LA Session Week of 10/26/2020 - Counting Solutions

1) 9 x 9 x 8 x 7 x 6 x 5 x 4 (9 choices for first digit, everything but 0, then for each subsequent digit you can't choose anything that was chosen before.

2) 9^7, first time you can't choose 0. Rest of the digits you just can't choose the previous one.

3) 26^3 - 26 - 26 + 1 (all 3 letter strings minus those with AA in slots 1,2, minus those with AA in slots 2,3, and add back in AAA since that was subtracted out twice.)

4) 26 + 26 + 26^2 + 26^2 + 26^3. (In pals of length 1 and 2, you only get to choose the first letter, in pals of length 3 and 4, you get to choose the first two letters and in pals of length 5 you get to choose the first 3 letters)

5) This question considered permutations of "RONALDMCDONALD".

Solution

RONALDMCDONALD has 2 As, 1 C, 3 Ds, 2 Ls, 1 M, 2 Ns, 2 Os, and 1 R, so a total of 14 letters.

(a) How many permutations are there total?

There are $\frac{14!}{2!3!2!2!2!}$ permutations of these letters.

(b) How many permutations start and end with vowels?

Our vowels are 2 As, and 2 Os. We can have our permutation start and end with A, start and end with O, start with A end with O or start with O and end with A. Let's tackle the first two options. In these options, the 2 As or 2 Os are set. The ways to permute the remaining letters is $\frac{12!}{2!3!2!2!}$. Multiply this number by 2 to account for both cases, so we have $\frac{12!}{2!3!2!}$ ways where the permutations start and end with A or start and end with O. Now, let's say start with A and end with O. We still have 12 letters left to permute and can do so in $\frac{12!}{2!3!2!}$ ways. Multiply this by 2 to cover both of these cases to get $\frac{12!}{2!3!}$ permutations that either start with A end with O or start with O end with A. Our final result is: $\frac{12!}{2!3!2!} + \frac{12!}{2!3!} = \frac{12!}{24} + \frac{12!}{12} = \frac{3(12!)}{24} = \frac{12!}{8}$.

(c) How many permutations do NOT have consecutive vowels in them?

Place our 10 consonants out with 11 gaps: _ C _ D _ D _ D _ L _ L _ M _ N _ N _ R _. We can choose 4 slots for our vowels out of the 11. Once we make that choice, we can permute the vowels is $\frac{4!}{2!2!}$ ways. Finally, we can permute our consonants in $\frac{10!}{3!2!2!}$ ways. Thus, the total number of permutations without consecutive vowels is $\binom{11}{4}\frac{4!10!}{2!2!3!2!2!} = \binom{11}{4}\frac{10!}{8}$.

(d) How many permutations contain the vowels in order (all As before all Os)?

There are 6 permutations of 2 As and 2 Os. Of these, only 1 has all of the vowels in order. Thus, if we count the total number of permutations, we can just divide this number by 6 to get the total number of permutations with the vowels in order. (To see this, note that for each permutation where the position of the vowels is fixed, there are 6 different orderings of the vowels, each of which is paired with each possible ordering of the consonants, since each of these are paired with the same thing, exactly 1/6th of all permutations have the vowels in order.) Thus, there are a total of $\frac{14!}{2!3!2!2!2!6}$ permutations where the vowles are in order.

Note: an alternate way to solve this is to choose 4 slots out of 14 for the vowels, and then permute the consonants in $\frac{10!}{3!2!2!}$ ways. The vowels, once their slots are chosen, are fixed. Thus, the total number of permutations with the vowels in order is $\binom{14}{4}\frac{10!}{3!2!2!} = \frac{14!10!}{10!4!3!2!2!} = \frac{14!}{4!3!2!2!}$. A quick inspection of both denominators (from this answer and the previous one) shows that the two answers are the same number.

(e) How many permutations contain the substring "OLAND"? (Note: This one is hard!)

Let's make a super letter out of "OLAND". This leaves 1 A, 1 C, 2 Ds, 1 L, 1 M, 1 N, 1 O, and 1 R. There are 10 "letters" total here, with one letter, D, appearing twice. This gives us $\frac{10!}{2!}$ permutations with OLAND in them.

But…some permutations have been double counted because there are strings with 2 OLAND's in them. Every string with 2 OLAND's in them were counted twice, once when the super letter came first and once when the super letter came last. So, we must count the number of permutations that have 2 occurrences of OLAND and subtract this value out of our original answer since these were counted twice. So, if we have two super letters "OLAND" we are left with 1 C, 1 D, 1 M, and 1 R. Thus, we have 6 letters total with one repeated letter, "OLAND". There are $\frac{6!}{2!}$ such permutations.

Thus, the total number of permutations that contain the substring "OLAND" is $\frac{10!-6!}{2!} = \mathbf{1814040}$.

Note: To double check the answer, I wrote a python program. It's slow because Python is slow, but it does indeed verify the answer…