Spring 2020 COT 3100 Final Exam

1) (12 pts) For one mile, Percy travels at an average speed of one mile an hour. For the subsequent mile he travels at an average speed of two miles an hour. He continues this upward trend, for each subsequent mile he increases his average speed from the previous mile by one mile per hour. Let k be the smallest integer such that after traveling for k miles, Percy's average speed from the beginning of the trip is greater than 2 miles an hour. What is k? Express Percy's average speed for this value of k as a fraction in lowest terms.

2) (12 pts) Define a function f as follows: f(1) = 2. For all integers n > 1, $f(n) = (f(n-1))^{2^n}$.

What is the value of $log_{65536}(log_{65536}(f(4))$?

Express your answer in the form $2^a - 2^b$, where both a and b are integers. Note that $65536 = 2^{16}$.

3) (12 pts) Consider the two following logical expressions:

(1) $(p \rightarrow q) \rightarrow r$
(2) $p \rightarrow (q \rightarrow r)$

Let the first expression be X and the second expression be Y. With proof, determine which of the following (if any) is a tautology? You may use any valid proof technique taught in class for your justification. (Note: it's possible that none of these two assertions are true, or exactly one is true, or both are true. Please clearly explain for each separately whether or not it is true, with proof.)

(a) $X \rightarrow Y$
(b) $Y \rightarrow X$

4) (a) (8 pts) Let A and B be sets such that $|A \cup B| = 10$, $|A \cap B| = 4$ and $|A| = 6$. What is the cardinality of the set $\wp(A \cup B) - \wp(A) - \wp(B)$? Recall that $\wp(X)$, represents the power set of the set X. Please simplify your answer to an integer.

(b) (4 pts) Disprove the following claim for sets A, B, C and D with a single counter-example: if A, B, C and D are sets such that A x B = C x D, then either A = C or B = D.

5) (a) (6 pts) How many integers in between 1 and 10^6 have an even number of divisors? Show work proving your answer. Express your answer in prime factorized form.

(b) (6 pts) With proof, determine all integer solutions to the following equation:

$$1935x + 2322y = 177$$

6) (12 pts) Let a be a non-negative integer. Use induction on n to prove that for all integers $n \geq a$:

$$\sum_{i=a}^{n} \binom{i}{a} = \binom{n+1}{a+1}$$

(Note: You may use well-known identities about combinations in your proof, but state when you use the identity and where the identity comes from.)

7) (a) (2 pts) How many permutations are there of the word QUARANTINE?

(b) (4 pts) How many of the permutations of QUARANTINE do NOT the same letter appearing consecutively? (Thus, any permutation with the substring "AA" should not be counted and any permutation with the substring "NN" should not be counted.)

(c) (6 pts) A mountain permutation is defined as one where the first portion of the permutation has letters in strictly increasing alphabetical order, and the second portion with the letters in strictly decreasing alphabetical order. One mountain permutations of the letters in QUARANTINE is AINTURQNEA. Notice that only a set of letters with a unique "maximum" alphabetic letter have mountain permutations. For this permutation, A < I < N < T < U, and U > R > Q > N > E > A. How many mountain permutations are there of the letters in the word QUARANTINE?

8) (12 pts) Consider all possible non-negative integer solutions to the equation a + b + c + d = 42. One of these solutions is chosen at random. What is the probability that all four values of a, b, c and d in this randomly selected solution are even? Express your final result as a fraction in lowest terms.

9) (12 pts) Define the relation R over integers as follows:

R = { (a, b) | |a - b| < 5 }

With proof, determine if R is (a) reflexive, (b) irreflexive, (c) symmetric, (d) anti-symmetric and (e) transitive.

10) (12 pts) Let $P(x) = x^5 + ax^4 + bx^3 + cx^2 + dx + e$. P(4) = P(5) = P(6) = P(7) = P(8) = 0. What is the value of a - b + c - d + e? Numerically the answer is somewhat large, but it can be represented easily as a sum of terms, where the terms are a product of 1 or more small integers. You may leave your answer in this form, instead of calculating it out. (Note: This is a challenging question. One little hint is that if r and s are roots of a polynomial, then (x - r)(x - s) is a factor of that polynomial.)

11) (5 pts) Former UCF Knights, Shaquem and Shaquill Griffin will give a virtual commencement address for the 2020 Spring graduating class of UCF. Both moved on from playing football for the Knights to the Seattle Seahawks. In what city do the brothers now live?