Integer Factorization in Problems, Algebraic Expressions

Restrictions based on integers

In many problems the intended solution(s) are restricted by being integers or positive integers. It's fairly easy to see that this restriction can have a big effect on the possibilities for answers. Consider the following query:

Find all possible solutions to the equation $x + y = 3$, where x and y are positive integers with $x < y$.

If we don't restrict x and y to be positive integers, it's clear there are an infinite number of solutions. (Namely, we can plug in any real number for x, and then we can assign $y = 3 – x$.) However, with this restriction, there are only two possible solutions for the ordered pair (x, y). These are (1, 2) and (2, 1). Since we are given that $x < y$, we know there is a single unique solution: **(1, 2).**

Many math problems add in these sorts of restrictions in just a few words or symbols, and it's only with these restrictions that a unique answer can be found.

Restrictions based on integer products

Now, let's consider the product of two integers, which is highlighted in this problem:

Let x and y be positive integers with $xy = 625$ and $1 < x < y$. Find x and y.

In questions of this nature, we want to express the right hand side as the product of 2 integers. For many cases of specific values, there aren't too many ways to do this. Let's try for 625:

625 = 1 x 625
625 = 5 x 125
625 = 25 x 25

For our purposes, the first product doesn't work because we require x to be greater than 1. In addition, the last product doesn't work because $x < y$.
It follows that there's a unique solution: **x = 5 and y = 125.**

Use of Prime Factorization

Let's consider the use of prime factorization in a problem:

Find the smallest integer x such that $90x = y^4$, where y is another positive integer. First, let's prime factorize $90 = 2 \times 3^2 \times 5$. If we want 90x to be a perfect fourth power, then $90x = 2^a3^b5^c$, with a, b and c being positive multiples of 4, because we know those exponents in the product are greater than 0. (Since we are trying to minimize x, there's no point in introducing new prime factors, such as a multiplicative factor of 7^4.) It follows that the smallest solution for x corresponds to $y = 2 \times 3 \times 5 = 30$, and that $x = 2^3 \times 3^2 \times 5^3$ = **9000.**

Factoring Algebraic Expressions

Let's say we're looking for all positive integer solutions (x, y) to

$xy + 2x + 3y = 215$

It's tempting to start plugging in values for x and solving for y. While this will work, recognizing an opportunity to factor is key. If we look at the expression on the left, it looks "close" to the form

$(x + a)(y + b)$

for some constants a and b. Specifically, a = 3 and b = 2. Let's try it out:

$(x + 3)(y + 2) = xy + 2x + 3y + 6$

In fact, to get that factorization, all we have to do is add 6 to both sides! Let's do it!

$xy + 2x + 3y + 6 = 215 + 6$
$(x + 3)(y + 2) = 221$
$(x + 3)(y + 2) = 13 \times 17$

There are only two ways to match up these factors. (Notice that 13 and 17 are prime so the only other way we could have written it is 1 x 221, but that would make either x or y negative, which isn't allowed.) So we could have:

$x + 3 = 13 \rightarrow x = 10$	OR	$x + 3 = 17 \rightarrow x = 14$
$y + 2 = 17 \rightarrow y = 15$		$y + 2 = 13 \rightarrow y = 11$

Notice here, had we restricted x and y further by saying that $x < y$, we would have only one unique solution.

In general, expressions of the form $(x + a)(y + b) = xy + ax + by + ab$, so if you see terms that fit the first 3 of this right hand side, it's easy in an equation to add ab to both sides and factorize the expression. With a little creativity, we can also see different forms of this factorization, with coefficients in front of x or y.

Notice what happens with three terms and coefficients of 1:

$(x + 1)(y + 1)(z + 1) = xyz + xy + xz + yz + x + y + z + 1$

One other thing to note here is that if x, y and z are roots of a cubic equation, then the expressions

$xyz, \quad xy + xz + yz, \quad x + y + z$, and 1

are all related to the coefficients of the cubic. (See roots of poly notes.)

Common Factorization Formulas
$x^2 - y^2 = (x - y)(x + y)$
$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$
$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$
$x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \ldots xy^{n-2} + y^{n-1})$

Fact About Number of Divisors of an Integer
From one of our previous examples, we see that the number of divisors of an integer typically come in pairs (as we express the product of the number). Let's look at a couple examples:

36 = 1 x 36	60 = 1 x 60
= 2 x 18	60 = 2 x 30
= 3 x 12	60 = 3 x 20
= 4 x 9	60 = 4 x 15
= 6 x 6	60 = 5 x 12
	60 = 6 x 10

As we can see, as we write an integer n as a product of two integers, d and n/d, these numbers are unique making the number of divisors of an integer even, **UNLESS** d = n/d for one of the divisors d. In that case, the same number is written down twice, but should be counted only once, since it's the same number. In order for d = n/d, it must be the case that $n = d^2$, which means that n is a perfect square.

Thus, we've shown (informally), that the number of divisors of an integer, n, is odd **if and only if** n is a perfect square. (Thus, the number of divisors is even with n is not a perfect square.)

Fact About Prime Numbers
A simple fact about prime numbers is that the only even one is 2. Thus, if we know that the sum of four distinct prime numbers is 31, this restricts what the numbers could be. Remember that two odd numbers add to an even number, so four odd numbers would add to an even number. Since 31 is odd, this means that of the prime numbers, 3 must be odd and 1 must be even. This means one of the four prime numbers is 2 and we're looking for three unique odd primes that add to 29. List the odd primes to 23:

3, 5, 7, 11, 13, 17, 19, 23

(Note that 29 is obviously too big because we need to have all three sum of 29.)

It's pretty easy to see that 23 is too big, since 3 + 5 + 23 > 29

Let's try listing solutions from the largest integer to the smallest:

19 pair with 3 and 7 (no other pair works because 5 and 5 aren't distinct)
17 pair with 5 and 7
13 pair with 5 and 11

Note that 11 + 7 + 5 < 29, so all solutions must have a prime greater than 11.

Thus, our only solutions for 4 distinct primes that add to 31 are

(2, 3, 7, 19),   (2, 5, 7, 17),   (2, 5, 11, 13)

Prime Factorizing n!

The nice thing about prime factorizing n! for small n is that n! is naturally factorized:

n! = 1 x 2 x 3 x … x n

If n is large enough and we don't want to prime factorize each integer upto n, then we can use the method taught in the lecture notes (NumTheory03) to determine the number of times a particular prime number divides evenly into n!