Fall 2018 COT 3100 Final Exam

Version A Solutions

1) For how many combinations of variable settings for p, q and r does the expression $\bar{p} \wedge (\overline{q \vee r})$ evaluate to true?

a) 0 **b) 1** c) 2 d) 3 e) None of the above

To make an and true, both operands have to be true. This forces p to be False. Then, we must analyze how to make $\overline{q \vee r}$ True as well. This is equivalent to making $q \vee r$ False, which can only happen with both q and r are set to False. It follows that there's ONLY one combination of variable settings for p, q, and r that makes the given expression true: (T, F, F).

2) Which of the following Boolean expressions is equivalent to $(\overline{p \wedge q}) \wedge ((\bar{p} \vee r) \vee \bar{q})$?

a) $p \vee q$ b) $p \wedge q$ c) $\bar{p} \wedge \bar{q}$ d) r **e) None of the above**

$(\overline{p \wedge q}) \wedge ((\bar{p} \vee r) \vee \bar{q}) \leftrightarrow (\overline{p \wedge q}) \wedge ((\bar{p} \vee \bar{q}) \vee r)$, Commutative Law
$\leftrightarrow (\overline{p \wedge q}) \wedge ((\overline{p \wedge q}) \vee r)$, DeMorgan's Law
$\leftrightarrow (\overline{p \wedge q})$, Absorption Law
$\leftrightarrow \bar{p} \vee \bar{q}$, De Morgan's Law

From here, we see that none of the choices (a) through (d) are logically equivalent to this one, so the right answer is choice (e).

3) Consider the following open statement: $P(x, y) = \forall x[\exists y | xy = 0]$. Assuming that the universe of values that x and y can be selected from are the same, for which of the following sets is the open statement true?

a) Z^+ b) Z^- c) {1,2,3} **d) Z** e) Q^+

We need to be able to select y = 0 to make the statement true for all x. The only set of those listed that contains 0 is Z.

4) Let A = {1, 2, 3, 4} and B = {2, 8, 9}. What is the value of $|\wp(A \cup B)|$?

a) 6 b) 7 **c) 64** d) 128 e) None of the above

There are 6 elements in $A \cup B$. It follows that there are $2^6 = 64$ elements in $\wp(A \cup B)$.

5) Let A = {1, 2, 3, 4} and B = {2, 8, 9}. Which of the following is an element of A x B?

a) 1 **b) (4,8)** c) (2,1) d) (8,4) e) None of the above

4 is an element of A, 8 is an element of B, thus, by the definition of Cartesian Product, (4, 8) is an element of A x B.

6) Assume that sets A, B and C are finite and that there exist some elements in the intersection of all three sets and that the number of elements in the intersection of each pair of these sets is strictly greater than the number of elements in the intersection of all three sets. Which of the following quantities is the smallest?

a) $|A - (B - C)|$ b) $|A \cup B \cup C|$ **c) $|A - B - C|$**

d) $|A \cup (B \cap C)|$ e) $|A \cap (\mathcal{U} - B)|$

We have to answer based on the choices given. We can see that choice (c) has fewer elements than A while choices (b) and (d) have MORE elements than A. So, we can eliminate choices (b) and (d).

Any element that is in A and C is included in the count for (a) but not (c), and we are guaranteed that this number is positive, so we can also eliminate choice (a).

Finally, notice that choice (e) is equivalent to |A – B|, because we intersect A with all elements not in B, which effectively is doing the same thing as removing from A all elements that are in B. Again, since there exist elements that belong precisely to just A and C and not B, these elements will belong to (e) but not (c).

It follows that (c) is the correct choice.

7) What is the remainder when 19^{12} is divided by 16?

a) 1 b) 3 c) 9 d) 11 e) None of the above

$19 \equiv 3 \ (mod\ 16)$, so we can equivalently find the remainder when 3^{12} is divided by 16. Let's build up a table of values for 3^x mod 16, where x is successive powers of 2:

Value	3^1	3^2	3^4	3^8
Mod 16	3	9	1	1

It follows that $3^{12} = 3^8 3^4 \equiv 1 \times 1 = 1$ (mod 16)

8) When writing out the steps to the Euclidean Algorithm to find the GCD between 173 and 47, how many equations are written with a positive remainder? (Note: assume that we have 173 by itself on the left hand side for the first equation of the algorithm.)

a) 2 b) 3 **c) 4** d) 5 e) None of the above

173 = 3 x 47 + 32
47 = 1 x 32 + 15
32 = 2 x 15 + 2
15 = 7 x 2 + 1

There are 4 steps with a positive remainder.

