LA Session - Proofs, Sets Solutions

1) For all integers n, prove that n(3n+1) is an even integer.

Solution

We'll do a proof by cases. All integers are either even or odd. So, we'll first prove the assertion for even integers, then we'll prove it for odd integers. Then we can conclude the claim is true for all integers.

Case 1: n is even. If n is even there exists an integer x such that n = 2x.

n(3n+1) = 2x(3(2x)+1) = 2x(6x+1), since x is an integer, x(6x+1) is an integer, thus n(3n+1) is an even integer because we expressed it as an integer times 2.

Case 2: n is odd. If n is odd, there exists an integer x such that n = 2x + 1.

n(3n+1) = (2x+1)(3(2x+1)+1) = (2x+1)(6x+4) = 2(2x+1)(3x+2),

since x is an integer, (2x+1)(3x+2) is also. It follows in this case as well that n(3n+1) is even because we expressed it as an integer times 2.

2) Given a set of n positive real numbers a_1, a_2, …, a_n, where n > 1, with an average of b, prove that the value of the largest element in the set is strictly less than bn.

Solution

We'll use proof by contradiction to prove the claim.
Assume to the contrary, that the largest element was bn or greater.
Without loss of generality, assume that this largest element is a_n. Then we have:

$a_1 + a_2 + \ldots + a_n \geq a_1 + a_2 + \ldots + a_{n-1} + bn$

Since, each element is positive, this quantity is

> bn, because n > 1 and the terms other than bn, of which there is at least 1, must be positive.

But, this contradicts the given information. Namely, if the average of the terms is b, we have $\frac{a_1+a_2+\cdots a_n}{n} = b$, meaning that $a_1 + a_2 + \cdots a_n = bn$.

This our initial assumption is incorrect, we can conclude that the largest term is strictly less than bn.

3) Let S = {1, 4, 7, 9} and T = {1, 2, 7, 8}. Explicitly list the members of the following sets: $S \cup T$, $S \cap T$. $S - T$, $S \times T$, $T \times S$, $\wp(S)$ and $\wp(T)$.

Solution

$S \cup T = \{1,2,4,7,8,9\}$
$S \cap T = \{1,7\}$
$S - T = \{4,9\}$
$S \times T$
$= \{(1,1), (1,2), (1,7), (1,8), (4,1), (4,2), (4,7), (4,8), (7,1), (7,2), (7,7), (7,8), (9,1), (9,2), (9,7), (9,8)\}$

$T \times S$
$= \{(1,1), (1,4), (1,7), (1,9), (2,1), (2,4), (2,7), (2,9), (7,1), (7,4), (7,7), (7,9), (8,1), (8,4), (8,7), (8,9)\}$

$\wp(S)$
$= \{\emptyset, \{1\}, \{4\}, \{7\}, \{9\}, \{1,4\}, \{1,7\}, \{1,9\}, \{4,7\}, \{4,9\}, \{7,9\}, \{1,4,7\}, \{1,4,9\}, \{1,7,9\}, \{4,7,9\}, \{1,4,7,9\}\}$

$\wp(T)$
$= \{\emptyset, \{1\}, \{2\}, \{7\}, \{8\}, \{1,2\}, \{1,7\}, \{1,8\}, \{2,7\}, \{2,8\}, \{7,8\}, \{1,2,7\}, \{1,2,8\}, \{1,7,8\}, \{2,7,8\}, \{1,2,7,8\}\}$

4) Use set laws to prove that the two following sets are equivalent.

(1) $A \cup B$ (2) $(A \cap B) \cup (A \cap \bar{B}) \cup (\bar{A} \cap B)$

Solution

Start with the left-hand side:

$(A \cap B) \cup (A \cap \bar{B}) \cup (\bar{A} \cap B) =$

$\left(A \cap (B \cup \bar{B})\right) \cup (\bar{A} \cap B) =$	Distributive Law
$(A \cap U) \cup (\bar{A} \cap B) =$	Inverse Law
$A \cup (\bar{A} \cap B) =$	Identity Law
$(A \cup \bar{A}) \cap (A \cup B) =$	Distributive Law
$U \cap (A \cup B) =$	Inverse Law
$(A \cup B)$	Identity Law

5) Let A, B and C be arbitrary sets taken from the positive integers.

Prove or disprove: If $A \cap B \cap C = \emptyset$, then $(A \subseteq \overline{B}) \vee (A \subseteq \overline{C})$.

Solution
This is false. Consider the following counter-example:

A = {1, 2}
B = {1, 3}
C = {2, 3}

In this example, we have $A \cap B \cap C = \emptyset$, since no one element is in all three sets, but both $A \subseteq \bar{B}$ and $A \subseteq \bar{C}$ are false because $1 \in A \wedge 1 \in B$ and $2 \in A \wedge 2 \in C$.