LA Session - Induction Preparation Solutions

1) An arithmetic sequence $a_1, a_2, a_3, \ldots$ is such that $a_{10} = 40$ and $a_{25} = 10$. Determine both a_1 and the sum of the first 50 terms of the sequence.

Let d be the common difference of the sequence.

$15d = a_{25} – a_{10} = 10 – 40$
$15d = - 30$
$d = -2$
$a_1 = a_{10} + (1-10)(d) = 40 – 9(-2) = \underline{\mathbf{58}}$
$a_{50} = a_1 + 49(d) = 58 + 49(-2) = 58 – 98 = -40$
$S_{50} = \frac{(a_1+a_{50})}{2} \times 50 = (58-40) \times 25 = \mathbf{450}$

2) An infinite geometric sequence $a_1, a_2, a_3, \ldots$ has a sum of 10. If the odd indexed terms (a_1, a_3, etc.) have a sum of 7, what is the common ratio of the sequence? What is the first term of the sequence?

Let r be the common ratio of the sequence. The even indexed terms plus the odd indexed terms sum to 10, so the even indexed terms sum to 10 – 7 = 3. Each even indexed term is r times the corresponding odd indexed term, thus the sum of the even indexed terms is simply 7r. It follows that $r = \mathbf{\frac{3}{7}}$.

3) Determine the following summation in terms of n: $\sum_{i=1}^{2n}(3i^2 - 4i)$.

$$\sum_{i=1}^{2n}(3i^2 - 4i) = 3\sum_{i=1}^{2n} i^2 - 4\sum_{i=1}^{2n} i$$
$$= \frac{3(2n)(2n+1)(4n+1)}{6} - \frac{4(2n)(2n+1)}{2}$$
$$= n(2n+1)(4n+1) - 4n(2n+1)$$
$$= n(2n+1)(4n+1-4)$$
$$= \boldsymbol{n(2n+1)(4n-3)}$$

4) Determine the following infinite summation: $\sum_{i=1}^{\infty}(2i-1)(\frac{1}{2})^i$.

$S = \sum_{i=1}^{\infty}(2i-1)(\frac{1}{2})^i$

$\frac{1}{2}S = \sum_{i=1}^{\infty}(2i-1)(\frac{1}{2})^{i+1} = \sum_{i=2}^{\infty}(2i-3)(\frac{1}{2})^i$, using an index shift.

Now, subtract the second equation from the first:

$$S - \frac{S}{2} = \sum_{i=1}^{\infty}(2i-1)\left(\frac{1}{2}\right)^i - \sum_{i=2}^{\infty}(2i-3)(\tfrac{1}{2})^i$$

$$\frac{S}{2} = \frac{1}{2} + \sum_{i=2}^{\infty}[(2i-1)\left(\frac{1}{2}\right)^i - (2i-3)\left(\frac{1}{2}\right)^i]$$

$$\frac{S}{2} = \frac{1}{2} + \sum_{i=2}^{\infty}[((2i-1)-(2i-3))\left(\frac{1}{2}\right)^i]$$

$$\frac{S}{2} = \frac{1}{2} + \sum_{i=2}^{\infty}[2(\tfrac{1}{2})^i]$$

$$\frac{S}{2} = \frac{1}{2} + \sum_{i=1}^{\infty}[(\tfrac{1}{2})^i]$$

$$\frac{S}{2} = \frac{1}{2} + \frac{\frac{1}{2}}{1-\frac{1}{2}}$$

$$\frac{S}{2} = \frac{1}{2} + 1$$

$S = \mathbf{3}$

5) Determine the following matrix sum: $\begin{bmatrix} 6 & 3 & -2 \\ 5 & -9 & 7 \end{bmatrix} + \begin{bmatrix} -2 & 12 & 3 \\ 6 & 4 & 8 \end{bmatrix}$.

$$\begin{bmatrix} 6 & 3 & -2 \\ 5 & -9 & 7 \end{bmatrix} + \begin{bmatrix} -2 & 12 & 3 \\ 6 & 4 & 8 \end{bmatrix} = \begin{bmatrix} \mathbf{4} & \mathbf{15} & \mathbf{1} \\ \mathbf{11} & \mathbf{-5} & \mathbf{15} \end{bmatrix}$$

6) Determine the following matrix product, $\begin{bmatrix} 2^n & 1-2^n \\ 3 & 2^{n-1} \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}$, in terms of n.

$$\begin{bmatrix} 2^n & 1-2^n \\ 3 & 2^{n-1} \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3(2^n)+1-2^n & 2(2^n)-(1-2^n) \\ 3\times 3+2^{n-1} & 3\times 2-2^{n-1} \end{bmatrix}$$

$$= \begin{bmatrix} 2(2^n)+1 & (2^{n+1})-1+2^n \\ 2^{n-1}+9 & 6-2^{n-1} \end{bmatrix} = \begin{bmatrix} \mathbf{2^{n+1}} & \mathbf{3(2^n)-1} \\ \mathbf{2^{n-1}+9} & \mathbf{-2^{n-1}+6} \end{bmatrix}$$

7) Let the Fibonacci Sequence be defined as follows: $F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}$, for all integers n ≥ 2. Determine and simplify the following matrix product, in terms of n:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{bmatrix} = \begin{bmatrix} F_n + F_{n-1} & F_{n-1} + F_{n-2} \\ F_n - F_{n-1} & F_{n-1} - F_{n-2} \end{bmatrix} = \begin{bmatrix} \boldsymbol{F_{n+1}} & \boldsymbol{F_n} \\ \boldsymbol{F_{n-2}} & \boldsymbol{F_{n-3}} \end{bmatrix}$$