Logarithms

In short, the logarithm function is the inverse of the exponential function. We learn in grade school that multiplication is repeated addition and exponentiation is repeated multiplication. (So, there are two levels of abstraction here from the intuitive idea of addition.) Thus, something like

$b^n = b \times b \times b \ldots \times b$ (listing b exactly n times)

literally means, "multiply b by itself n times." So, "solving" this problem would be like asking, "What number do you get when you multiply b by itself n times?"

Now, let's say we knew the answer to the exponentiation problem was a, giving us the statement:

$b^n = a$

Someone could turn the question around and ask, "How many times must I multiply b by itself to obtain a?" If I were given the equation above, by definition, the answer to that question is n. This is precisely how we define the logarithm. Given the exponent statement above, we define

$\log_b a = n$

Thus, if you are given any logarithm statement such as the one above, by definition, we know that $b^n = a$. Similarly, if we are given this exponent statement, by definition of the logarithm, we know that $\log_b a = n$.

Let's derive some log rules, assuming that we already have knowledge of some exponent rules:

<u>Log Addition</u>
Let $c^x = A$ and $c^y = B$. By definition of logarithm, these two statements are equivalent to saying

$x = \log_c A$ and $y = \log_c B$.

Now, consider the product AB:

$AB = (c^x)(c^y) = c^{x+y}$

Now, convert this exponent statement, $AB = c^{x+y}$ to a corresponding log statement:

$\log_c AB = x + y$

Now, just substitute by definition for x and y:

$\log_c AB = \log_c A + \log_c B$

So, the sum of two logs (using the same base), is a single log with the same base that is the product of the two values. This looks weird until you plug some numbers into it:

Consider $c = 2$, $A = 2^3$ and $B = 2^4$

$\log_2 (2^3)(2^4)$ means how many times do I have to multiply two, to obtain the product of multiplying 2 three times and then multiplying 2 four times. Just by definition, it seems pretty obvious that the answer is 3 + 4, because the total times we multiplied 2 is 7, the sum of the number of times we multiplied it in both values. Now, we can express 3, by definition as $\log_2 2^3$ and we can represent 4, by definition as $\log_2 2^4$.

<u>Log Power Rule</u>

Consider an expression of the form $\log_b A^n$. To more fully appreciate an expression of this form, let's first plug in $n = 2$:

$\log_b A^2 = \log_b AA = \log_b A + \log_b A = 2\log_b A$.

So, now we know that $\log_b A^2 = 2\log_b A$.

Now, let's plug in $n = 3$:

$\log_b A^3 = \log_b A^2A = 2\log_b A + \log_b A = 3\log_b A$.

Hopefully, you get the picture by now. When we raise A to the power n, we are saying, "multiply A by itself n times." But if $\log_b A$ represents how many times we have to multiply b by itself to obtain A, then multiplying this by n will represent how many times we have to multiply b by itself to get n copies of A. Hopefully this illustration with $b = 2$, $A = 2^3$ and $n = 4$ will make it clear:

$\log_2 (2^3)^4 = \log_2 (2^3 2^3 2^3 2^3) = \log_2 2^3 + \log_2 2^3 2^3 2^3 = \log_2 2^3 + \log_2 2^3 + \log_2 2^3 2^3 =$

$\log_2 2^3 + \log_2 2^3 + \log_2 2^3 + \log_2 2^3 = 4 \log_2 2^3$

Thus, in general, the rule is $\log_b A^n = n\log_b A$.

<u>Subtraction Rule</u>

We can do roughly the same proof as the addition proof (but we divide instead of multiply) to obtain this log rule:

$\log_c A - \log_c B = \log_c \frac{A}{B}$.

