**Function Practice Problems from old AHSME/AMC/AIME**

1) (1993 AHSME #12) If , for all x > 0, what is 2f(x)?

**Solution**

Plug in x/2: . Thus, 2f(x) = .

2) (1995 AHSME #14) If f(x) = ax4 – bx2 + x + 5 and f(-3) = 2, then what is f(3)?

**Solution**

f(-3) = a(-3)4 – b(-3)2 + -3 + 5 = 2, so 81a - 9b + 2 = 2 → 81a-9b = 0

f(3) = a(3)4 – b(-3)2 + 3 + 5 = 81a – 9b + 8 = 0 + 8 = 8

**f(3) = 8.**

3) (1988 AHSME #15) If a and b are integers such that x2 – x – 1 is a factor of ax3 + bx2 + 1, then what is b?

**Solution**

For some function f(x), we have: ax3 + bx2 + 1 = (x2 – x – 1)f(x)

Notice that for the first term to match, f(x) = ax + c, for constants a and c:

ax3 + bx2 + 1 = (x2 – x – 1)(ax+c)

ax3 + bx2 + 1 = ax3 – ax2 – ax + cx2 – cx – c

ax3 + bx2 + 1 = ax3 + (c– a)x2 – (a+c)x – c

Equate coefficients.

When equating the constant coefficients we have +1 = -c, so c = -1.

When equating coefficients for x we have 0 = -(a+c), so c = -a, a = 1

When equating coefficients for x2 we have b = c – a = -1 – 1 = -2.

Thus, **b = -2.**

4) (1984 AHSME #16) The function f(x) satisfies f(2 + x) = f(2 – x) for all real numbers x. If the equation f(x) = 0 has four distinct real roots, what is the sum of those roots?

**Solution**

Let two of those distinct roots be a and b, where both a > 2 and b > 2. If a is a root greater than 2, then we can write f(a) = f(2 + x), so a = 2 + x and x = a – 2. 0 = f(2 + x) = f(2 – x) = f(2 – (a – 2)) = f(4 – a).

Thus, if a is one root, 4 – a is another distinct root.

Similarly, if b is one root, 4 – b is another distinct root.

Thus, the sum of these four roots must be a + (4 – a) + b + (4 – b) = **8.**

5) (1983 AHSME #18) Let f be a polynomial function such that for all real x, . For all real x, what is ?

**Solution**

There exist constants a and b such that

Equate the coefficient for x2, so we have 2 + a = 5, so a = 3.

Equate constant coefficients, so we have a + b + 1 = 3. Since a=3, b=-1. It follows that f(x) = x2 + 3x -1.

Thus,

6) (1999 AHSME #17) Let P(x) be a polynomial such that when P(x) is divided by x – 19, the remainder is 99, and when P(x) is divided by x – 99, the remainder is 19. What is the remainder when P(x) is divided by (x – 19)(x – 99)?

**Solution**

There exists polynomials S(x) and T(x), with T(x) being degree one such that:

P(x) = S(x)(x – 19)(x – 99) + T(x)

We know that P(19) = 99 and P(99) = 19.

P(19) = S(19)(0) + T(19) = 99

P(99) = S(99)(0) + T(99) = 19

We also know that T(x) = ax + b, for constants a and b, since T(x) has to be a function with a degree lower than (x – 19)(x – 99).

T(19) = 19a + b = 99

T(99) = 99a + b = 19

Subtract top equation from bottom to yield 80a = - 80, a = -1, b = 118

Thus, the remainder is **–x + 118.**

7) (1991 AHSME #21) If , for all ,1 and , then what is ?

**Solution**

I need to figure out what to plug into the expression inside of the f to make it equal to x.

So we want to solve for y in terms of x in the equation below.

So,

Now we can solve for :

8) (1986 AHSME #24) Let p(x) = x2 + bx + c, where b and c are integers. If p(x) is a factor of both x4 + 6x2 + 25 and 3x4 + 4x2 + 28x + 5, what is p(1)?

**Solution**

If p(x) is a factor of both polynomials it would also be a factor of 3f(x) – g(x).

3f(x) = 3x4 + 18x2 + 75

-g(x) = 3x4 + 4x2 + 28x + 5

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3f(x)-g(x) = 14x2 – 28x + 70 = 14(x2 – 2x + 5)

p(x) must divide into 14(x2 – 2x + 5) evenly, which means that p(x) = x2 – 2x + 5. It follows that p(1) = 1 – 2 + 5 **= 4.**

9) (2003 AMC A #20) If f(x) = ax3 + bx2 + cx + d and f(-1) = 0, f(0) = 2 and f(1) = 0, what is b?

**Solution**

f(-1) = -a + b – c + d = 0

f(0) = d = 2

f(1) = a + b + c + d = 0

f(-1) = -a + b – c + 2 = 0

f(1) = a + b + c + 2 = 0

Now add the two equations:

f(-1) + f(1) = 2b + 4 = 0. It follows that b = **-2.**

10) (1993 AIME #5) Let P0(x) = x3 + 313x2 – 77x – 8. For integers n ≥ 1, define Pn(x) = Pn-1(x - n). What is the coefficient of x in P20(x)?

**Solution**

Pn(x) = Pn-1(x - n) = Pn-2(x – n – (n – 1)) = … P0(x – (n + (n-1)+…1))

Pn(x) =

So, P20(x) =

Now, we must find the coefficient of x in what is above:

3(210)2 – 2(313)(210) – 77

(630) (210) – (626)(210) – 77 = (210)(630 – 626) – 77 = 4(210) – 77 = 840 – 77 = **763.**

11) (1986 AIME #11) The polynomial 1 – x + x2 – x3 + … + x16 – x17 may be written in the form , where y = x+1 and all the ai’s are constants. Find the value of a2.

Substitute y = x + 1, means that x = y – 1:

f((y-1)) = 1 - (y-1) + (y-1)2 - (y-1)3 + … + (y-1)16 - (y-1)17

Notice that –(y-1) = 1 – y, so we have

f(1-y) = 1 + (1 – y) + (1 – y)2 + (1 – y)3 + … + (1 – y)17

Coefficient of y2 =

Using the hockey stick identity by repeatedly applying Pascal’s Triangle we find that this sum equals

Exercise left to the reader: Use induction on n to prove the following for all integers greater than or equal to a. (Note: a is a fixed integer.)

**Functions – Some Key Points**

1) You can plug in anything you want for x in f(x) to obtain the information that you need.

2) Exploit the fact that x2n = (-x)2n, when evaluating f(x) and f(-x). So, of you are give f(x) and most of the terms have even powers, it helps you find f(-x), since those even powered terms stay the same.

3) When you have two different expressions for the same polynomial, equate coefficients.

4) When there is a vertical line of symmetry, roots come in pairs (except if the line of symmetry contains a root. Thus, the sum of the roots is whatever that line of symmetry is (x = c for some constant c) times the number of roots.

5) When given that the remainder of P(x) divided by x – r is q, this means that P(r) = q.

6) Just like with integers, if a polynomial divides into two other polynomials, it also divides into any linear combination of those two polynomials.