9) Alice averages 15 miles per hour driving from Orlando to Tampa. She averages 45 miles per hour on her drive back from Tampa to Orlando. What was her average speed, in miles per hour, for the round trip?

a) 20 b) 25 c) 30 d) 40 **e) None of the above**

Let D be the distance traveled. The total time she spends going to Tampa is $\frac{D}{15}$ hours. The total time she spends traveling back is $\frac{D}{45}$. Her total time spent is $\frac{D}{15}+\frac{D}{45}=\frac{3D+D}{45}=\frac{4D}{45}$. The total distance she traveled for the round trip was 2D. It follows that her average speed was $\frac{2D}{\frac{4D}{45}}=\frac{45}{2}$ miles per hour. This is not an answer choice, so the correct selection is (e).

10) Which of the following is equal to $\sum_{i=0}^{n}(2i+1)$, assuming that n is a positive integer?

a) n^2 **b) n^2+2n+1** c) $2n^2-n$ d) $2n^2+n$ e) None of the above

$$\sum_{i=0}^{n}(2i+1)=[2\sum_{i=0}^{n}i]+[\sum_{i=0}^{n}1]=\frac{2n(n+1)}{2}+(n+1)=(n+1)(n+1)=n^2+2n+1$$

11) Let matrix A have 3 rows and 7 columns. Let matrix B have 7 rows and 5 columns. What are the dimensions of the product A x B? (Dimensions are listed as ordered pairs with rows followed by columns.)

a) (3, 5) b) (3, 7) c) (5, 3) d) (7, 7) e) None of the above

A matrix with dimensions r x c multiplied by another matrix with dimensions c x d results in an answer with dimensions r x d.

12) Which of the following integrals provides an upper bound to the sum $\sum_{i=1}^{n} \ln(i)$?

a) $\int_1^n \frac{1}{x}dx$ b) $\int_1^n lnxdx$ **c) $\int_1^n \ln(x+1)dx$** d) $\int_{.5}^{n-.5} lnxdx$ e) None of the above

Notice that each integral in choices (a) to (c) go from 1 to n, and in some sense have one fewer "term" than a sum with n terms, since we're going to replace each term in the sum with an integral over a width 1 unit in x. Since ln (1) = 0, it's likely that this is the term that we are "ignoring." Thus, we seek to find an integral such that the integral from 1 to 2 is greater than the second term in the sum, ln(2). (a) is definitely out because the function decreases. (b) is also out because the integral from 1 to 2 of ln x is strictly less than ln 2 (draw the graph to see this.) But (c) is valid. The integral of ln(x+1) from x = 1 to 2 has to be greater than ln (2), because the graph of the function in that domain is ln (2) or greater. It follows that the correct choice is (c).

13) A bag contains 5 green Skittles, 3 red Skittles and 7 yellow Skittles. In how many different ways can you choose 3 Skittles, one of each color?

a) 3 b) 5 c) 7 d) 15 **e) 105**

5 x 3 x 7 = 105, for each choice of a green Skittle, you can pair it with any of the red Skittles, which can be paired with any of the yellow ones.

14) John gets a 100% on his geography quizzes 80% of the time. In the third nine weeks he took 5 geography quizzes, which of the following is the probability that he got a 100% on three of them?

a) $(.8)^3$ b) $5(.8)^3(.2)^2$ **c) $10(.8)^3(.2)^2$** d) $(.8)^5$ e) None of the above

This is a binomial distribution question (and perhaps it should be worded to say exactly 3 of them), but the inference here is that it's testing this particular concept from class.

Plugging into the binomial distribution formula we get $\binom{5}{3}(.8)^3(.2)^2 = 10(.8)^3(.2)^2$.