<u>Log Change Base Rule</u>

This is perhaps the most important log rule. If we have logs of two different bases, none of the previous rules apply. Thus, it becomes necessary to be able to change the base of a logarithm so that we can transform a calculation into an equivalent form where all of the logs are the same base. Here is the rule:

$$log_c A = \frac{log_b A}{log_b c}$$

Let's prove this one. Let x = $\log_c A$ and let y = $\log_b c$ and let z = $\log_b A$. The three equivalent exponent equations are:

$A = c^x$, $c = b^y$ and $b^z = A$

Equate the two expressions for A in the first and last equations:

$c^x = b^z$

but we have that $c = b^y$, so just substitute that in for c:

$(b^y)^x = b^z$

$b^{yx} = b^z$

It follows that yx = z. Solve for x to obtain:

$x = \frac{z}{y}$

Substitute in for x, y and z and we have:

$log_c A = \frac{log_b A}{log_b c}$

Here is a quick example on changing a base from 4 to 2:

$\log_4 x = \frac{log_2 x}{log_2 4} = \frac{log_2 x}{2}$

We will see this rule used often in problems where the given logs are in different bases.

<u>$f^{-1}(f(x)) = x$, as applied to logs</u>
Most students easily see the following

$$log_b b^x = x$$

This is like saying, "What power do I have to raise b to, in order to obtain b^x?" It's a self-referential question! The answer is in the question, it's x. Most students don't have trouble with this not because they see this as self-referential, but because of the power rule and knowing that $\log_b b = 1$.

Now, let's flip the order of applying the functions to get:

$$b^{log_b x} = x$$

This says, "What answer do I get when I raise b to the power such that if I were to raise b to that power, I would get x?" Again, though that's wordy, the answer is in the question: x.

But, because the power rule doesn't seem to be in there, students have trouble applying this rule. If you see both this rule and the previous one as being self-referential, because both involve applying a function and then its inverse, then you should be able to understand and apply both rules easily. In some sense, "The b's cancel!" But…beware of getting too giddy canceling b's. make sure that what you are plugging into makes sense with the definitions provided.

One Other Trick

Here is one other rule that is pretty interesting and allows us to "exchange" the base of an exponential statement:

$$a^{log_b c} = c^{log_b a}$$

Here is the proof. Let's just start with the left hand side and call this quantity x:

$$x = a^{log_b c}$$

Now, rewrite this as a log statement with base a, as it's currently an exponent statement with base a:

$$log_a x = log_b c$$

Now, let's use a common base. Since which common base we use doesn't matter, I'll use the base of the natural logarithm, e. This base is very common in mathematics so the expression $log_e x$ has the shorthand lnx. I'll use this shorthand to save some writing. We will change the base of both logs to e:

$$\frac{lnx}{lna} = \frac{lnc}{lnb}$$

Swap the terms for ln a and ln c in this equality to get:

$$\frac{lnx}{lnc} = \frac{lna}{lnb}$$

Then, use the common base rule backwards to get back to logs with different bases:

$$log_c x = log_b a$$

Finally, rewrite this statement as an equivalent exponential statement using c as the base to get:

$$c^{log_b a} = x$$

Of course, we had started with $x = a^{log_b c}$. It follows that $a^{log_b c} = c^{log_b a}$, as desired.

Incorrect Steps Students OftenTake

One of the biggest issues with logs is that students often make incorrect steps. I suspect they do this because (a) they've forgotten the exact valid formulas and the incorrect steps look pretty close to the correct ones, (b) they do NOT expand out the meaning of a log or exponent to verify if their step makes mathematical sense.

Here are three common steps that students take that are not correct:

$$log_c(A + B) \neq log_cA + log_cB$$

$$log_c(\frac{A}{B}) \neq \frac{log_cA}{log_cB}$$

$$[log_cA]^n \neq nlog_cA$$

Notice that one of the two sides of each of these questions IS equal to one of the sides of one of the correct rules, BUT, the other side is NOT equal. To debunk each of these, let's plug in some numbers and the conceptually explain why the two sides aren't usually equal:

Try c = 2, A = 4 and B = 4 for the first rule.

LHS = $log_2(4 + 4) = log_28 = 3$, since $2^3 = 8$

RHS = $log_24 + log_24 = 2 + 2 = 4$, since $2^2 = 4$.

