COT 3100 Final Exam - Part A (Relations, Functions) - 25 pts (12/3/2024)

Last Name: ___________________ , First Name: _____________________

1) (5 pts) How many anti-symmetric relations over the set A = {1, 2, 3, 4, 5, 6, 7, 8} contain the ordered pairs (1, 1), (2, 3), (6, 4), (7, 7), and (8, 3)? Please leave your answer as a product of terms, each of which is written in exponential form.

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2) (6 pts) Let R = { (a, b) | $\exists c \in Z^+ \text{ such that } ab = c^2$} be a relation defined over the positive integers. Prove that R is transitive. (You may use the fact that if a rational number squared is an integer, then that rational number itself is also an integer.)

3) (8 pts) Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be functions where f is a surjective function and g is an injective function.

(a) Prove or disprove: the composition function, $g \circ f$ is a surjective function.
(b) Prove or disprove: the composition function, $g \circ f$ is an injective function.

For each part, clearly state whether the assertion is true or not, followed by a proper justification.

4) (6 pts) Let r, s and t be roots of the cubic polynomial $f(x) = 3x^3 - 5x^2 + 7x - 9$. What is the cubic equation, with **leading coefficient 9,** with the roots $\frac{1}{r}$, $\frac{1}{s}$, and $\frac{1}{t}$?

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COT 3100 Final Exam - Part B - 100 pts (12/3/2024)

Last Name: ___________________ , First Name: _____________________

1) (8 pts) Please complete the truth table below.

p	q	r	$p \to q$	$\bar{q} \to r$	$(p \to q) \to (\bar{q} \to r)$
F	F	F			
F	F	T			
F	T	F			
F	T	T			
T	F	F			
T	F	T			
T	T	F			
T	T	T			

2) (10 pts) Let A, B and C be finite sets of integers. Prove the following assertion **via direct proof:**

$$\text{If } C \subseteq A \cap B, \text{ then } A - B \subseteq A - C.$$

3) (10 pts) Find all ordered pairs of integers, (x, y), that satisfy the equation $275x + 64y = 1$. Put a box around your final answer.

4) (10 pts) **Prove via induction on n**, that for all non-negative integers n, $7^n - (-1)^n 3^{2n}$ is divisible by 16.

5) (10 pts) **Prove via induction on n,** that for all integers n ≥ 2, $\sum_{i=2}^{n} \frac{i}{i^2-1} = \frac{1}{2}(H_{n-1} + H_{n+1}) - \frac{3}{4}$, where H_n represents the n^{th} Harmonic number. (For reference, $H_n = \sum_{i=1}^{n} \frac{1}{i}$.) As a courtesy, completing this problem requires a partial fraction decomposition, which some students might not have learned. To that end, please feel free to use the fact that for all positive real numbers a,

$$\frac{a+1}{a^2+2a} = \frac{1}{2a} + \frac{1}{2(a+2)}$$

6) (10 pts) Cassie reads her digital clock and notices that the digits from left to right are in strictly increasing order. (Examples of what she could have read are 2:37 and 12:59.) Assuming that Cassie read her clock in between 12:00 PM and 11:59 PM, inclusive, at how many different times/minutes could she have read her clock? Another way to ask the question is: for how many different valid settings of the 3 or 4 digits on a digital clock are all the digits in strictly increasing order? Note: This problem requires some casework. There's no elegant solution that I know of. I'll be impressed if someone finds one!

7) (10 pts) Consider a variant of the Monty Hall problem with 10 doors, of which 3 have cars behind them and 7 have goats. Once the contestant picks a door, two other doors are revealed to have goats behind them.

(a) (2 pts) What is your probability of winning a car if you stay with the door you originally chose?

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(b) (8 pts) What is your probability of winning a car if you randomly switch to one of the other unrevealed doors?

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8) (10 pts) Let X be a continuous random variable defined by the probability density function below:

$$p(x) = cx^2, 0 \leq x \leq 3$$
$$p(x) = 0, otherwise$$

(a) (3 pts) Find the value of c. Please express your answer as a fraction in lowest terms.

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(b) (3 pts) Determine the probability that $1 \leq X \leq 2$. Please express your answer as a fraction in lowest terms.

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(c) (4 pts) Determine the value of E(X). Please express your answer as a fraction in lowest terms.

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9) (10 pts) Ava lives D miles away from work. When she drives to work at an average speed of 45 miles per hour, she arrives to work exactly on time. If she were to leave at the exact same time, but drive an average of 40 miles per hour, she would arrive at work two and a half minutes late.

(a) (6 pts) How many **miles** away from work does she live?

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(b) (4 pts) If she were to leave at the exact same time but drive at an average speed of 50 miles per hour, how many minutes early would she arrive at work?

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10) (10 pts) Elijah invests $1000 dollars each year in an account that accrues 6% interest. He puts his investment in once a year at the same time. Thus, if he's made n investments (over a total duration of n – 1 years), we can think of the current amount of money he has invested as investing $1000 for n – 1 years, investing another $1000 for n – 2 years, investing another $1000 for n – 3 years, … and investing the last $1000 for 0 years.

(a) (1 pt) Write down the value of $1000 invested for precisely k years (at 6% annual growth) in terms of k.

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(b) (2 pts) Using the expression from (a), noting that the scenario described can be modeled as n separate investments, write down a summation equal to the total current value of Elijah's investment. (Hint: Your sum should go from k = 0 to k = n – 1.)

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(c) (3 pts) Using the geometric sum formula, determine the value of the sum above in terms of n.

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(d) (4 pts) Using the result from part (c), determine the minimum number of consecutive annual $1000 investments Elijah must make so that the total value of his investment exceeds $100,000. In particular, for full credit, you must solve for this minimum value n and get an answer of the form $\log_a b$, where a is a real number greater than 1 and b is a positive integer. (Since n must be integral, the answer to the question will simply be the smallest integer greater than this expression.) Put a box around your answer. (You'll need a calculator for this part.)

11) (2 pts) Moana 2 recently opened in theaters across the US.
How many Moana movies have been made so far? ________________

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