15) How many permutations of the letters in ACEVEDO contain consecutive vowels?

a) $\frac{7!}{2} - 3(4!)$ b) $7! - 3(4!)$ c) 72 d) $4(6!)$ e) None of the above

The total number of permutations is $\frac{7!}{2}$, since there are 2 Es (only repeat). Of these, the only ones that DON'T contain consecutive vowels are the ones with the vowels in positions 1, 3, 5, 7, as shown in the question. We can arrange the vowels in $\frac{4!}{2}$ ways and we can arrange the consonants in 3! ways, keeping each category in the same set of slots. Thus, we must subtract out $\frac{4!3!}{2} = 3(4!)$ combinations that we shouldn't count form $\frac{7!}{2}$. Thus, the final answer is $\frac{7!}{2} - 3(4!)$.

16) A chicken store sells four types of items: nuggets, strips, filets and legs. You've decided that you want to buy exactly 12 items and that you want at least 4 strips. Assume that the store has plenty of each type of item in stock. How many different orders can you make? (Two orders are different if they have a different quantity of at least one type of item. One valid order for this query is 3 nuggets, 5 strips, 2 filets and 2 legs.)

a) $\binom{12}{4}$ b) $\binom{12}{8}-\binom{8}{4}$ c) $\binom{11}{4}$ d) $\binom{8}{4}$ **e) None of the above**

Buy the 4 strips, so you have 8 items left to buy. So this is combinations with repetition with n = 8 and r = 4. It follows that we can make the order in $\binom{8+4-1}{4-1}=\binom{11}{3}$ ways. None of the choices are equal to this value, so the correct answer is (e).

17) John is late to school 5% of the time. Given that he is late, he receives a detention 20% of the time. Given that he is on time, he still receives a detention 10% of the time. John has received a detention on a particular day. What is the probability that he was late on that day?

a) $\frac{1}{10}$ b) $\frac{1}{19}$ c) $\frac{2}{19}$ **d) $\frac{2}{21}$** e) None of the above

p(late and detention) = .05 x .2 = .01
p(on time and detention) = .95 x .1 = .095

p(late | detention) = $\frac{p(late\ and\ detention)}{p(detention)}=\frac{.01}{.01+.095}=\frac{10}{105}=\frac{2}{21}$

18) Trisha rolls two dice – one is a regular six-sided die with labels 1 through 6, and the other is a four-sided die with labels 1 through 4. Both dice are fair, so each of the possible labels on a single die is equally likely to show on any given roll. Trisha rolls both dice simultaneously. What is the probability that the sum of the labels showing is less than 7?

a) $\frac{7}{8}$ **b) $\frac{7}{12}$** c) $\frac{3}{4}$ d) $\frac{7}{24}$ e) None of the above

Sample space = 6 x 4 = 24.
Ways to roll 7 or greater: (6, 1), (6, 2), (6, 3), (6, 4), (5, 2), (5, 3), (5, 4), (4, 3), (4, 4) and (3, 4).
Thus the number of ways to roll less than 7 is 24 – 10 = 14.
The desired probability is $\frac{14}{24}=\frac{7}{12}$.

19) Ten years ago, John was twice as old as Mary. In five years, John will only be 50% older than Mary. What is the sum of their ages right now?

a) 45 **b) 65** c) 75 d) 85 e) None of the above

Let J be John's age now and M be Mary's age now.

Using the given information, we have:

(J – 10) = 2(M – 10)
J + 5 = 1.5(M + 5)

J – 10 = 2M – 20, so J = 2M – 10. We can substitute this into the other equation:

2M – 10 + 5 = 1.5M + 7.5
2M – 5 = 1.5M + 7.5
.5M = 12.5
M = 25 → J = 2(25) – 10 = 40

Thus, currently, the sum of their ages is 25 + 40 = 65.

20) John can build a house in 15 days. Jim can build a house in 20 days. They hire Jonah and all together, the three of them build a house in 5 days. How many days would it take Jonah to build the house by himself?

a) 10 b) 15 c) 16 d) 18 **e) None of the above**

In 5 days, John builds $\frac{5}{15} = \frac{1}{3}$ of the house. In 5 days, Jim builds $\frac{5}{20} = \frac{1}{4}$ of the house. This means that in 5 days, Jonah built $1 - \frac{1}{3} - \frac{1}{4} = \frac{12-4-3}{12} = \frac{5}{12}$ of the house. This means in one day, Jonah would be able to build $\frac{5}{12} \div 5 = \frac{1}{12}$ of a house. It follows that he could build a house on his own in $\frac{1}{\frac{1}{12}} = 12$ days.