The main problem here is that when I add two values that has nothing to do with how many times I need to multiply the base to get that sum. Notice that if I were to have plugged in A = 8 and B = 2, then the sum 10, isn't a perfect power of 2, even though both 8 and 2 are. So, the left hand side would be some irrational value in between 3 and 4, while the right hand side would be an integer since both components are integers.

Try the same set of values for the second equation:

LHS = $log_2\left(\frac{4}{4}\right) = log_21 = 0$, since $2^0 = 1$.

RHS = $\frac{log_24}{log_24} = \frac{2}{2} = 1$

The inequivalence becomes even more obvious if we start plugging in larger numbers for B, say A = 4 and B = 16. In this scenario, the LHS becomes negative but the RHS can't be negative since both components are positive. In general, there is no good rule if I am dividing two logs. The LHS side here corresponds properly to the subtraction rule.

Finally, when I raise a log to a power, that means multiplying the log by itself over and over again, it has NOTHING to do with multiplying A by itself over and over again. As a simple example, try c = A = 2 and n = 1000.

LHS = $[log_2 2]^{1000} = 1^{1000} = 1$

RHS = $1000 log_2 2 = 1000(1) = 1000$

So in this example, the RHS is much bigger than the LHS, but I can easily get the opposite behavior by plugging in c = 2, A = 4 and n = 1000:

LHS = $[log_2 4]^{1000} = 2^{1000}$

RHS = $1000 log_2 4 = 1000(2) = 2000$

2^{1000} has 309 digits while 2000 only has 4 digits!!!

Historical Sidenote

Just these rules, greatly advanced astronomy in the 17th century. Without calculators, multiplication by hand was extremely tedious. With the advent of logs, multiplication became easier. Say I wanted to multiply the two following numbers

$(2.345 \times 10^5) \times (3.152 \times 10^3)$

using log rules what we do is take the log of this whole expression (I'll use base 10):

$\log_{10} (2.345 \times 10^5) \times (3.152 \times 10^3) = \log_{10} 2.345 + \log_{10} 10^5 + \log_{10} 3.152 + \log_{10} 10^3$

$= \log_{10} 2.345 + 5 + \log_{10} 3.152 + 3$, via log definition

$= 8 + \log_{10} 2.345 + \log_{10} 3.152$

Now, when we get to this part, it turns out that several people, including the Scottish man John Napier, spent many years of their lives creating logarithm tables. These tables had entries for the logs of many values within some specified range. (For this example, assume that the table with from 1 to 10.) Thus, at this point, the person making the calculation would look up the values of $\log_{10} 2.345$ and $\log_{10} 3.152$, obtaining .3701 and .4986. Then he or she would add these two numbers to get 0.8687. Finally, there would be a different inverse table that would give the value of $10^{.8687}$, which is roughly 7.391. It would follow that the desired product would be about 739,100,000. (The tables had more decimal places than I've given in this example. For this example, the exact product is 739,144,000.

Practice Log Problems

Now, we will apply these rules to solve some problems!

1) Solve for x in the following equation: $\boldsymbol{log_2\left(log_2(log_2(x))\right) = 2.}$

This one isn't too bad. Just convert the given log statement to an exponential to get:

$$\left(log_2(log_2(x))\right) = 2^2 = 4$$

Now, we just repeat the process to get:

$$log_2(x) = 2^4 = 16$$

And…one more time, to solve for x:

$$x = 2^{16} = 65536$$

2) The sequence $\log_{12} 162$, $\log_{12} x$, $\log_{12} y$, $\log_{12} z$, $\log_{12} 1250$ is an arithmetic progression. What is x?

