Competitive Training Camp
Lecture 6

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Game Theory

- The Basics
- The Game of Nim
- Composite Games - Grundy Number

The Basics: Main Idea

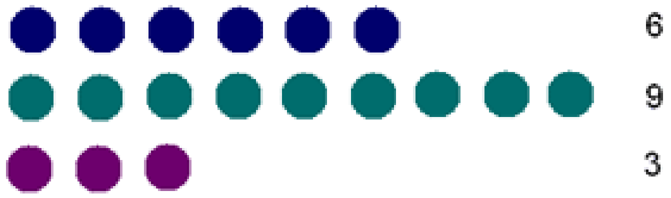
- All terminal positions are losing.
- If a player is able to move to a losing position then he is in a winning position.
- If a player is able to move only to the winning positions then he is in a losing position.

The Basics: Implementation

```
boolean isWinning(position pos) {
    moves[] // possible positions to which
            // I can move from the position pos
    for (all x in moves)
        if (!isWinning(x)) return true;
    return false;
}
```

The Game of Nim: Problem Statement

- Problem Statement: There are n piles of coins. When it is a player’s turn he chooses one pile and takes at least one coin from it. If someone is unable to move he **loses** (so the one who removes the last coin is the **winner**).



The Game of Nim: Main Idea

- Let $n_1, \ldots, n_k$ be the pile sizes.
- Crucial observation: It is a **losing** position for the current player

if and only if

$n_1 \text{ xor } \ldots \text{ xor } n_k = 0$.

WHY?

The Game of Nim: Main Idea

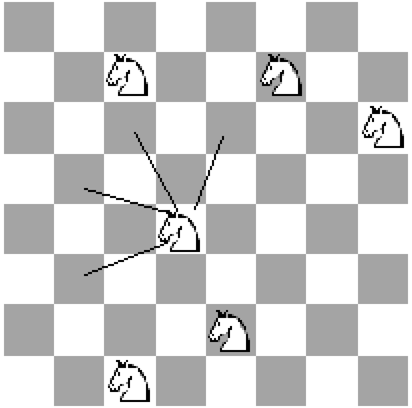
- **From the losing positions we can move only to the winning ones:**
  - if xor of the sizes of the piles is 0 then it will be changed after our move (at least one 1 will be changed to 0, so in that column will be odd number of 1s).

The Game of Nim: Main Idea

- **From the losing positions we can move only to the winning ones:**
  - if xor of the sizes of the piles is 0 then it will be changed after our move (at least one 1 will be changed to 0, so in that column will be odd number of 1s).
- **From the winning positions it is possible to move to at least one losing:**
  - if xor of the sizes of the piles is not 0 we can change it to 0 by finding the leftmost column where the number of 1s is odd, changing one of them to 0 and then by changing 0s or 1s on the right side of it to gain even number of 1s in every column.

Composite Games - Grundy Numbers

- Problem Statement: N x N chessboard with K knights on it. Unlike a knight in a traditional game of chess, these can move only as shown in the picture below (so the sum of coordinates is decreased in every move).



Composite Games - Grundy Numbers

- There can be more than one knight on the same square at the same time. Two players take turns moving and, when it is a player's, turn he chooses one of the knights and moves it. A player who is not able to make a move is declared the loser.
- This is the same as if we had K chessboards with exactly one knight on every chessboard. This is the ordinary sum of K games and it can be solved by using the **grundy numbers**. We assign grundy number to every subgame according to which size of the pile in the Game of Nim it is equivalent to. When we know how to play Nim we will be able to play this game as well.

Composite Games - Grundy Numbers

```
int grundyNumber(position pos) {
    moves[] // possible positions to which
            // I can move from pos
    set s;
    for (all x in moves)
        insert into s grundyNumber(x);
    // return the smallest non-negative
    // integer not in the set s;
    int ret = 0;
    while (s.contains(ret)) ret++;
    return ret;
}
```

Composite Games - Grundy Numbers

0	0	1	1	0	0	1	1
0	0	2	1	0	0	1	1
1	2	2	2	3	2	2	2
1	1	2	1	4	3	2	3
0	0	3	4	0	0	1	1
0	0	2	3	0	0	2	1
1	1	2	2	1	2	2	2
1	1	2	3	1	1	2	0

Exercise #1

- https://community.topcoder.com/stat?c=problem_statement&pm=6856

Solution

- https://www.topcoder.com/tc?module=Static&d1=match_editorials&d2=srm330

Exercise #2

- https://community.topcoder.com/stat?c=problem_statement&pm=7424&rd=10662

Solution

- https://www.topcoder.com/tc?module=Static&d1=match_editorials&d2=srm338

Exercise #3

- https://community.topcoder.com/stat?c=problem_statement&pm=6239

Solution

- https://www.topcoder.com/tc?module=Static&d1=match_editorials&d2=srm309

Further Readings

- **USACO Guide**: https://usaco.guide/adv/game-theory?lang=cpp
- **CP Algorithms**: https://cp-algorithms.com/game_theory/games_on_graphs.html

- *Winning Ways for Your Mathematical Plays*, Conway, et. al.
- *Game Theory*, Tirole, et. al.