21) Let f(x) = 3x + 4 and g(x) = $(x – 2)^2$. What is g(f(x))?

a)$3(x – 2)^2 + 4$ b) $3x^2 + 12x + 4$ **c) $9x^2 + 12x + 4$**

d) $3x^2 + 2$ e) None of the above

$g(f(x)) = g(3x+4)= (3x+4 – 2)^2 = (3x + 2)^2 = 9x^2 + 12x + 4$

22) Let R be a relation over A x A where A = {1, 2, 3, 4}.
In particular, let R = {(1, 2), (2, 2), (2, 3), (3, 2), (3, 3), (3, 4), (4, 1), (4, 4)}. What ordered pair must be added to R to make it reflexive?

a) (1, 1) b) (2, 2) c) (1, 4) d) (2, 4) e) None of the above

The only missing ordered pair of the form (a, a) is (1, 1).

23) Let A = {1, 2, 3, 4, 5} and B = {a, b, c}. How many surjective functions are there with the domain of A and co-domain of B?

a) 3^5 b) $3^5 – 2^5$ c) $3^5 – 3(2^5)$ **d) $3(3^4 – 2^5 + 1)$** e) None of the above

There are a total of 3^5 functions from A to B. Of these, we want to subtract out all of the ones that don't have all 3 output values represented. First, we subtract out all of the functions that map to {a, b} only (there are 2^5 of these), then all those that map to {a,c} only (there are also 2^5 of these), and those that map to {b,c} only (also 2^5 of these). But, in doing so, we subtracted out the functions that only map to {a}, only map to {b} and only map to {c} 2 times each. Thus, we have to add 3 back in because we subtracted out 3 too many. Thus, our final count is:

$3^5 – 2^5 – 2^5 – 2^5 + 3 = 3^5 – 3(2^5) + 3 = 3(3^4 – 2^5 + 1)$.

24) What is the name given to a relation that is reflexive, anti-symmetric and transitive?

a) equivalence relation **b) partial ordering relation** c) quiggy relation

d) symbiotic relation e) None of the above

This is a partial ordering relation.

25) James Madison is known as the "Father of the Constitution." What major document which outlines the basis of the United States government did Madison help write?

a) Constitution b) Declaration of Independence c) Gettysburg Address

d) Missouri Compromise e) Farmer's Almanac

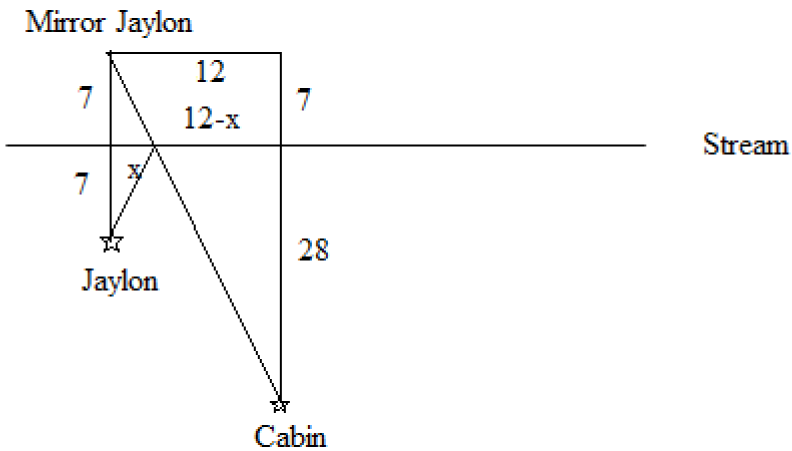
<u>Fall 2018 COT 3100 Section 2 Final Exam - Free Response Solutions</u>

1) (8 pts) Jaylon is 7 miles south of a stream that flows due east. He is also 12 miles west and 21 miles north of his apartment. He wishes to go to the stream to collect some water and then return to his apartment. What's the minimum distance he can travel to accomplish this task? (Hint: the result is an integer number of miles.)

