<u>Polynomials and Roots of Polynomials</u>

The standard form of a polynomial is something like $f(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$, where each of the c_i's are a given constant.

To add or subtract polynomials, simply add/subtract the appropriate link terms:

$(3x^3 - 2x^2 + 4) + (x^4 + 7x^2 - 3x + 1) = x^4 + 3x^3 + 5x^2 - 3x + 5$

$(3x^3 - 2x^2 + 4) - (x^4 + 7x^2 - 3x + 1) = -x^4 + 3x^3 - 9x^2 + 3x + 3$

To multiply two polynomials, you have to multiply each combination of terms (choosing one term from the first polynomial and one term from the second polynomial), and then combine like terms. Here is a graphic for the following product:

$$(3x^3 - 2x^2 + 4)(x^4 + 7x^2 - 3x + 1)$$

term\term	x^4	$7x^2$	$-3x$	1
$3x^3$	$3x^7$	$21x^5$	$-9x^4$	$3x^3$
$-2x^2$	$-2x^6$	$-14x^4$	$6x^3$	$-2x^2$
4	$34x^4$	$28x^2$	$-12x$	4

Like terms are highlighted, so we can see that

$$(3x^3 - 2x^2 + 4)(x^4 + 7x^2 - 3x + 1) = 3x^7 - 2x^6 + 21x^5 + 11x^4 + 9x^3 + 26x^2 - 12x + 4$$

By hand, it's typical to order the terms as they are in the table, from top to bottom and then left to right within rows.

To divide a polynomial by a monomial (something of the form x – a), we can use synthetic division. If we're dividing by a polynomial of degree greater than 1, then we have to write out something similar to long division. For brevity's sake, this long division will be skipped.

Here is how to use synthetic division to calculate $x^4 + 7x^2 - 3x + 1$ divided by $x - 3$:

Write out all coefficients of the polynomial into which we are dividing, including 0 coefficients:

1 0 7 -3 1

First, drop the most significant coefficient down, and then multiply it by the root delineated by the term we are dividing by (thus, if we are dividing by $x - a$, then we multiply by a.), and then put that below the next term:

1	0	7	-3	1
	3			
1				

From, here we'll add the two terms in column 2 and put the result right below. Then we repeat the same steps into column 3, then 4, etc. Here is what we get:

1	0	7	-3	1
	3	9	48	135
1	3	16	45	136

This means that: $\frac{x^4+7x^2-3x+1}{x-3} = x^3 + 3x^2 + 16x + 45 + \frac{136}{x-3}$.

The quotient is a polynomial with degree one less with the coefficients on the bottom row, and the last term on that row is the remainder, which we write in the numerator of a fraction with the denominator equal to the term we're dividing by.

Also, the remainder theorem tells us that this remainder, 136, equals f(a), in this case f(3).

Sure enough, $f(3) = 3^4 + 7(3^2) – 3(3) + 1 = 81 + 63 – 9 + 1 = 136$.

If this computation yields a remainder of 0, then we've found a root of the original polynomial. The rational root theorem says that all rational roots of a polynomial can be expressed as $\pm\frac{p}{q}$, where p is a divisor of c_0, the constant coefficient of the polynomial and q is a divisor of c_n, the coefficient of the most significant term.

Viewing a Polynomial through its roots

Another way of expressing a polynomial is via its roots. Let's say a polynomial has roots $r_1, r_2, \ldots, r_n$ and a leading coefficient of 1. Note that some of the r_i's can be the same as that simply means that the multiplicity of a root (how many times a root appears as a root), can be more than 1. Then, we can express the polynomial as follows:

$$(x - r_1)(x - r_2) \ldots (x - r_n)$$

Now, let's see what happens when we equate this view with the coefficient view of a polynomial for a quadratic (degree 2):

$$(x - r_1)(x - r_2) = x^2 + c_1x + c_0$$

$$x^2 - (r_1 + r_2)x + r_1r_2 = x^2 + c_1x + c_0$$

In order for these quadratics to be equal, we must have:

$$c_1 = -(r_1 + r_2)$$
$$c_0 = r_1r_2$$

Thus, without factoring a quadratic or using the quadratic equation, we have some partial knowledge about the roots of the quadratic just by looking at the coefficients.

