Fall 2018 COT 3100 Section 1 Homework 8 Solutions

1) Consider an ant that is walking on a Cartesian grid, starting at (0,0) and ending at (15, 18). The ant always chooses to walk exactly one unit either up or to the right (towards his destination) whenever he arrives at a Lattice point. (A Lattice point is a point with integer coordinates.) Thus, from (0,0) he either walks to (1, 0) or (0, 1). If the ant is not allowed to go to the points (6, 8) and (11, 15), how many different paths can he take on his walk?

Solution

Without any forbidden points, we know the ant must take 33 "steps" of which he must choose 15 to go to the right (positive x-axis). Thus, with the forbidden positions there are $\binom{33}{15}$ ways for the ant to walk the desired path. From this number, we must subtract out the number of forbidden paths.

The number of forbidden paths are the number of paths through (6, 8) or (11, 15). We can count these paths separately, but in doing so, we will have counted paths that go through both points twice, so we must subtract out all of those paths that go through both paths twice, via the Inclusion-Exclusion principle. In general, given a point (x, y) where $0 \le x \le 15$ and $0 \le y \le 18$, there are $\binom{x+y}{x}$ ways to get to point (x, y) and $\binom{33-x-y}{15-x}$ ways to get from (x, y) to (15, 18). Since we can pair up any of the first set of paths with any of the second pair of paths, the product of these two terms is the total number of paths from (0, 0) to (15, 18) that go through (x, y). It follows that the number of paths through (6, 8) is $\binom{14}{6}\binom{19}{9}$ and the number of paths through (11, 15) is $\binom{26}{11}\binom{7}{4}$. Finally, we must calculate the number of paths from (0, 0) to (15, 18) that go through both (6, 8) and (11, 15), since these were subtracted out twice and need to be added back in. All of these paths can be broken down into 3 sub-paths, ones that go from (0, 0) to (6, 8), then (6, 8) to (11, 15) and then from (11, 15) to (15, 18). We want to count the number of ways to take each of these subpaths and multiply all three of the terms. This yields: $\binom{14}{6}\binom{12}{5}\binom{7}{3}$ paths that go through both forbidden points, which now need to be added in. Our final result is:

$$\binom{33}{15} - \binom{14}{6}\binom{19}{9} - \binom{26}{11}\binom{7}{4} + \binom{14}{6}\binom{12}{5}\binom{7}{3}$$

2) A class contains 18 girls and 14 boys. For all parts of this question, each boy and girl are distinguishable from one another. Answer the following questions:

Solution

(a) In how many ways can a committee of one boy and one girl be chosen?

We can choose a committee of one boy and one girl in 18 x 14 = **252 ways.**

(b) In how many ways can a committee of five students be chosen?

We can choose a committee of 5 students in $\binom{32}{5}$ ways.

(c) In how many ways can a committee of four girls and three boys be chosen?

We can choose our 4 girls in $\binom{18}{4}$ ways and our boys in $\binom{14}{3}$ ways. Since we can pair up any choice of girls with boys, we multiply these to get a total of $\binom{18}{4}\binom{14}{3}$ possible committees.

(d) In how many ways can a committee of six students be chosen such that all the students on the committee are the same sex?

Just add the two options: all boys and all girls: $\binom{18}{6} + \binom{14}{6}$.

(e) In how many ways can the girls and boys form a line where no two boys are standing next to one another?

We place the girls in 18! ways, with 19 gaps. We then permute the boys in ${}_{19}P_{14}$ ways amongst the gaps. In factorials this is: $\frac{18!19!}{5!}$ orderings for the line. (Normally, we would choose 14 slots for the boys out of 19 and then permute the boys in 14! ways. This is equivalent to directly permuting the 14 boys in 19 slots.)

(f) How many committees of seven students contain at least two girls?

Count all committees of size 7 and then subtract out ones with 0 or 1 girls:

$$\binom{32}{7} - \binom{14}{7} - \binom{18}{1}\binom{14}{6}$$

3) How many solutions does the equation a + b + c + d + e + f = 30 have if each variable must be a non-negative integer and $a \le 3$, $b \le 7$ and $d \ge 8$?

Solution

If we force d to be 8 or greater, let's just create d' = d + 8 and restrict d' to be non-negative. Now, we count the number of solutions to:

$$a + b + c + d' + e + f = 22$$

with $a \le 3$, $b \le 7$.

Without the two restrictions there are $\binom{22+6-1}{6-1} = \binom{27}{5}$ solutions to the equation. Now, we'll subtract out the solutions where a > 3, then we'll subtract out the solutions were b > 7.

To count the number of solutions where a > 3, just set a' = a + 4 and find the number of non-negative solutions to a' + b + c + d' + e + f = 18. There are $\binom{18+6-1}{6-1} = \binom{23}{5}$ solutions to this equation.