<u>Solution</u>

This problem is identical to the cowboy problem from the second to last recitation, with different substituted numbers. Here is the solution for convenience:

An equivalent problem would be if Jaylon was 7 miles ***<u>north</u>*** of the stream. Consider this picture:



Basically, since Jaylon has to get to the stream anyway, it doesn't really matter which side of the stream he comes from. Any path he could take to the stream from his original point, "Mirror Jaylon" can take an equivalent path of the same length and end at the exact same spot as our original cowboy. From there, their straight line paths to the cabin are identical.

It's fairly obvious that "Mirror Jaylon's" best path is the straight line that goes through the stream straight to the cabin which has length $\sqrt{12^2 + 35^2} = \mathbf{37}$ **miles.** Thus, our regular cowboy can simply aim to hit the stream at the same exact point and also obtain a 37 mile path. Any other path pursued by the regular cowboy will be longer. We can prove this by drawing the equivalent path of "Mirror Cowboy" which will end up being 2 sides of a triangle with 17 being the third side. (The triangle inequality shows that the sum of those two sides is greater than 17, and not the ideal path for the cowboy.)

Grading: For the solution path above: 7 pts to get to sqrt(1369), 1 pt to simplify to 37.

Any solution that picks some random point on the stream and calculates the correct traveling distance is worth 4 points. (1 or 2 pts off for incorrect calculation.)

The correct Manhattan distance (47 miles) gets 2 pts of credit. Setting up the correct Calculus equation to minimize gets 3 pts out of 8.

2) (8 pts) Find the ordered pair (x,y) that satisfy the pair of equations shown below:

$$log_2 x^3 + log_4 y^2 = 6$$

$$log_4 x^4 + log_2 y^6 = 20$$

Solution

Let A = $log_2 x$ and B = $log_2 y$. Note the log base change rule, namely that $log_a b = \frac{log_c b}{log_c a}$. It follows that $log_4 x = \frac{log_2 x}{log_2 4} = \frac{log_2 x}{2} = \frac{A}{2}$. Similarly, $log_4 y = \frac{B}{2}$. Now take our given equations and apply the power rule to get:

$$3log_2 x + 2log_4 y = 6$$

$$4log_4 x + 6log_2 y = 20$$

Substituting so that our equations are in A and B now, we get:

$$3A + \frac{2B}{2} = 6$$

$$\frac{4A}{2} + 6B = 20$$

Now, solve the system:

3A + B = 6, so B = 6 - 3A

2A + 6B = 20

Substitute our expression for B from equation 1 into equation 2:

2A+6(6 - 3A) = 20
2A + 36 - 18A = 20
16A = 16
A = 1

It follows that B = 6 - 3(1) = 3.

To solve the problem, recall that A = $log_2 x$, so x = 2^1 = 2 and B = $log_2 y$, so y = 2^3. The desired solution is **(2, 8).**

Grading: Base change idea = 2 pts
Setting up system of equations = 1 pt
Solving system = 3 pts
Extracting final answer = 2 pts

3) (12 pts) Recall that $\frac{d}{dx}\sin(x) = \cos(x)$ and $\frac{d}{dx}\cos(x) = -\sin(x)$. We denote the n^{th} derivative of a function as $\frac{d^n y}{dx^n}$. Let $y = \sin(2x)$. Conjecture a guess for the function $\frac{d^{4n} y}{dx^{4n}}$, using the function y given, for all non-negative integers n. (Thus, your formula will be a formula for the 0^{th}, 4^{th}, 8^{th}, 12^{th}, etc. derivatives of $\sin(2x)$.) Prove your guess via mathematical induction on n.

Solution

The fourth derivative of sin(x) is sin(x). Using this understanding and the effect of the chain rule, we can surmise that for the given function, $\frac{d^{4n} y}{dx^{4n}} = 16^n \sin(2x)$.

Base case: n = 0, LHS = sin(2x), RHS = 16^0sin(2x) = sin(2x), this the base case holds.

