Fall 2016 COT 3100 Final Exam

VERSION A

<u>Please double check that you bubbled your PID and exam version on your scantron. If you do not, you will get a ZERO on this exam.</u>

1) For how many of the 8 possible settings of the Boolean variables x, y and z is the logical expression $\bar{x} \wedge (y \vee \bar{z})$ true?

a) 1 b) 2 c) 3 d) 4 e) None of the Above

2) Which rule of inference is being used in the following logical argument?

"We know that (1) Bob either has a crush on Pamela or Shelly, and we also know that (2) either he doesn't have a crush on Pamela or he does have a crush on Emily. We can deduce that he must have a crush on either Shelly or Emily."

a) Modus Tollens b) Rule of Resolution c) Rule of Conditional Proof

d) Rule of Proof by Cases e) None of the Above

3) Let both x and y be chosen from the universe of real numbers. Which of the following is a true statement?

a)$\forall x \exists y[xy > yx]$ b))$\exists x \exists y[xy > yx]$ c))$\exists x \forall y[xy > yx]$

d))$\forall x \forall y[xy > yx]$ e) None of the Above

4) 27 students own a car, bicycle or skateboard. 7 students own both a car and bicycle, 8 students own both a car and skateboard, and 9 students own both a bicycle and skateboard. 3 students own all 3. Out of the 27 students, the same number own a car as own a bicycle as own a skateboard. How many students own a car in the group?

a) 16 b) 20 c) 24 d) 48 e) None of the Above

5) Consider using a direct proof to prove the following assertion about arbitrary finite sets A, B and C:

$$\text{if } B \cap C = A \cap B, \text{ then } B \cap C \subseteq A \cap C$$

First, we should state what needs to be proved. In this case, we would state that we must show for an arbitrarily selected element x, if $x \in B \cap C$, then we must show $x \in A \cap C$. Which of the following would be the very first step in a direct proof ***after*** stating what is above?

a) Let x be an arbitrarily chosen element such that $x \in B \cap C$.
b) Assume to the contrary that $x \notin A \cap C$.
c) Assume that $B \cap C = A \cap B$.
d) Assume that $C = A$.
e) None of the Above

6) Assume that A and B are finite sets of integers. If we know that $A - B = B - A$, what can we ascertain about the sets A and B?

a) They are equal sets.
b) They are mutually exclusive sets.
c) Both A and B are the empty set.
d) Set A is larger than set B.
e) None of the Above

7) What is the quotient when 137 is divided by 15?

a) 2 b) 8 c) 9 d) 15 e) None of the Above

8) What is the greatest common divisor of 2760 and 990?

a) 1 b) 9 c) 10 d) 90 e) None of the Above

9) Let x be an integer such that $x \equiv 5 \ (mod\ 6)$ and $x \equiv 14 \ (mod\ 15)$. Which of the following ***must be*** true about x?

a) $x \equiv 29 \ (mod\ 30)$ b) $x \equiv 29 \ (mod\ 90)$ c) $x \equiv 14 \ (mod\ 30)$

d) $x \equiv 14 \ (mod\ 90)$ e) None of the Above

10) Consider using induction to prove that the following statement is true for all non-negative integers n: $\sum_{i=1}^{3n} i^2 = \frac{n(3n+1)(6n+1)}{2}$. Which of the following would be a valid expression of the inductive step?

a) Prove for n = k+1 that $\sum_{i=1}^{3k+1} i^2 = \frac{(k+1)(3k+2)(6k+2)}{2}$.

b) Prove for n = k+1 that $\sum_{i=1}^{3k+3} i^2 = \frac{(k+1)(3k+2)(6k+2)}{2}$.

c) Prove for n = k+1 that $\sum_{i=1}^{3k+3} i^2 = \frac{(k+1)(3k+4)(6k+4)}{2}$.

d) Prove for n = k+1 that $\sum_{i=1}^{3k+3} i^2 = \frac{(k+1)(3k+4)(6k+7)}{2}$.

e) None of the Above

11) Consider proving an inductive step where you must show that $5 \mid (3^{2k+2} + 4^{k+2})$, for an arbitrary non-negative integer k. (The inductive hypothesis is assuming that $5 \mid (3^{2k} + 4^{k+1})$ for an arbitrary non-negative integer k.) Which of the following represents the best first step in decomposing the expression $3^{2k+2} + 4^{k+2}$?

