LA Session - Logic Solutions

1) Fill out the following truth table:

p	q	r	$(\neg q \vee r)$	$\neg(\neg q \vee r)$	$(p \wedge \neg(\neg q \vee r))$	$p \wedge q$	$(p \wedge \neg(\neg q \vee r)) \vee (p \wedge q)$
F	F	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	T	F	F	T	F	F	F
F	T	T	T	F	F	F	F
T	F	F	T	F	F	F	F
T	F	T	T	F	F	F	F
T	T	F	F	T	T	T	T
T	T	T	T	F	F	T	T

2) Use the laws of logic to prove the two following expressions are logically equivalent:

(a) $p \wedge (\overline{\bar{r} \wedge \bar{p}} \wedge (\bar{p} \rightarrow s))$ (b) p

Solution:

Starting with (a) we will attempt to simplify it so as to arrive at (b).

$p \wedge (\overline{\bar{r} \wedge \bar{p}} \wedge (\bar{p} \rightarrow s))$

$p \wedge ((r \vee p) \wedge (\bar{p} \rightarrow s))$ ⇔ De Morgan's Law, and two applications of Double Negation

$p \wedge ((r \vee p) \wedge (\bar{\bar{p}} \vee s))$ ⇔ Law of Implication

$p \wedge ((r \vee p) \wedge (p \vee s))$ ⇔ Double Negation

$p \wedge ((p \vee r) \wedge (p \vee s))$ ⇔ Commutative Law

$p \wedge (p \vee (r \wedge s))$ ⇔ Distributive Law

p ⇔ Law of Absorption

Note: There are many, many ways to show the equivalence of these two expressions.

3) Use the laws of implication to complete the following argument:

1. $p \vee q$
2. $p \rightarrow s$
3. $q \rightarrow r$
4. $\bar{r}$
5. $s \rightarrow (t \wedge u)$

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t

(Note: Numbers 1 – 5 are the premises and proposition below is what is to be deduced from those premises.)

Solution:

1. $q \rightarrow r$ $(premise)$
2. $\bar{r}$ $(premise)$
3. $\bar{q}$ $(Modus\ Tollens\ with\ 1\ and\ 2)$
4. $(p \vee q)$ (premise)
5. p $(disjunctive\ elimination\ using\ 3\ since\ \bar{q}\ \ is\ true\ therefore\ q\ is\ false)$
6. $p \rightarrow s$ $(premise)$
7. s $(Modus\ Ponens\ with\ 5\ and\ 6)$
8. $s \rightarrow (t \wedge \mathrm{u})$ (premise)
9. $t \wedge u$ $(Modus\ Ponens\ with\ 7\ and\ 8)$
10. t $(Rule\ of\ Conjunctive\ Simplification)$

4) Prove or disprove the following statements. Use the domain of real numbers for each variable, unless otherwise stated in the problem.

Solution:

a) $\exists x[x^2 + 4x + 3 > 2x^2 + 5x + 6]$

$\exists x[x^2 + x + 3 < 0]$ (bring all terms to one side of the inequality)

$$\exists x\left[(x)^2 + 2x(1)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 3 < 0\right]\ (using\ completing\ the\ square, factorize\ the\ equation)$$

$$\exists x\left[\left(x + \frac{1}{2}\right)^2 + \frac{11}{4} < 0\right]$$

$$\exists x\left[\left(x + \frac{1}{2}\right)^2 < -\frac{11}{4}\right]$$

Since there are no real solutions to the following inequality, we can conclude that the given statement is False.

b) $\forall x \in Z^+[(x+1)^2 > x^2]$
$\forall x \in Z^+[(x^2 + 2x + 1) - x^2 > 0]$
$\forall x \in Z^+[2x + 1 > 0]$
$\forall x \in Z^+\left[x > -\frac{1}{2}\right]$

The above statement holds true for all positive values of x therefore, (b) is True.

c) $\exists x[\forall y(y^2 - 3xy + x^2 \geq 0)]$
The statement holds true when x = 0.
$\exists x[\forall y(y^2 - 3(0)y + (0)^2 \geq 0)]$
$\exists x[\forall y(y^2 \geq 0)]$
$y^2 \geq 0 \; holds\ true\ for\ all\ real\ values\ of\ y$ therefore, (c) is True.

d) $\exists x[\forall y(y^2 - 3xy + x^2 > 0]$
We can attempt this problem by completing the square.
$\exists x[\forall y(y^2 - 3xy + x^2 > 0]$
$$\exists x[\forall y((y)^2 - 2y\frac{3x}{2} + \left(\frac{3x}{2}\right)^2 - \left(\frac{3x}{2}\right)^2 + x^2 > 0)]$$
$$\exists x[\forall y(\left(y - \frac{3x}{2}\right)^2 - \frac{5x^2}{4} > 0)]$$

The minimum point for the above expression is $\left(\frac{3x}{2}, -\frac{5x^2}{4}\right)$. $-\frac{5x^2}{4} > 0$ is not true for any value of x therefore we conclude that (d) is indeed False.

e) $\exists x \in Z^+[\exists y \in Z^+(x^y\ has\ precisely\ 7\ divisors)]$
In order to prove this statement, we need a pair of x and y that satisfy the given condition. When x = 64 and y = 1, the divisors are then: 1, 2, 4, 8, 16, 32, 64. Since there is a total of 7 divisors, (e) is True. (Note: any prime number raised to the sixth power will suffice as a valid case to prove the assertion.)