Inductive hypothesis: Assume for an arbitrary non-negative integer n = k that

$$\frac{d^{4k} y}{dx^{4k}} = 16^k \sin(2x).$$

Inductive step: Prove for n = k + 1 that $\frac{d^{4(k+1)} y}{dx^{4(k+1)}} = 16^k \sin(2x)$.

$$\frac{d^{4(k+1)} y}{dx^{4(k+1)}} = \frac{d^4 y}{dx^4}\left(\frac{d^{4k} y}{dx^{4k}}\right)$$

$$= \frac{d^4 y}{dx^4}\left(16^k \sin(2x)\right), \text{ using the I. H.}$$

$$= \frac{d^3 y}{dx^3}\left(16^k (2)\cos(2x)\right)$$

$$= \frac{d^2 y}{dx^2}\left(-16^k (2)(2)\sin(2x)\right)$$

$$= \frac{dy}{dx}\left(-16^k (2)(2)(2)\cos(2x)\right)$$

$$= (-1)(-1)16^k (2)(2)(2)(2)\sin(2x)$$

$$= 16^k (16)\sin(2x)$$

$$= 16^{k+1} \sin(2x)$$

Grading Criteria:

Guess = 4 pts (can give partial, 2 pts for sin(2x), 2 pts for 16^n or 2^{4n})
Base case = 1 pt (n=0 or n = 1 permitted)
IH = 1 pt
IS = 1 pt
Do 4 derivatives (2 pts) - note either order
Do IH (2 pts) - note either order
Simplify (1 pt)

4) (15 pts) A number of electronic coins have recently gained value. In particular, a bytecoin is worth 10 cents, a megacoin is worth one dollar, an opticoin is worth one dollar and finally a knightcoin is also worth one dollar. How many different combinations of bytecoins, megacoins, opticoins and knightcoins are worth exactly 20 dollars? Please leave your answer in powers, combinations, factorials, etc. and carefully explain what each expression in your answer and work represent.

Solution

Let x equal the number of bytecoins, y equal the number of megacoins, z equal the number of opticoins and w equal the number of knightcoins. Using the given info, we want the number of non-negative integer solutions to the equation:

$$10x + 100y + 100z + 100w = 2000$$

Divide this equation by 10:

$$x + 10y + 10z + 10w = 200$$

Since $x = 200 - 10y - 10z - 10w$, and y, z and w are integers, it follows that $10 \mid x$. Create a new integer variable x' such that $x = 10x'$, so our new equation reads, since x is divisible by 10:

$$10x' + 10y + 10z + 10w = 200$$

So now we want the number of non-negative integer solutions to the equation:

$$x' + y + z + w = 20$$

using the formula for combinations with repetition, the result is $\binom{20+4-1}{4-1} = \binom{23}{3}$.

An alternate way to solve the problem is to realize that since x can be anything in the equation $x + 10y + 10z + 10w = 200$, that any solution to $10y + 10z + 10w \le 200$, since we can treat x as our "overflow" variable for amount less than 200. Now, we want the number of solutions to $10y + 10z + 10w \le 200$. So, now just divide by 10 to get $y + z + w \le 20$. Finally, we learned that the way to solve this in class is to introduce a "slack" variable to account for the sum less than 20, so we are really looking for the total number of solutions to $y + z + w + x' = 20$, which is the same as above. (So, the result as well as the equation for which we are solving are the same, but this just shows an alternate line of reasoning to get there.)

Grading: 3 pts for setting up initial equation
2 pts for dividing by 10
4 pts for strategy to deal with x (either say mult 10 or slack var)
3 pts to reduce problem applying strategy
3 pts for applying combo with rep formula correctly

Valid summation = 10/15

5) (10 pts) What is the probability that a randomly chosen positive divisor of 15^{79} is an integer multiple of 45^{30}? Please express your answer as a fraction in lowest terms.

Solution

Note that $15^{79} = 3^{79}5^{79}$. It follows that the number of divisors this number has is (79+1)(79+1) = 80^2. (Recall that all divisors are of the form 3^a5^b with $0 \le a,b \le 79$. So the number of divisors is equal to the number of ordered pairs (a, b) that satisfy the constraints given. Since the choices of a and b are independent, we simply multiply the number of possible choices of by the number of possible choices by b. For both, the possible number of choices equals the number of integers in between 0 and 79, inclusive, which is 80.)

Thus, the sample space for our problem is 80^2, since there are 80^2 divisors of 15^{79}.

