COT 3100 Fall 2018 Homework #6 Solutions

1) Use mathematical induction on n to prove the following assertion for all positive integers n: $\sum_{i=1}^{n} \frac{i(i+1)}{2} = \frac{n(n+1)(n+2)}{6}$.

Solution

Base case: n = 1. LHS = $\sum_{i=1}^{1} \frac{i(i+1)}{2} = \frac{1(2)}{2} = 1$, RHS = $\frac{1(1+1)(1+2)}{6} = \frac{6}{6} = 1$. It follows that the given statement is true for n = 1 and the base case holds.

Inductive hypothesis: Assume for an arbitrary positive integer = k that

$$\sum_{i=1}^{k} \frac{i(i+1)}{2} = \frac{k(k+1)(k+2)}{6}$$

Inductive step: Prove for n = k + 1 that

$$\sum_{i=1}^{k+1} \frac{i(i+1)}{2} = \frac{(k+1)(k+2)(k+3)}{6}$$

$$\sum_{i=1}^{k+1} \frac{i(i+1)}{2} = \sum_{i=1}^{k} \frac{i(i+1)}{2} + \frac{(k+1)(k+2)}{2}$$

$$= \frac{k(k+1)(k+2)}{6} + \frac{(k+1)(k+2)}{2}$$

$$= \frac{k(k+1)(k+2)}{6} + \frac{3(k+1)(k+2)}{6}$$

$$= \frac{(k+1)(k+2)[k+3]}{6}$$

This proves the inductive step, as desired. (Note: the inductive hypothesis was used on the second line of the proof, when substituting for the summation from i=1 to k.)

2) Use induction on n to prove that $4^{2n} - 15n - 1$ is divisible by 225 for all non-negative integers n.

Solution

Base case: n=0. The expression evaluates to $4^{2(0)} - 15(0) - 1 = 0$, which is divisible by 225

Inductive hypothesis: Assume for an arbitrary integer n=k that $4^{2k} - 15k - 1$ is divisible by 225. Namely, assume that $4^{2k} - 15k - 1 = 225a$ for some integer a.

Inductive step: Prove for n=k+1 that $4^{2(k+1)} - 15(k+1) - 1$ is divisible by 225. Namely, show that $4^{2(k+1)} - 15(k+1) - 1 = 225b$, for some integer b.

$$\begin{aligned}
4^{2(k+1)} - 15(k+1) - 1 &= 4^{2k+2} - 15k - 15 - 1 \\
&= 4^2 4^{2k} - 15k - 16 \\
&= 16(4^{2k}) - 15k - 16 \\
&= 16(4^{2k}) - (16)15k - 16 + 15(15k), \text{ add/subtracting } 15(15k) \\
&= 16(4^{2k} - 15k - 1) + 225, \text{ factoring out 16 from the first 3 terms} \\
&= 16(225a) + 225, \text{ using the inductive hypothesis} \\
&= 225(16a+1),
\end{aligned}$$

Since a is an integer, let b = 16a + 1, and we are done. Thus, for all non negative integers n, $4^{2n} - 15n - 1$ is divisible by 225.

3) Use induction to show that $\begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ -2^n+1 & 2^n \end{pmatrix}$ for all positive integers n.

Solution

Base case: n=1 LHS = $\begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}^1 = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$, RHS = $\begin{pmatrix} 1 & 0 \\ -2^1+1 & 2^1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$,

so the base case is true. (1 pts)

Inductive hypothesis: Assume for an arbitrary integer n=k that $\begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}^k = \begin{pmatrix} 1 & 0 \\ -2^k+1 & 2^k \end{pmatrix}$. (1 pt)

Inductive step: Prove for n=k+1 that $\begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & 0 \\ -2^{k+1}+1 & 2^{k+1} \end{pmatrix}$ (1 pts)

$\begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}^k \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$ (2 pts)

$= \begin{pmatrix} 1 & 0 \\ -2^k+1 & 2^k \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$, using the inductive hypothesis (2 pts)

$= \begin{pmatrix} 1 & 0 \\ -2^k+1-2^k & 2(2^k) \end{pmatrix}$, multiplying out the matrices (1 pt)

$= \begin{pmatrix} 1 & 0 \\ -2^{k+1}+1 & 2^{k+1} \end{pmatrix}$, using exponent rules for simplication. (2 pt)

Thus, it follows that the assertion is true for all positive integers n.