Week 5-1

Sets

Union - $\cup$, Intersection - $\cap$, Complement - $\overline{A}$

Set Difference : $A - B$, Cartesian Product = $A \times B$

Power Set : $P(A)$, Subset : $\subseteq$, Proper Subset : $\subset$

$A \times B = \{(a,b) \mid a \in A \wedge b \in B\}$

$A = \{1,2\}$ $B = \{a,b,c\}$

$A \times B = \{(1,a), (1,b), (1,c), (2,a), (2,b), (2,c)\}$

$|A \times B| = |A| \times |B|$, multiplication principle

$A \subseteq B$ iff for all $a \in A$, $a \in B$.

[B contains every item in A]

$A \subset B$ iff $A \subseteq B \wedge A \neq B$.

[B contains every item in A, plus at least one more unique item not in A]

$P(A) = \{S \mid S \subseteq A\}$

$A = \{1,2,3\}$

$P(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

$|P(A)| = 2^{|A|}$

$P(A)$ = each possible subset of the set A

Week 5 -2

000 = $\emptyset$
001 = $\{1\}$
010 = $\{2\}$
011 = $\{1,2\}$
100 = $\{3\}$
101 = $\{3,1\}$
110 = $\{3,2\}$
111 = $\{3,2,1\}$

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for(i=0; i<(1<<n); i++) {
  //i represents a subset
  //use binary to identify membership
}
```

Set Proofs

1. Direct Proof
2. Direct Proof Contrapositive
3. Pf by Contradiction

(1) $p \to q$ (Direct Proof)

(a) Assume p is true.

(b) Take logical deductions

$p \to p_1$
$p_1 \to p_2$
$\vdots$

(c) Arrive at q

$p_3 \to p_4$
$p_4 \to q$

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(2) Direct Proof of Contrapositive ($p \to q$)

(a) Take contrapositive ($\overline{q} \to \overline{p}$.)

(b) Assume $\overline{q}$

(c) Take logical steps

(d) Arrive at $\overline{p}$: $\overline{q} \to q_1 \to q_2 \dots \to \overline{p}$.

Proof by Contradiction

1. $p \to q$
   - (a) Assume $\bar{q}$.
   - (b) Take logical steps
   - (c) We may invoke p somewhere
   - (d) Arrive at a contradiction

$$\bar{q} \to q_1 \to q_2 \xrightarrow{\text{use } p} q_3 \to F$$

$$(p \wedge \bar{q}) \xrightarrow{\cdots} F \quad (\underset{q|p}{\cancel{p|q}} \wedge p \% q = 5)$$

Direct

Week 5-4

If $C \subseteq A \cap B$ then $A - B \subseteq A - C$

1. Assume $C \subseteq A \cap B$
2. Prove $A - B \subseteq A - C$

$\Rightarrow$ if $x \in A - B$, then $x \in A - C$.

(2a) Assume $x \in A - B$

(2b) $x \in A \wedge x \notin B$ (def set diff)

(2c) if $x \in A \wedge x \notin B \Rightarrow x \notin A \cap B$. def intersection.

(2d) Since $C \subseteq A \cap B \wedge x \notin A \cap B$, then $x \notin C$. If $x \in C$, that would violate the subset rule.

(2e) $x \in A \wedge x \notin C$, so by def set diff. $x \in A - C$. $\boxtimes$ Done!

Orig: If $C \subseteq A \cap B$, then $A - B \subseteq A - C$

Contrapositive if $A - B \not\subseteq A - C$, then $C \not\subseteq A \cap B$

Assume $A - B \not\subseteq A - C$

Prove $C \not\subseteq A \cap B$

$\rightarrow$ if $A - B \not\subseteq A - C$, $\exists\, x$, such that $x \in A - B \wedge x \notin A - C$.

$\Rightarrow$ $\underline{x \in A} \wedge x \notin B \wedge \underline{x \notin A - C}$.

$\Rightarrow$ if $x \notin A - C$, then $\boxed{x \in A \wedge x \notin C \text{ is false.}}$ but $x \in A$, so it follows that $x \notin C$ is false.

$\Rightarrow$ So $x \in C$.

Thus, $x \in A$, $x \notin B$, $\wedge$ $x \in C$

$\Rightarrow$ $x \notin A \cap B$. $\wedge$ $x \in C$

It follows that $C \nsubseteq A \cap B$.

Pf by Contradiction

Orig: if $C \subseteq A \cap B$, then $A - B \subseteq A - C$

Assume the opposite, that $A - B \nsubseteq A - C$.

Then there exists an x such that $x \in A - B \wedge x \notin A - C$.

$\Rightarrow$ $x \in A \wedge x \notin B \wedge x \notin A - C$

$\Rightarrow$ Since $(x \in A \wedge x \notin C)$ is false it follows that $x \notin C$ is false, so $x \in C$, and $x \notin A \cap B$

$\Rightarrow$ But this contradicts our fact that $C \subseteq A \cap B$.

$\rightarrow$ ~~S~~ it follows that our initial assumption was incorrect and $A - B \subseteq A - C$.

$R = \{ (x,y) \mid \gcd(x,y) > 1 \}$, Set ints greater than 1.

Ref: For all $a > 1$, $(a,a) \in R$ because $\gcd(a,a) = a \geq 1$.

Symm: if $(a,b) \in R$, then $(b,a) \in R$.

$(a,b) \in R \Rightarrow \gcd(a,b) > 1$

$\Rightarrow \gcd(b,a) > 1$

$\Rightarrow (b,a) \in R$.

Anti Sym false $(2,4) \in R \wedge (4,2) \in R$, and $2 \neq 4$.
since $\gcd = 2$

trans false $(2,6) \in R$, $(6,9) \in R$ but $(2,9) \notin R$.
$\gcd = 2$ $\gcd = 3$

Irrefl false because $(2,2) \in R$.
$\gcd = 2$

$R \subseteq A \times B$, $S \subseteq A \times B$, $T \subseteq B \times C$

$T \circ R \cup T \circ S = T \circ (R \cup S)$

(1) if $(x,z) \in T \circ R \cup T \circ S$, then $(x,z) \in T \circ (R \cup S)$

(2) if $(x,z) \in T \circ (R \cup S)$, then $(x,z) \in T \circ R \cup T \circ S$.

$\rightarrow (x,z) \in T \circ R \cup T \circ S$

$\Rightarrow (x,z) \in T \circ R$ or $(x,z) \in T \circ S$

$\exists y$ s.t $(x,y) \in R \wedge \underline{(y,z) \in T}$	$\exists y$ s.t $(x,y) \in S \wedge \underline{(y,z) \in T}$
$\Rightarrow$ if $(x,y) \in R$, $\underline{(x,y) \in R \cup S}$.	if $(x,y) \in S$, $\underline{(x,y) \in R \cup S}$.
$\Rightarrow (x,z) \in T \circ (R \cup S)$	$\Rightarrow (x,z) \in T \circ (R \cup S)$.