<u>LA Session - Probability 2 Solutions</u>

1) In a tennis league, there are six teams of 5 players each. A set of 7 players is selected randomly out of the 30 players in the league. What is the probability that at least one player from each team will be selected?

Solution

Our sample space is $\binom{30}{7}$, since we can choose any set of 7 players out of 30. We must count the number of these selections that contain at least 1 player from each team.

Since there are 6 teams, we can ONLY have one team that has 2 players from it. We can select this team (we'll call it the over-represented team) in 6 ways.

Once we select the over-represented team, we can choose 2 team members from it in $\binom{5}{2} = 10$ ways.

For each of the other 5 teams, we can select a member from those teams in $\binom{5}{1} = 5$ ways.

It follows that the desired probability is $\frac{6\times10\times5^5}{\binom{30}{7}}$~0.09210138520483348.

2) You roll a pair of standard fair six-sided dice three times, and record the sum of the two faces showing each time. Let these three sums be s_1, s_2, and s_3. What is the probability that s_2 is strictly bigger than both s_1 and s_3?

Solution

There is nothing unique about s_2 being the unique largest sum, so this probability is the same as s_1 being the unique largest sum or s_3 being the unique largest sum. Assign this probability to be x.

Our strategy is as follows: (1) figure out the probability that the largest sum is NOT unique. Call this y. (2) Our final answer will be (1-y)/3, since 1-y is the probability that the largest sum is unique, and it's equally likely that it's any of the three sums.

One way in which the largest sum isn't unique is if all three sums are the same. This probability is as follows: $2(\frac{1}{36})^3 + 2(\frac{2}{36})^3 + 2(\frac{3}{36})^3 + 2(\frac{4}{36})^3 + 2(\frac{5}{36})^3 + +(\frac{6}{36})^3 = \frac{2+16+54+128+250+216}{36^3} = \frac{666}{36^3} = \frac{37}{2592}$.

The other way would be if two of the sums are the same and the third is lower. Choose which sum of the three is lower in 3 ways. Then, go through each possible largest sum, and go through the probability that two sums are equal to that value and the third sum is strictly lower. For example, for the largest sum k = 9, the desired term is $3 \times \frac{26}{36} \times (\frac{4}{36})^2$, because the probability of rolling less than a 9 is 26/36 (add up the desired probabilities from the distribution of 2 dice rolls) and the probability of rolling a 9 is 4/36. We square the latter since it occurs on two pairs or rolls.

In full, this can happen with probability:

$$3[\frac{1}{36} \times (\frac{2}{36})^2 + \frac{3}{36} \times (\frac{3}{36})^2 + \frac{6}{36} \times (\frac{4}{36})^2 + \frac{10}{36} \times (\frac{5}{36})^2 + \frac{15}{36} \times \left(\frac{6}{36}\right)^2 + \frac{21}{36} \times \left(\frac{5}{36}\right)^2 + \frac{26}{36} \times \left(\frac{4}{36}\right)^2 + \frac{30}{36} \times \left(\frac{3}{36}\right)^2 + \frac{33}{36} \times \left(\frac{2}{36}\right)^2 + \frac{35}{36} \times \left(\frac{1}{36}\right)^2]$$

Simplifying, we get:

$$\frac{4+27+96+250+540+525+416+270+132+35}{36\times 36\times 12}=\frac{2295}{36\times 36\times 12}=\frac{85}{576}$$

It follows that the probability of there being a repeated largest value is

$$\frac{37}{2592}+\frac{85}{576}=\frac{74+765}{5184}=\frac{839}{5184}$$

This the probability that there IS a unique largest value is $1-\frac{839}{5184}=\frac{4345}{5184}$.

It follows that the answer to the question is this value divided by 3 or $\frac{4345}{15552}$.

Note: It may very well be the case that the direct approach of trying each possible sum for s_2 and multiplying by the probability the other two sums are smaller would have been faster than this one!

3) Suppose we roll a fair 6 sided die with the numbers [1,6] written on them. After the first die roll we roll the die k times where k is the number on the first die roll. The number of points you score is the sum of the face-values on all die rolls (including the first). What is the expected number of points you will score?