a) $3^{2k+2} + 4^{k+2} = 9 \times 3^{2k} + 16 \times 4^{k}$

b) $3^{2k+2} + 4^{k+2} = 9 \times 3^{2k} + 4 \times 4^{k+1}$

c) $3^{2k+2} + 4^{k+2} = 3 \times 3^{2k+1} + 4 \times 4^{k+1}$

d) $3^{2k+2} + 4^{k+2} = 3 \times 3^{2k+1} + 16 \times 4^{k}$

e) None of the Above

12) Consider the following inductive proof where the following statement is proved for all positive integers n: $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}$. How many of the missing signs (denoted by a question mark) are equal signs (instead of ≤ or < signs)?

Base case: n = 1. LHS = $\sum_{i=1}^{1} \frac{1}{i^2} = 1$, RHS =$2 - \frac{1}{1} = 1$, inequality holds.

Assume for an arbitrary positive integer n = k that $\sum_{i=1}^{k} \frac{1}{i^2} \leq 2 - \frac{1}{k}$.

Prove for n = k+1 that $\sum_{i=1}^{k+1} \frac{1}{i^2} \leq 2 - \frac{1}{k+1}$.

$$\begin{aligned}
\sum_{i=1}^{k+1} \frac{1}{i^2} \ &? \ \left(\sum_{i=1}^{k} \frac{1}{i^2}\right) + \frac{1}{(k+1)^2} \\
&? \ 2 - \frac{1}{k} + \frac{1}{(k+1)^2} \\
&? \ 2 - \frac{(k+1)^2 - k}{k(k+1)^2} \\
&? \ 2 - \frac{k^2 + 2k + 1 - k}{k(k+1)^2} \\
&? \ 2 - \frac{k^2 + k + 1}{k(k+1)^2} \\
&? \ 2 - \frac{k^2 + k}{k(k+1)^2} \\
&? \ 2 - \frac{k(k+1)}{k(k+1)^2} \\
&? \ 2 - \frac{1}{(k+1)}
\end{aligned}$$

a) 5 b) 6 c) 7 d) 8 e) None of the Above

13) How many different strings of 6 letters contain either the letter 'a' or the letter 'b'? For the purposes of this problem, there are 26 possible letters.

a) $\frac{26!}{6!20!}$ b) $\frac{26!}{20!}$ c) 2^6 d) $2 \times 25^6 - 24^6$ e) None of the Above

14) How many permutations of the letters LASERPRINTER do not contain consecutive vowels (A, E, I, O, U)?

a) 12! b) $2 \times 8! \times \binom{9}{4}$ c) $\binom{9}{4}\binom{8}{3}$ d) $\binom{9}{4}\frac{8!}{3!}$ e) None of the Above

15) Yelena is buying sodas for a holiday party. She wants to buy 14 six-packs and she has the following six-packs to choose from: Coke, Diet Coke, Sprite, Fanta and Mountain Dew. She must buy at least 4 six-packs of Coke and there are only 3 six-packs of Mountain Dew left at the store. In how many ways can she buy the six-packs? (Two combinations are different if there is a different number of six-packs of at least one type of soda between the two combinations.)

a) $\binom{14}{4} - \binom{11}{4}$ b) $\binom{14}{4} - \binom{10}{4}$ c) $\binom{14}{4}$ d) $\binom{18}{4}$ e) None of the Above

16) A fair standard six-sided die is rolled three times. What is the probability that all three rolls landed with a different number of dots showing?

a) $\frac{1}{6}$ b) $\frac{4}{9}$ c) $\frac{1}{2}$ d) $\frac{5}{9}$ e) None of the Above

17) In a video game, Sean gets assigned challenge A $\frac{3}{5}$ of the time and he gets assigned challenge B $\frac{2}{5}$ of the time. His chance of winning challenge A is 20% and his chance of winning challenge B is 60%. Yesterday, when Sean played the game once, he won his challenge. What is the probability that he was assigned challenge A?

a) $\frac{1}{4}$ b) $\frac{1}{3}$ c) $\frac{1}{2}$ d) $\frac{2}{3}$ e) None of the Above

18) Here is a description of a discrete random variable X:

$$X = \begin{cases} 2, \text{ with probability } \frac{1}{6} \\ 5, \text{ with probability } \frac{1}{3} \\ 6, \text{ with probability } \frac{1}{2} \end{cases}$$

