Fall 2017 COT 3100 Final Exam Part A

Last Name: ___________________________ , First Name: ____________________

1) (8 pts) It takes Bob six days to paint their townhouse, and it takes Carol ten days to paint their townhouse. For the purposes of this problem, assume a day is equal to 8 hours. (No one wants to paint "all" day!!!) They invite Seema over to help them paint their townhouse and complete the task in one day (eight hours). If Seema were painting the townhouse on her own, how many hours total would it have taken her to complete the task? Express your answer as a fraction in lowest terms.

2) (8 pts) Use an efficient technique via hand to calculate the remainder when 4^{20} is divided by 47. (You may follow the book's technique or come up with an ad hoc technique that follows valid mod rules to minimize work by hand.) Please show each step.

3) (15 pts) Find all ordered pairs of integers, (x, y), which satisfy the equation, $51x + 192y = 39$.

4) (12 pts) Prove, for all positive integers n, using induction on n that

$$\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}^n = \frac{1}{4}\begin{bmatrix} 5^n + 3 & 1 - 5^n \\ 3(1 - 5^n) & 3(5^n) + 1 \end{bmatrix}.$$

5) (18 pts) Janice is trying to buy ornaments for her Christmas tree. The store she went to has 7 types of ornaments available. She'd like to buy a minimum of 5 ornaments and a maximum of 30 ornaments. In addition, she wants to buy at least two cupcake ornament. There are only 10 ball ornaments available and 15 candy cane ornaments available. The store has more than 30 of all of the other five types of ornaments. In how many different ways can she buy ornaments to satisfy all of her restrictions as well as the availability of ornaments at the store she went to? (Since this question is worth a lot of points, please carefully show and explain your work so that partial credit can be awarded.)

6) (12 pts) Bob is considering buying raffle tickets for four prizes. The table below summarizes the current information about each of the prizes, right before Bob buys raffle tickets:

Prize	Value	Price for 1 Raffle Ticket	Tickets Already Purchased
Wine	$800	$5	88
Beach	$500	$10	19
Vacation	$1000	$20	17
Dinner	$250	$15	6

Bob has decided that he's going to spend exactly $60 on raffle tickets and that he's only going to buy raffle tickets for a single prize. Also, assume the raffle closes right after Bob makes his purchase. If Bob wants to maximize his expected winnings, which item should he buy raffle tickets for? (Note: No credit is given for the correct answer. All of the credit is given for the justification of the correct answer.) For example, if Bob chooses Beach, then he purchases 6 raffle tickets for it, since each costs $10. After his purchase, there are 25 raffle tickets for the item, of which Bob has 6, so his chance of winning the Beach prize would be 24%. If he chooses the item which maximizes his expected winnings, what are his expected net winnings? (Note: The net winnings are just his expected prize value minus how much he spent on the raffle tickets to obtain that expected prize value.)

7) (12 pts) Let R, S and T be relations such that $R \subseteq A \times B$, $S \subseteq B \times C$ and $T \subseteq B \times C$.

(a) Prove or disprove: $(S \cap T) \circ R \subseteq S \circ R \cap T \circ R$
(b) Prove or disprove: $S \circ R \cap T \circ R \subseteq (S \cap T) \circ R$

8) (14 pts) Let the functions f and g be as follows: f: A → B and g: B → C, with $g \circ f$ being an injective function.

(a) Prove or disprove: f must be an injective function.
(b) Prove or disprove: g must be an injective function.

9) (1 pt) What color is the inside of a pink grapefruit? ________

Scratch Page - Please clearly mark any work on this page you would like graded.

Fall 2017 COT 3100 Final Exam Part B

Last Name: ___________________________ , First Name: ____________________

1) (15 pts) Using induction on n, prove for all integers $n \geq m$, where m is a fixed positive integer, and j is another fixed non-negative integer such that $j < m$, that

$$\sum_{i=m}^{n} \binom{i}{j} = \binom{n+1}{j+1} - \binom{m}{j+1}$$

2) (10 pts) Consider the following algorithm to determine if an integer is divisible by 11 or not:

Go through the digits from least significant to most significant. Keep a counter that you start at zero and alternately add and subtract each digit from the counter. At the end, determine if the counter is divisible by 11. If it is, the original number is divisible by 11. If it isn't, the original number isn't divisible by 11.

Here is the algorithm being implemented on the number 76180258:

1. Add 8, counter is 8
2. Subtract 5, counter is 3
3. Add 2, counter is 5
4. Subtract 0, counter is 5
5. Add 8, counter is 13
6. Subtract 1, counter is 12.
7. Add 6, counter is at 18.
8. Subtract 7, counter is at 11.

Since 11 is divisible by 11, 76180258 is divisible by 11.

Prove that this algorithm works. Namely, show that the calculation prescribed is equivalent to the value of the original number mod 11. (A natural consequence of proving this is the correctness of the algorithm.)