Solution

Using the linearity of expectation, the expected sum on k die rolls is 3.5k. Thus, if we roll k first, our expected total is k + 3.5k = 4.5k.

Each value of k is equally likely, so we use the expectation formula with probability 1/6 for each value of k to yield the answer:

$$\sum_{k=1}^{6}\frac{1}{6}(4.5k)=\frac{3}{4}\sum_{k=1}^{6}k=\frac{3}{4}(21)=\mathbf{\frac{63}{4}}$$

4) Balls are randomly removed from a bag without replacement. If the probability that the first five balls withdrawn are all green is one-half, what is the fewest possible number of balls in the bag at the start?

Solution

Let n be the number of balls in the bag. Let k equal the number of green balls. The probability the first five balls are green is

$$\frac{k}{n}\times\frac{(k-1)}{(n-1)}\times\frac{(k-2)}{(n-2)}\times\frac{(k-3)}{(n-3)}\times\frac{(k-4)}{(n-4)}=\frac{1}{2}$$

It's clear that in order for this to be true, lots of things must cancel. Perhaps the easiest way to imagine this occurring is that k cancels with n-1, because then k-1 cancels with n-2, and so forth. So consider the scenario where k = n-1:

$$\frac{(n-1)}{n} \times \frac{(n-2)}{(n-1)} \times \frac{(n-3)}{(n-2)} \times \frac{(n-4)}{(n-3)} \times \frac{(n-5)}{(n-4)} = \frac{1}{2}$$

$$\frac{n-5}{n} = \frac{1}{2}$$

$$n = 2n - 10$$

$$n = 10$$

Now, our job is to prove that no smaller value of n (and positive integer k) satisfy the given equation. Thus, we must only rule out n = 6, 7, 8, 9 and 10.

It's clear that k < n. If k = n, then the probability would be 1 to pull all green balls. Next, note that if k ≤ n-1 as we know is true, then the ensuing fraction is no more than $\frac{n-5}{n}$. But for n < 10, this fraction is less than ½, meaning that there is no valid solution to the original equation in positive integers n and k with n < 10.

5) A Bubble Sort is a common algorithm taught to students that sorts a list of numbers (https://en.wikipedia.org/wiki/Bubble_sort). Given a random permutation of the integers 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10, what is the probability that the permutation will be sorted **just one pass** of Bubble Sort? (For example, the permutation 3, 1, 2, 7, 4, 5, 6, 10, 8, 9 would get sorted in a single pass. 3 would swap with 1 and 2. Then 7 would sway with 4, 5 and 6, followed by 10 being swapped by 8 and 9. But, the permutation 3, 1, 2, 7, 4, 5, 8, 6, 10, 9 would not get sorted by one pass of the algorithm. After one pass, the array would be 1, 2, 3, 4, 5, 7, 6, 8, 9, 10.)

Solution

Note that in a single pass of Bubble Sort, we do 9 comparisons, after each one, we either swap or don't swap. Thus there are 2^9 possible sets of actions that could occur on a single pass of Bubble Sort. For each different choice of action, exactly 1 original ordering will cause that set of actions to result in a sorted array. Thus, the desired probability is $\frac{2^9}{10!} = \frac{2}{14175}$.

To see an illustration of this consider the sequence S,S,N,N,N,S,S,N,S, where S represents swap and N represents no swap.

Given the input array:

3, 1, 2, 4, 5, 8, 6, 7, 10, 9

A single pass of Bubble Sort will perform the operations listed above leaving the array sorted. This is the only array for which this is true.

Another way to see this is that for any array that will be sorted by one pass of Bubble Sort, we can partition it into segments

S1 | S2 | S3 | S4 …

Where each item in S1 < each item in S2 < each item in S3, etc.

and within each segment, the largest item is first and the rest follow it in order.

Here's another sample array:

4, 1,2, 3, | 9, 5, 6, 7, 8, | 10

There are 9 locations we can choose to put these splits between the 10 numbers. For each split location, we can either choose to place it or not choose to place it, so there are 2^9 ways to form the partition, each one corresponding to exactly one array that gets sorted by a single pass of Bubble Sort.