What is the variance of X?

a) 1 b) 2 c) 3 d) 5 e) None of the Above

19) How many reflexive relations can be defined on a set A where |A| = 5?

a) 2^5 b) 2^{10} c) 2^{15} d) 2^{25} e) None of the Above

20) Let R be the following relation defined over the set of positive integers:

$$R = \{(x, y) \mid \left\lfloor \frac{x}{y} \right\rfloor = 1\}$$

Note: $\lfloor a \rfloor$ represents the largest integer less than or equal to a. Which of the following properties does R have?

a) reflexive b) symmetric c) transitive d) irreflexive e) None of the Above

21) Let R be a relation defined on the set A = {1, 2, 3, 4, 5}.

In particular, let R = {(1, 1), (2, 3), (2, 5), (3, 2), (3, 4)}.

which of the following is the symmetric closure of R?

a) {(1, 1), (2, 3), (2,5), (3, 2), (3, 4), (2, 2), (3, 3), (4, 4), (5, 5)}

b) {(1, 1), (2, 3), (2,5), (3, 2), (3, 4), (4, 4), (4, 3), (5, 2), (5, 5)}

c) {(1, 1), (2, 2), (2, 3), (2, 5), (3, 2), (3, 3), (3, 4)}

d) {(1, 1), (2, 3), (2, 5), (3, 2), (3, 4), (4, 3), (5, 2)}

e) None of the Above

22) Let f(x) = 2x + 3 and g(x) = $4x^2$ + 5x + 1. Which of the following is g(f(x))?

a) $8x^2 + 10x + 5$ b) $16x^2 + 34x + 52$ c) $16x^2 + 58x + 52$

d) $8x^3 + 22x^2 + 17x + 3$ e) None of the Above

23) Let A and B be sets such that |A| = 5 and |B| = 7. How many injective functions can be defined with a domain of A and a co-domain of B?

a) 35 b) $\frac{7!}{2}$ c) 7! d) 7^5 e) None of the Above

24) Let A, B and C be finite sets such that f: A→B is an injective function and g: B→C is a surjective function. Which of the following properties **must be true** about the composition function $g^{o}f$? ***(Note: There is no choice E for this particular question.)***

a) injective b) surjective c) injective and surjective d) neither injective nor surjective

25) In what tasty beverage does the local joint Axum Coffee specialize?

a) coffee b) slurpees c) vodka tonics d) orange juice e) None of the Above

<u>**Fall 2016 COT 3100 Section 1 Final Exam - Free Response**</u>

Last Name: ___________________, First Name : _____________________

1) (8 pts) Sets

The Inclusion-Exclusion Principle for two sets, A and B, states: $|A \cup B| = |A| + |B| - |A \cap B|$.

Using this as a given fact, prove the Inclusion-Exclusion Principle for three sets.

(a) (1 pt) State the Inclusion-Exclusion Principle for 3 sets, A, B and C:

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(b) (7 pts) Do the proof of the Inclusion-Exclusion Principle for three sets using the result for two sets and other necessary set laws.

2) (6 pts) Induction

Using mathematical induction on n, prove the following for all non-negative integers n:

$$11 \mid (27 \times 23^n + 17 \times 10^{2n})$$

3) (6 pts) Probability

Box A has 2 shiny pennies and 8 dull pennies. Box B has 6 shiny pennies and 9 dull pennies. First, one of the two boxes is chosen, with Box A having a 30% probability of being chosen and Box B having a 70% probability of being chosen. Then, a penny is randomly selected from the chosen box. Given that the selected penny is shiny, what is the probability that it came from Box A? Please express your answer as a fraction in lowest terms (integers in numerator and denominator with a greatest common divisor of 1).

4) (5 pts) Relations

The transitive closure of a relation R is defined as $t(R) = R \cup R^2 \cup R^3 \cup \ldots$ In reality, we don't need to union an infinite number of terms if the relation R is on a finite set. Instead, if the relation is on a set of n items, we can just take the union of the first n terms of what's written above to obtain the transitive closure of a relation. Let A = {1, 2, 3, 4, 5, 6, 7, 8}. Let R be the following relation on A: { (1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 7), (7, 8), (8, 1) } How many terms are in t(R) for this particular relation? Please provide proof of your answer.