Graphs: A Powerful Abstract Representation of Data

Definition

A graph is a collection of dots, called vertices, and connections between those dots, called edges. There are two categories of adjectives to describe different types of graphs:

unweighted vs. weighted
undirected vs. directed

In a weighted graph, each connection between vertices has an associated number, called an "edge weight". In an undirected graph, no such number is associated and by default, we typically assign 1.

In a directed graph, the order of the two vertices in a connection matters. Thus, in a directed graph, an edge from vertex a to vertex b does not imply an edge from vertex b to vertex a. In an undirected graph, no order is given to the two vertices that are connected, so if vertex a and b are connected via an edge, one can go from a to b, OR b to a.

A multi-graph allows for more than one edge between the same two vertices. These are relatively rare in contests, however, when they appear, they are rather tricky, because many algorithms that assume only one connection between any pair of vertices tends to fail when this assumption isn't true.

How to Store a Graph

The easiest way to store a graph is a two dimensional integer array of size *n* x *n*, where *n* is the number of vertices in the graph:

```
int[][] adjmat = new int[n][n];
```

Typically, adjmat[i][j] would store the edge weight for the edge from vertex i to vertex j. If it's an unweighted graph, we store 1 if the edge exists. If no such edge exists, we can either store a large integer or 0 and code accordingly. Alternatively, we can store null and make it an array of type Integer (in Java).

This storage method is great for when you are first learning about graphs. It's often inefficient though, especially for sparse graphs, where a vast majority of possible edges don't exist. (Imagine a graph with 10^5 vertices and $3x10^5$ edges!!!)

The best way to store a graph for contests is an array of lists:

```
ArrayList[] adjList = new ArrayList[n];
for (int i=0; i<n; i++)
    adjList[i] = new ArrayList<Integer>();
```

adjList[i] would be a list storing all vertices, vertex i is connected to, filled in later.

Graph Traversal - Depth First Search

The goal of a graph traversal is simply to mark all vertices that can be visited, following edges from a particular vertex.

The general "rule" used in searching a graph using a depth first search is to search down a path from a particular source vertex as far as you can go. When you can go to farther, "backtrack" to the last vertex from which a different path could have been taken. Continue in this fashion, attempting to go as deep as possible down each path until each node has been visited. Here is some code for DFS assuming the more efficient graph storage - it just marks

```
public static void dfs(ArrayList[] graph, boolean[] visited,
int v) {

    visited[v] = true;
    for (Integer next : ((ArrayList<Integer>)graph)[v])
        if (!visited[next])
            dfs(graph, visited, next);
}
```

The running time of DFS is O(V+E). To see this, note that each edge and vertex is visited at most twice. In order to get this efficiency, an adjacency list must be used. (An adjacency matrix can not be used to complete this algorithm that quickly.)

Graph Traversal - Breadth First Search

The idea in a breadth first search is opposite to a depth first search. Instead of searching down a single path until you can go no longer, you search all paths at a uniform depth from the source before moving onto deeper paths. Once again, we'll need to mark both edges and vertices based on what has been visited.

In essence, we only want to explore one "unit" away from a searched node before we move to a different node to search from. All in all, we will be adding nodes to the back of a queue to be ones to searched from in the future. Thus, we start with our source vertex in the queue and then whenever we dequeue an item, we enqueue all of its "new" neighbors who are all one unit away, so the queue stores all items of distance 1 from the source before all items who are distance 2 from the source, and so forth.

The code on the following page runs a bfs from vertex v, marking the distance to all vertices from v (on an unweighted graph). It returns an array with these distances and a -1 to indicate unreachable vertices.

