf{h}^T A^T A \mathbf{h}.$$

Taking the derivative of  $f$  with respect to  $\mathbf{h}$  and setting the result to zero,

$$\begin{aligned} \frac{d}{d\mathbf{h}} f = 0 &= \frac{1}{2} (A^T A + (A^T A)^T) \mathbf{h} \\ 0 &= A^T A \mathbf{h}. \end{aligned}$$

$\mathbf{h}$  should equal the eigenvector of  $B = A^T A$  that has an eigenvalue of zero

$$B\vec{h} = \lambda\vec{h}$$

(or, in the presence of noise the eigenvalue closest to zero)


General form of total least squares

(Warning: change of notation.  $\mathbf{x}$  is a vector of parameters!)

$$E_{\text{TLS}} = \sum_i (\mathbf{a}_i \mathbf{x})^2$$

$$= \|\mathbf{A}\mathbf{x}\|^2 \quad \text{(matrix form)}$$

$$\|\mathbf{x}\|^2 = 1 \quad \text{constraint}$$

minimize	$\ \mathbf{A}\mathbf{x}\ ^2$		minimize	$\frac{\ \mathbf{A}\mathbf{x}\ ^2}{\ \mathbf{x}\ ^2}$
subject to	$\ \mathbf{x}\ ^2 = 1$			(Rayleigh quotient)

Solution is the eigenvector corresponding to smallest eigenvalue of

$$\mathbf{A}^\top \mathbf{A}$$

(equivalent)

Solution is the column of  $\mathbf{V}$  corresponding to smallest singular value

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top$$



# Homogeneous Linear Least Squares problem

$$A\mathbf{x} = \mathbf{0}$$

$$A = U\Sigma V^T = \sum_{i=1}^9 \sigma_i \mathbf{u}_i \mathbf{v}_i^T$$

- If the homography is *exactly determined*, then  $\sigma_9 = 0$ , and there exists a homography that fits the points exactly.
- If the homography is *overdetermined*, then  $\sigma_9 \geq 0$ . Here  $\sigma_9$  represents a “residual” or goodness of fit.
- We will not handle the case of the homography being *underdetermined*.



# Solving for H using DLT

Given  $\{x_i, x'_i\}$  solve for H such that  $x' = Hx$

1. For each correspondence, create 2x9 matrix  $A_i$
2. Concatenate into single  $2n \times 9$  matrix  $A$
3. Compute SVD of  $A = U\Sigma V^T$
4. Store singular vector of the smallest singular value  $h = v_{\hat{i}}$
5. Reshape to get  $H$





# Recap: Two Common Optimization Problems

## Problem statement

$$\text{minimize } \|\mathbf{Ax} - \mathbf{b}\|^2$$

least squares solution to  $\mathbf{Ax} = \mathbf{b}$

## Solution

$$\mathbf{x} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{b}$$

```
import numpy as np
x, resid, rank, s = np.linalg.lstsq(A, b)
```

## Problem statement

$$\text{minimize } \mathbf{x}^T \mathbf{A}^T \mathbf{A} \mathbf{x} \text{ s.t. } \mathbf{x}^T \mathbf{x} = 1$$

non - trivial lsq solution to  $\mathbf{Ax} = 0$

## Solution

$$[\mathbf{v}, \lambda] = \text{eig}(\mathbf{A}^T \mathbf{A})$$

$$\lambda_1 < \lambda_{2..n} : \mathbf{x} = \mathbf{v}_1$$



# References

Basic reading:

- Szeliski textbook, Section 3.6.

Additional reading:

- Hartley and Zisserman, “Multiple View Geometry in Computer Vision,” Cambridge University Press 2004.  
a comprehensive treatment of all aspects of projective geometry relating to computer vision, and also a very useful reference for the second part of the class.
- Richter-Gebert, “Perspectives on projective geometry,” Springer 2011.  
a beautiful, thorough, and very accessible mathematics textbook on projective geometry (available online for free from CMU’s library).



# Questions?