If there is a leading coefficient of c_2, then we have

$$\frac{c_1}{c_2} = -(r_1 + r_2)$$
$$\frac{c_0}{c_2} = r_1r_2$$

We can generalize the results to any degree of polynomial. In general, we'll have the following for a polynomial of degree n:

$$r_1 + r_2 + \cdots + r_n = -\frac{c_{n-1}}{c_n}$$

$$r_1 r_2 \ldots r_n = (-1)^n \times \frac{c_0}{c_n}$$

In addition, we can equate all of the other coefficients and get equations for the sum of the products of roots by 2 items, 3 items, etc. That will be beyond the scope of this lecture though.

But, we can quickly look at the relationships for a cubic at least:

$$r_1 + r_2 + r_3 = -\frac{c_2}{c_3}$$

$$r_1 r_2 + r_1 r_3 + r_2 r_3 = \frac{c_1}{c_3}$$

$$r_1 r_2 r_3 = -\frac{c_0}{c_3}$$

Finally, let's take a look at a couple problems that show how we can utilize this information:

1) What is the sum of the roots of the equation $3x^2 - 15x + 7 = 0$?

Solution
The sum of the roots of the quadratic $ax^2 + bx + c = 0$ is -b/a. Thus, for this equation the sum of the roots is $\boldsymbol{-\left(-\frac{15}{3}\right) = 5.}$

2) What is the product of the roots of the equation $x^2 + 8x + 10 = 0$?

Solution
The product of the roots of the quadratic $ax^2 + bx + c = 0$ is c/a. Thus, for this equation the sum of the roots is $\frac{10}{1} = \mathbf{10}$.

3) What is the following product: $(2x^4 + x - 7)(3x^3 + 4x + 6)$?

Solution
$(2x^4 + x - 7)(3x^3 + 4x + 6) =$ $(6x^7 + 8x^5 + 12x^4) +$
$(3x^4 + 4x^2 + 6x) +$
$(-21x^3 - 28x - 42)$

$= \mathbf{6x^7 + 8x^5 + 15x^4 - 21x^3 + 4x^2 - 22x - 42}$

4) Let r and s be the roots of the equation $x^2 - 4x + 2 = 0$. What is the quadratic equation with leading coefficient 1 with roots r^2 and s^2?

Solution

If an equation has roots r^2 and s^2, then the sum of those roots is $r^2 + s^2$ and the product of those roots is r^2s^2. If we can find the values of these two expressions, we can recreate the desired quadratic.

Using the given information, we see that:

$r + s = 4$
$rs = 2$

So, take the first equation and square it and simplify:

$(4)^2 = (r + s)^2 = r^2 + 2rs + s^2 = r^2 + 2(2) + s^2$

Now, solve for $r^2 + s^2$:

$16 = r^2 + 4 + s^2$
$12 = r^2 + s^2$

Since $rs = 2$, it follows that $(rs)^2 = r^2s^2 = (2)^2 = 4$.

Thus, the desired quadratic is $\mathbf{x^2 - 12x + 4 = 0}$**.**

5) If $x + \frac{1}{x} = 10$, what is $x^3 + \frac{1}{x^3}$?

Solution

Cube the given expression:

$$(x + \frac{1}{x})^3 = 10^3$$

$$x^3 + 3x + \frac{3}{x} + \frac{1}{x^3} = 1000$$

$$x^3 + 3(x + \frac{1}{x}) + \frac{1}{x^3} = 1000$$

But we know what $x + \frac{1}{x}$ is, so let's substitute for that:

$$x^3 + 3(10) + \frac{1}{x^3} = 1000$$

$$x^3 + \frac{1}{x^3} = \mathbf{970}$$