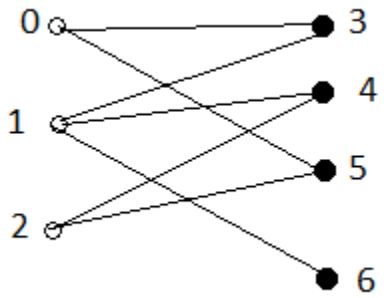
```
public static int[] bfs(ArrayList[] graph, int v) {

    int n = graph.length;
    int[] distance = new distance[n];
    Arrays.fill(distance, -1);
    visited[n] = true;
    ArrayDeque<Integer> q = new ArrayDeque<Integer>();
    q.offer(v);

    while (q.size() > 0) {
        int cur = q.poll();
        for (Integer next : ((ArrayList<Integer>)graph)[cur]) {
            if (distance[next] == -1) {
                distance[next] = distance[cur]+1;
                q.offer(next);
            }
        }
    }

    return distance;
}
```

Basically, we need two data structures: an array that keeps track of where we've been (and how far away those vertices are) AND the queue to keep track of the locations from which we still need to explore. When we dequeue, we basically just add all relevant (previously unvisited vertices) vertices to our queue. Note that as soon as we do this, we MUST mark these new vertices as visited. We can't wait until we dequeue them to do so, can you see why?

Two Coloring – Application of BFS

Coloring an unweighted undirected graph is the problem of assigning each vertex in a graph a color such that no two vertices connected via an edge are colored the same color. Trivially, a complete graph of n vertices (a graph that has an edge between all pairs of vertices), requires exactly n different colors to color.

A graph that is "two colorable" is a graph that requires only 2 colors to color in this fashion. Below is an example of a graph that is two-colorable, with the colors for each vertex shown.



A common problem is determining if a graph is two-colorable or not. This can be solved via adapting BFS. Notice how if a graph is two-colorable, we can split the vertices into two "sides", all the vertices of one color on the left, and all the vertices of the other color on the right. (In this picture, left = white, right = black.) When doing so, ALL edges must connect one vertex from the left with one vertex on the right, since no two vertices on the left are allowed to be connected to each other, and same for the right.

Thus, we can solve the two-color problem as follows:

1) Start a BFS at an unvisited node, say node 0. Go ahead and mark this as visited and as one of your two colors.

2) As you do the BFS, you attempt to enqueue all the neighbors of the current node. If the neighbor isn't visited yet, assign it the opposite color of the node you are coming from and add it to the queue. **The modification is that if you see that a neighbor IS visited, don't just skip it, verify that it's assigned color is opposite of the color of the node you are coming from. If it isn't, then the graph isn't two colorable.**

Thus, this problem can be solved in O(V+E) time with a minor addition of code to a regular BFS, in the case where a neighboring node to a vertex has previously been visited. This works because when colors are assigned to nodes the first time they are visited, then these colors are "forced" as the only possible viable colors.

If a BFS completes without contradiction but there are more unvisited nodes, that means the graph has separate components and you have to continue another BFS from the next unvisited node. At the end of the algorithm, either you find out that a graph is NOT two-colorable, or you have determined a valid two-coloring of the graph.

Simple Cycle Detection in a Specific Directed Graph

Consider a special directed graph where each node has out-degree one. This means that each node "points to" another node. (Imagine a situation where each sign on campus tells you to go to another sign on campus!) Clearly this sort of graph must have some cycles, because you can start at any node and follow the edges forever. To detect these cycles, do the following:

a) Start at an unvisited node.

b) Follow the only edge out of that node to where it goes, and as you do so, add each vertex into a stack. For example, we might do something like : 1 → 3 → 9 → 5 →.

c) As you are doing this, keep a set of items that are in the stack, so that you can detect the first time you hit a repeat: 1 → 3 → 9 → 5 → 6 → 9.

d) When you hit the repeat, start popping items off the stack until you get to the first instance of the repeated item in the stack: pop 6, pop 5, pop 9 → stop. Thus, the cycle is 9 → 6 → 5. It also follows that 1, and 3 aren't in a cycle but just lead into the 9→6→5 cycle. Repeat all the steps above for any reoccurring unvisited nodes.