Now, let's prime factorize $45^{30} = (3^2)^{30}5^{30} = 3^{60}5^{30}$. In order for one of the divisors of 15^{79} to be a multiple of 45^{30}, we require it to be of the form 3^a5^b where $a \ge 60$ and $b \ge 30$. So, we must count the number of divisors of 15^{79} of the form 3^a5^b where $a \ge 60$ and $b \ge 30$. Recall that we also have $a \le 79$ and $b \le 79$. Thus, we simply desire the number of ordered pairs (a, b) with $60 \le a \le 79$, and $30 \le b \le 79$. Again, the choices of a and b are independent, so we can count the number of ordered pairs by simply multiplying the number of integers a that satisfy the given inequality by the number of integers b that satisfy its given inequality. There are (79 - 60 + 1) = 20 possible values of a and (79 - 30 + 1) = 50 possible values of b, for a total of 20 x 50 possible divisors of 15^{79} that are also multiples of 45^{30}.

It follows that our desired probability is $\frac{20\times50}{80\times80} = \frac{10}{64} = \mathbf{\frac{5}{32}}$.

Grading: 4 pts for sample space (can give partial here if progress is made).
4 pts for figuring out which items in the sample space are multiples of 45^{30}.
1 pt for writing as a fraction (with smaller # over bigger #)
1 pt for reducing to lowest terms

6) (10 pts) Let $f(x) = \frac{3x-2}{x+5}$, with a domain of all reals except x = -5. Determine $f^{-1}(x)$ as well as the domain and range of $f^{-1}(x)$.

Solution

Switch x and y and solve for y:

$$x = \frac{3y-2}{y+5}$$
$$x(y+5) = 3y-2$$
$$xy + 5x = 3y - 2$$
$$5x + 2 = 3y - xy$$
$$5x + 2 = y(3-x)$$
$$y = \frac{5x+2}{3-x}$$

It follows that $\boldsymbol{f^{-1}(x) = \frac{5x+2}{3-x}}$. The domain of this inverse function is the range of the original function, it's **all reals except for x = 3.** The range of this function is the domain of the original function which is **all reals except for y = -5.**

Grading: 2 pts to switch x and y
6 pts to solve for y (give partial as you see fit)
1 pt for domain (has to be 100% correct to get the point)
1 pt for range (has to be 100% correct to get the point)

7) (12 pts) Define a relation R over the set of rational numbers as follows: R = { (p, q) | p - q has a denominator with absolute value less than 200 when reduced to lowest terms }. Determine, with proof, whether or not R is (a) reflexive, (b) irreflexive, (c) symmetric, (d) anti-symmetric and (e) transitive.

Solution

The relation is reflexive. For any rational number p, $p - p = \frac{0}{1}$, thus, the corresponding denominator in question is less than 200 and (p, p) ∈ R, for all rational numbers p.

The relation is NOT irreflexive because $(\frac{1}{1}, \frac{1}{1})$ belongs to the relation (as an example of the general case previously stated.)

The relation is symmetric because if (p, q) is in the relation, then $p - q = \frac{m}{n}$, where |n| < 200. We can multiply this equation through by -1 to yield $q - p = \frac{-m}{n}$. Since m is an integer, -m is also an integer and $\frac{-m}{n}$ is a rational number. Finally, since |n| < 200, this result is a rational number with a denominator less than 200. From this equation, it follows that (q, p) ∈ R, as desired to prove that R is symmetric.

R is NOT anti-symmetric. Note that $(\frac{2}{1}, \frac{1}{1}) \in R$ and $(\frac{1}{1}, \frac{2}{1}) \in R$, but that $\frac{2}{1} \neq \frac{1}{1}$, and that the difference between these fractions is $\frac{1}{1}$ and $\frac{-1}{1}$, respectively, both with denominators less than 200.

R is NOT transitive. Note that $(\frac{1}{21}, \frac{1}{35}) \in R$ and $(\frac{1}{35}, \frac{1}{20}) \in R$, but $(\frac{1}{21}, \frac{1}{20}) \notin R$. Notice that gcd(21, 35) = 7, so lcm(21, 35) = 105 and gcd(35, 20) = 5, so lcm(35, 20) = 140, but since gcd(21, 20) = 1, lcm(21, 20) = 420. (The denominator of the difference will generally equal the lcm of the original denominators.)

Grading: 2 pts for reflexive, irrflexive, symmetric, anti-symmetric, 4 pts for transitive. (1 pt for getting whether or not it is that quality right, 1 pt for proof of the first four, 3 pts for proof of the last one)