UNIVERSITY OF CENTRAL FLORIDA

Department of Mathematics/School of Electrical Engineering and Computer Science

MAP/COP 3930H Fall 2009

Mathematical Modeling & Scientific Computing

Home Work # 4

(Due October 29, 2009)

Instructions:

1. Use paper size (8.5 x 11) and write on one side only.
2. Show all relevant steps in arriving at your answer (or conclusion).
3. Clearly designate your answers (circle, underline, etc.)

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1. Solve the initial value Problem:

$$xy' + (x-2)y = 3x^3e^{-x}, \quad y(1) = 0$$

2. A thermometer is taken from an inside room to the outside where the air temperature is 5°F. After 1 minute the thermometer reads 55°F, and after 5 minutes the reading is 30°F. What was the initial temperature of the room?

3. The population $P(t)$ at time t in a suburb of a large city is governed by the initial-value problem

$$\frac{dP}{dt} = P(10^{-1} - 10^{-7}P), \quad P(0) = 5000,$$

where t is measured in months. What is the limiting value of the population? At what time will the population be equal to one-half of this limiting value?

4. *Controlling a population*—The fish and game department in a certain state is planning to issue hunting permits to control the deer population (one deer per permit). It is known that if the deer population falls below a certain level m, the deer will become extinct. It is also known that if the deer population rises above the carrying capacity M, the population will decrease back to M through disease and malnutrition.

   a. Discuss the reasonableness of the following model for the growth rate of the deer population as a function of time:

   $$\frac{dP}{dt} = rP(M-P)(P-m)$$

   where P is the population of the deer and r is a positive constant of proportionality. Include a phase line.

   b. Explain how this model differs from the logistic model $dP/dt = rP(M-P)$. Is it better or worse than the logistic model?

5. Consider launching a satellite into orbit using a single-stage rocket. The rocket is continuously losing mass, which is being propelled away from it at significant speeds. We are interested in predicting the maximum speed the rocket can attain.[2]

a. Assume the rocket of mass m is moving with speed v. In a small increment of time Δt it loses a small mass Δm_p, which leaves the rocket with speed u in a direction opposite to v. Here, Δm_p is the small propellant mass. The resulting speed of the rocket is $v + \Delta v$. Neglect all external forces (gravity, atmospheric drag, etc.) and assume Newton's second law of motion:

$$\text{force} = \frac{d}{dt}(\text{momentum of system})$$

where momentum is mass times velocity. Derive the model

$$\frac{dv}{dt} = \left(\frac{-c}{m}\right)\frac{dm}{dt}$$

where $c = u + v$ is the relative exhaust speed (the speed of the burnt gases relative to the rocket).

b. Assume that initially, at time $t = 0$, the velocity $v = 0$ and the mass of the rocket is $m = M + P$, where P is the mass of the payload satellite and $M = \epsilon M + (1 - \epsilon)M$ $(0 < \epsilon < 1)$ is the initial fuel mass ϵM plus the mass $(1 - \epsilon)M$ of the rocket casings and instruments. Solve the model in part (a) to obtain the speed

$$v = -c \ln \frac{m}{M - P}$$

c. Show that when all the fuel is burned, the speed of the rocket is given by

$$v_f = -c \ln\left[1 - \frac{\epsilon}{1 - \beta}\right]$$

where $\beta = P/M$ is the ratio of the payload mass to the rocket mass.

d. Find v_f if $c = 3$ km/sec, $\epsilon = 0.8$, and $\beta = 1/100$. (These are typical values in satellite launchings.)

e. Suppose scientists plan to launch a satellite in circular orbit h km above the earth's surface. Assume that the gravitational pull toward the center of the earth is given by Newton's inverse square law of attraction:

$$\frac{\gamma m M_e}{(h + R_e)^2}$$

where γ is the universal gravitational constant, m is the mass of the satellite, M_e is the earth's mass, and R_e is the radius of the earth. Assume that this force must be balanced by the centrifugal force $mv^2/(h - R_e)$, where v is the speed of the satellite. What speed must be attained by a rocket to launch a satellite into an orbit 100 km above the earth's surface? From your computation in part (d), can a single-stage rocket launch a satellite into an orbit of that height?