Spring 2014 COT 3100 Final Exam

Last Name: ___________________ , First Name : _____________________

1) (15 pts) PRF (Induction)

Let M = $\begin{bmatrix} a & 0 \\ 1 & 1 \end{bmatrix}$, where a is an arbitrary non-zero real number. Use induction on n to prove that $M^n = \begin{bmatrix} a^n & 0 \\ \sum_{i=0}^{n-1} a^i & 1 \end{bmatrix}$ for all positive integers n.

2) (10 pts) PRF (Logic)

There are three people, Adam(A), Belinda(B) and Celine(C), in a room. Each has a hat on and is holding a pet. The colors of the three hats are red(R), orange(O) and purple(P) and the pets are a dog(D), a guinea pig(G) and a hamster(H). Consider the following given information:

1) The person wearing the red hat is not holding the guinea pig.
2) Celine is not holding the hampster.
3) The person wearing the orange hat is holding the hampster.
4) Belinda is holding the dog.

Using any technique, determine which hats each of the individuals is wearing and which pets each of the individuals is holding.

Person	Hat Color	Pet
Adam		
Belinda		
Celine		

3) (10 pts) PRF (Sets)

Prove or disprove the following assertion about finite sets A and B:

$$P(A) \cup P(B) \subseteq P(A \cup B)$$

Recall that *P(A)* is simply the set of all subsets of A.

4) (15 pts) NTH (Number Theory)

This question will concern all positive integer solutions to the equation $a^2 + b^2 = c^2$ where $\gcd(a, b) = 1$, $\gcd(a, c) = 1$ and $\gcd(b, c) = 1$.

a)(5 pts) Let n be an even integer. Prove that $n^2 \equiv 0 \pmod 4$.

b)(5 pts) Let n be an odd integer. Prove that $n^2 \equiv 1 \pmod 4$.

c)(5 pts) Prove that for all solutions of $a^2 + b^2 = c^2$ where $\gcd(a, b) = 1$, $\gcd(a, c) = 1$ and $\gcd(b, c) = 1$, that c must be odd and that exactly 1 of a and b must be odd.

5) (15 pts) CTG (Counting)

A class has 10 freshmen, 20 sophomores, 30 juniors and 40 seniors. Note: For the purposes of this question we'll define upperclassmen to be juniors and seniors. (Note: Leave your answer in products, factorials, powers and combinations.)

a) (3 pts) If the class must form a group of 4 representatives, with one from each year in school, how many different sets of representatives can the class have?

b) (5 pts) How many committees can be formed with 3 non-upperclassmen and 4 upperclassmen from the 100 students in the class?

c) (7 pts) We define the composition of a committee to be the ordered quadruplet (a, b, c, d), where a represents the number of freshmen in the committee, b the number of sophomores, c the number of juniors and d the number of seniors. How many different compositions can there be of a committee of 15 students from the class?

6) (15 pts) PRB (Probability)

Each part will concern the situation of rolling three fair six-sided dice. Label these dice A, B and C. Note: Please simplify each of these answers as fractions in lowest terms.

a) (5 pts) What is the probability that the sum of the three dice is 5?

b) (5 pts) Given that die A shows a 3, what is the probability that the sum of dice A and B equals what is showing on die C?

c) (5 pts) Given that one of the three dice shows a 1 (note that we don't know which one of A, B and C), what is the probability that all three sum to three?

7) (10 pts) PRF (Functions)

Let $f(x) = 3x^2 - 12x + 7$ with a domain of $x \in (-\infty, 2]$. Determine $f^{-1}(x)$ and its domain and range.

8) (10 pts) PRF (Relations)

Recall that a bitwise and (&) between two integers is the result of taking the and of each pair of corresponding bits in the two operands. For example 11 & 13 is 9, because 11 is 1011 in binary, 13 is 1101 in binary, and both of these numbers have 1s in the first and last locations, thus their bitwise and is 1001, which is equal to 9. Define the following relation over **non-zero** 32-bit integers:

Let R = { (a, b) | a and b are 32-bit **non-zero** integers with (a & b) equal to 0 }

Determine, with proof, whether or not R is (a) irreflexive, (b) symmetric and (c) transitive.