In an arithmetic sequence, the difference between consecutive terms is the same. Let d be this common difference. The given sequence has five terms. Thus, the difference between the first two terms is d and the difference between the first and last terms is 4d. (This is because we add d four times to get from the first term to the fifth.) This gives us the following two equations:

$\log_{12} x - \log_{12} 162 = d$

$\log_{12} 1250 - \log_{12} 162 = 4d$

The second equation has only one variable, so let's focus on this one. Just use the log subtraction rule to get:

$$log_{12}\frac{1250}{162} = 4d$$

Let's divide both the numerator and denominator by 2 to reveal a better way of looking at that log:

$$log_{12}\frac{625}{81} = 4d$$

$$log_{12}\frac{5^4}{3^4} = 4d$$

$$log_{12}(\frac{5}{3})^4 = 4d$$

$$4log_{12}\frac{5}{3} = 4d$$

So, the power rule came in handy and now we can solve for d:

$$d = log_{12}\frac{5}{3}$$

Plugging into the first equation, we see:

$$log_{12}x - log_{12}162 = log_{12}\frac{5}{3}$$
$$log_{12}\frac{x}{162} = log_{12}\frac{5}{3}$$

Now, since both sides are log of something and equal to one another, and log is a monotonically increasing function, we know that both of the things we are taking log of are equal:

$$\frac{x}{162} = \frac{5}{3}$$

$$x = 270$$

This is the "proper" way to solve the problem. A student with an intuitive understanding would see that if we took the log of terms in a geometric series we would get an arithmetic series, so that the terms 162, x, y, z and 1250 form a geometric series and then would then solve for the common ratio and get 5/3 and multiply 162 by that common ratio.

3) What is the value of $(\mathbf{81}^{log_3 \mathbf{1234}})^{\mathbf{0.25}}$?

This looks daunting, but let's use some exponent rules first and go from there. 81 can be re-expressed as 3^4:

$$(81^{log_3 1234})^{0.25} = (3^{4log_3 1234})^{0.25}$$

Now, the exponent rule here is to multiply the 0.25 through the exponent to get:

$$(3^{4log_3 1234})^{0.25} = 3^{4(0.25)log_3 1234}$$

Now, we can just multiply 4 and 0.25. Convenient!

$$3^{4(0.25)log_3 1234} = 3^{log_3 1234}$$

This should look familiar; it's our self-referential question. The answer is just 1234.

$$3^{log_3 1234} = 1234$$

4) Determine the ordered pair, (a, b), that satisfies the following pair of equations:

$$\boldsymbol{log_{16}a^2 + log_8 b^3 = 11}$$

$$\boldsymbol{log_8 a^6 + log_{16}b^{10} = 32}$$

As we analyze this problem, we see that we must change bases. Since 8 and 16 are both powers of 2, 2 is a good choice:

$$\frac{log_2 a^2}{log_2 16} + \frac{log_2 b^3}{log_2 8} = 11$$

$$\frac{log_2 a^6}{log_2 8} + \frac{log_2 b^{10}}{log_2 16} = 32$$

Solve for the logs with constants only:

$$\frac{log_2 a^2}{4} + \frac{log_2 b^3}{3} = 11$$

$$\frac{log_2 a^6}{3} + \frac{log_2 b^{10}}{4} = 32$$

Now, use the power rule:

$$\frac{2log_2 a}{4} + \frac{3log_2 b}{3} = 11$$

$$\frac{6log_2 a}{3} + \frac{10log_2 b}{4} = 32$$

At this point we see the expressions $log_2 a$ and $log_2 b$. A very helpful idea when we have complicated repeated expressions in equations is to create a new variable to stand for them and rewrite the equations with those new variables. Let x = $log_2 a$ and y = $log_2 b$. Simplify fractions and now we get:

$$\frac{x}{2} + y = 11$$

$$2x + \frac{5y}{2} = 32$$

Now, we can easily see that this is just a regular system of 2 linear equations!

Use any method to solve. We can rewrite the first equation and solve for x, yielding x = 22 - 2y. Then we can substitute for x in the second equation:

$$2(22 - 2y) + \frac{5y}{2} = 32$$

Multiply everything by 2:

$$4(22 - 2y) + 5y = 64$$

$$88 - 8y + 5y = 64$$

$$88 - 3y = 64$$

$$3y = 24$$

$$y = 8$$

It follows that x = 22 - 2(8) = 6.