Note that what may happen is that you may start at a new unvisited node that leads into previously discovered nodes. For example, imagine that 2→1 and your very next search is from vertex 2. When you see that 1 is visited, you know what will happen…2 goes to 1 goes to 3 goes into the cycle…Thus, you stop your search and recognize that 2 is also not in a cycle.

Topological Sort

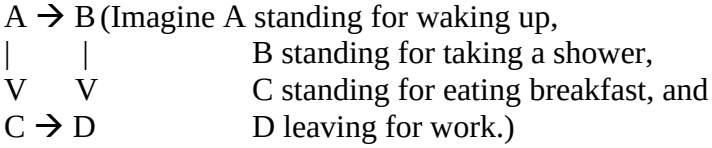
The goal of a topological sort is given a list of items with dependencies, (ie. item 5 must be completed before item 3, etc.) to produce an ordering of the items that satisfies the given constraints. In order for the problem to be solvable, there can not be a cyclic set of constraints. (We can't have that item 5 must be completed before item 3, item 3 must be completed before item 7, and item 7 must be completed before item 5, since that would be an impossible set of constraints to satisfy.)

We can model a situation like this using a directed acyclic graph. Given a set of items and constraints, we create the corresponding graph as follows:

1) Each item corresponds to a vertex in the graph.
2) For each constraint where item a must finish before item b, place a directed edge in the graph starting from the vertex for item a to the vertex for item b.

This graph is directed because each edge specifically starts from one vertex and goes to another. Given the fact that the constraints must be acyclic, the resulting graph will be as well.

Here is a simple situation:



(Imagine A standing for waking up,
B standing for taking a shower,
C standing for eating breakfast, and
D leaving for work.)

Here a topological sort would label A with 1, B and C with 2 and 3, and D with 4.

Let's consider the following subset of CS classes and a list of prerequisites:

CS classes: COP 3223, COP 3502, COP 3330, COT 3100, COP 3503, CDA 3103, COT 3960 (Foundation Exam), COP 3402, and COT 4210.

Here are a set of prerequisites:

COP 3223 must be taken before COP 3330 and COP 3502.
COP 3330 must be taken before COP 3503.
COP 3502 must be taken before COT 3960, COP 3503, CDA 3103.
COT 3100 must be taken before COT 3960.
COT 3960 must be taken before COP 3402 and COT 4210.
COP 3503 must be taken before COT 4210.

A goal of a topological sort then is to find an ordering of these classes that you can take.

Topological Sort – Iterative Version

Just as there is always a vertex in a directed acyclic graph (DAG) that has no outgoing edges, there must ALSO be a vertex in a DAG that has no incoming edges. This vertex corresponds to one that is safe to put in the *front* of the topological sort, since it has no prerequisites.

Thus, the algorithm is as follows for a graph, G, with n vertices:

1. Initialize TOP to be an empty list

2. While TOP has fewer than n items:

    a. Find a vertex v that is not in TOP that has an in degree of 0.
    b. Add v to TOP.
    c. Remove all edges in G from v.

In implementing the algorithm, store separately, the in degrees of each vertex. Every time you remove an edge in step 2c, update the corresponding in degree count.

For a sparse graph, in order to implement step a, use a Priority Queue. Keep in mind that every time you change an in-degree of a vertex, you have to delete the item from the priority queue and re-insert it.

Let's apply this algorithm to our class constraints:

COP 3223 must be taken before COP 3330 and COP 3502.
COP 3330 must be taken before COP 3503.
COP 3502 must be taken before COT 3960, COP 3503, CDA 3103.
COT 3100 must be taken before COT 3960.
COT 3960 must be taken before COP 3402 and COT 4210.
COP 3503 must be taken before COT 4210.

COP 3223 goes first, since it has no pre-requisites.
COP 3502 goes next, since it has no pre-requisites left.
COP 3330 goes next, since it has no pre-requisites left.
COT 3100 goes next, since it has no pre-requisites left.
COP 3503 goes next, since it has no pre-requisites left.
COT 3960 goes next, since it has no prerequisites left.
COP 3402 goes next, followed by
COT 4210.