To count the number of solutions where b > 7, just set b' = b + 8 and find the number of non-negative solutions to a + b' + c + d' + e + f = 14. There are $\binom{14+6-1}{6-1} = \binom{19}{5}$ solutions to this equation.

Finally, we double subtracted solutions where a > 3 AND b > 7. We must add these back in. To do this, set a' = a + 4 AND b' = b + 8 and find the number of solutions to a' + b' + c + d' + e + f = 10. There are $\binom{10+6-1}{6-1} = \binom{15}{5}$ solutions to this equation.

Putting together all of these pieces with the Inclusion/Exclusion Principle, we have a total of

$$\binom{27}{5} - \binom{23}{5} - \binom{19}{5} + \binom{15}{5}$$

solutions to the given equation that adhere to the restrictions give.

4) How many solutions does the equation $a + b + c + d + e + f + g + h \le 40$ have if each variable must be a non-negative integer?

Solution

Let's create a new variable i, which is non-negative. If the sum of a through h is less than 40, i can simply carry the "slack" of this sum (difference from 40). So, we can get the answer to this question by finding the number of non-negative integer solutions to the equation:

$$a + b + c + d + e + f + g + h + i = 40$$

Basically, each solution to this equation in non-negative values is in one-to-one correspondence to a solution of the original equation with the inequality. (In some sense, the values of a through h "force" the value of i, so for each valid a through h, there is precisely one i that fits it in the second equation, and for every solution to the second equation it naturally maps to a single solution to the first equation.)

Thus, we can just count the number of solutions to this equation. There are $\binom{40+9-1}{9-1} = \binom{48}{8}$ such solutions.

5) A class has 2n students who must be split up into pairs. We consider two sets of pairs S and T different if at least one pair in the set S isn't a pair in the set T. A pair is unordered, so we consider the pair (1, 2) and (2, 1) to be the same pair. How many different sets of pairs can the class be split up into, in terms of n? (For example, with n = 2, the answer is 3. Sets are {(1, 2), (3, 4)}, {(1, 3), (2, 4)} and {(1, 4), (2, 3)}.)

Solution

Consider carrying out this process by always pairing up the lowest numbered student who isn't yet paired. At first, we'd have to pair up #1. There would be 2n - 1 choices to pair up #1 with. Next, we'd either have to pair up #2 or #3. (If 2 was paired with 1, then we'd have have to pair up 3 next. If 1 wasn't paired with 2, then we'd have to pair up 2 next.) Luckily, no matter which person we are pairing up, we are guaranteed that there are 2n-3 choices for which to pair that person up with. For the third person to pair up, no matter who it is (could be 3, 4 or 5), we are guaranteed precisely 2n-5 choices, and so forth. It follows that the solution is $\boldsymbol{\prod_{i=1}^{n}(2i-1)}$. There are many ways to express this quantity. One nice way to do so is $\frac{(2n)!}{2^n n!}$. We can derive this by noting that (2n)! has all positive numbers from 1 to 2n multiplied together and from this we want to cross out the even values 2, 4, 6, …, 2n. But, all of these even values can be written as a product of 2 and each integer from 1 to n. Regrouping, we get n copies of 2, and each integer from 1 to n in our product, which is just n!.

6) How many integers in between 1 and 10^7 are divisible by 15, 35, or 77?

Solution

Let set A be the set of integers in between 1 and 10^7 are divisible by 15.
Let set B be the set of integers in between 1 and 10^7 are divisible by 35.
Let set C be the set of integers in between 1 and 10^7 are divisible by 77.
Note that there are $\left\lfloor \frac{n}{k} \right\rfloor$ integers in between 1 and n, inclusive that are divisible by k, since if we group the integer in groups (1, k), (k+1, 2k) and so forth, the last integer in each set is divisible by k.

If we want to know the number of integers divisible by both a and b, note that both a and b divide into lcm(a, b) and both divide into any smaller number. Thus, if we want to answer the query for how many integers from 1 to n are divisible by both a and b, the result is $\left\lfloor \frac{n}{\mathrm{lcm(a,b)}} \right\rfloor$.

Utilizing this logic and the Inclusion-Exclusion Principle, we have the number of integers in set A or set B or set C to be:

$$\left\lfloor \frac{10^7}{15} \right\rfloor + \left\lfloor \frac{10^7}{35} \right\rfloor + \left\lfloor \frac{10^7}{77} \right\rfloor - \left\lfloor \frac{10^7}{105} \right\rfloor - \left\lfloor \frac{10^7}{1155} \right\rfloor - \left\lfloor \frac{10^7}{385} \right\rfloor + \left\lfloor \frac{10^7}{1155} \right\rfloor = \mathbf{961038}$$