Now that we have x and y, we can find a and b:

$x = 6 = log_2 a$, so a = 2^6 = 64

$y = 8 = log_2 b$, so b = 2^8 = 256

5) What is the following sum: $\sum_{i=1}^{89} log_{10}(\tan i°)$?

This is again a problem that looks daunting. Also, it does require some trig information. Specifically that tan x° and tan (90-x)° are reciprocals of one another. This information can easily be seen in a right triangle with acute angles x° and (90-x)°. Let's say the side opposite to angle x° has length a and the side opposite to angle (90-x)° has length b. Then, by definition of tangent, we have $tanx° = \frac{a}{b}$ and $\tan(90 - x)° = \frac{b}{a}$. Multiplying these two fractions gives an answer of 1.

Now, when we look at this sum, let's "expand" it. It says find:

$$log_{10}(\tan 1°) + log_{10}(\tan 2°) + \cdots + log_{10}(\tan 88°) + log_{10}(\tan 89°)$$

If we pair up the first and last terms, and we use the fact about the product of the tangents of complementary angles we just derived, we have:

$$log_{10}(\tan 1°) + log_{10}(\tan 89°) = log_{10}((\tan 1°)(\tan 89°)) = log_{10}1 = 0$$

Now, we can continue to form pairs of angles for x = 2, 3, …, 44, each of which, when we sum the corresponding two terms of the sum, will sum to 0.

This leaves one unmatched term:

$$log_{10}(\tan 45°) = log_{10}1 = 0$$

But, as we can see, this is no problem.

It follows that the value of the sum is 0. Notice that the base of the logarithm never really came into play in this problem. This sum is 0 regardless of the base chosen for the log.

6) Let A, B and C be three positive integers such that gcd(A, B, C) = 1 and

$$\boldsymbol{A log_{200} 5 + B log_{200} 2 = C}$$

What are A, B and C?

We first use the power rule backwards:

$$log_{200}5^A + log_{200}2^B = C$$

Then, let's do the log sum rule backwards:

$$log_{200}5^A 2^B = C$$

Now, we can write the corresponding exponent statement, by definition:

$$200^C = 5^A 2^B$$

Prime factorize 200…

$$(2^3 5^2)^C = 5^A 2^B$$

$$2^{3C}5^{2C} = 5^{A}2^{B}$$

For these to be equal, we must equate coefficients. Now, we have

A = 2C and B = 3C.

We can set C = 1, A = 2 and B = 3 and this is the unique positive integer solution where A, B and C don't share any common factors.

7) What is the value of the following sum: $\sum_{i=2}^{100} \frac{1}{log_i 100!}$

This looks a bit like the tangent problem in that the terms are scary, but maybe some nice simplification will occur. The big problem is that the log bases are all different. We might want the bases to all be the same. Let's just change a base to a common base. It doesn't really matter which one, as we will soon see. I'll just call it b for now:

$$\sum_{i=2}^{100} \frac{1}{log_i 100!} = \sum_{i=2}^{100} \frac{1}{\frac{log_b 100!}{log_b i}} = \sum_{i=2}^{100} \frac{log_b i}{log_b 100!}$$

Now, the big next step is realizing that we can use the change of base log rule BACKWARDS!!! To simplify the sum:

$$\sum_{i=2}^{100} \frac{log_b i}{log_b 100!} = \sum_{i=2}^{100} log_{100!} i$$

This is really shorthand for $log_{100!}2 + log_{100!}3 + \cdots + log_{100!}99 + log_{100!}100$.

Since our bases are all the same, we can just use the log addition rule over and over again, to get:

$$\sum_{i=2}^{100} log_{100!} i = log_{100!}(2 \times 3 \times \ldots \times 99 \times 100)$$

But…WAIT…that product inside the log is just a factorial…in fact, it's 100! So we now have

$$\sum_{i=2}^{100} log_{100!} i = log_{100!}(2 \times 3 \times \ldots \times 99 \times 100) = log_{100!}100! = 1$$

How elegant!!!