**Distance = Rate x Time Notes**

A common Algebra I topic that many students have trouble with is distance problems. The reason students have trouble is because they want to use their intuition, but occasionally, their intuition leads them astray. It's critical in solving these problems simply to use the formula that is the title of the notes and "think slowly and carefully." In addition, it's important to know ***what the question is asking.*** It's shocking how many students do mathematically correct work but answer the wrong question. In addition, it's important to identify all of the unknowns in a problem and extremely early on, assign variables to all of the relevant unknown quantities. Often times, to simplify work, try to remove variables early on in the process. For example, if the distance to Orlando to Tampa is 90 miles, and you stop somewhere in between and we want to identify the two distances Orlando -> stop, stop -> Tampa, it's probably better to let the first distance equal x, which makes the other distance 90 - x. It's better to directly set this as 90 - x than to set it to y, since it reduces the number of variables in the problem from the get go.

So, to summarize, there are four key tips I recommend for D = RT problems:

(1) Carefully read the problem so that you know what it is asking for.

(2) Don't use your intuition, carefully plug into the formula D = RT or one of its other associated forms. (R = D/T or T = D/R.)

(3) Early in the process, identify all unknown quantities conceptually and then give variable names to some of these (the important ones), being cognizant of trying to limit the number of unique variables that get assigned while capturing all relevant information in the problem.

(4) After solving for the various variables you set up, ***reread*** the full question to make sure you are answering the correct question.

Example #1: Classic Error Prone Example

Here is the most classic example that is prone to errors:

Doris drives from point A to point B averaging 40 miles per hour.
She drives back from point B to point A averaging 60 miles per hour.
What is the average speed of her round trip?

Everyone wants to answer 50 miles per hour, since 50 is the average of 40 and 60. But this is wrong. To see this, note that if we changed it to 0 and 60, 30 wouldn't be the answer. If we averaged 0 mph, we would never make it from point A to point B =) (There are better ways to visualize this, but I hope this extreme case shows the absurdity of taking the numeric averages of the two average speeds.)

So, let's forget about our intuition and set up some variables and then some equations.

Let D = distance between points A and B.
Let t1 = time on the drive from A to B.
Let t2 = time on the drive from B to A.
Let r = the average speed of the round trip.

We have three given pieces of information. Plug into the equation D = RT with these three pieces of info:

For the first two pieces of info, we get these two equations:

D = (40 mph) * t1
D = (60 mph) * t2

The last piece of info is that the round trip had average speed of r. This is the corresponding equation:

2D = r * (t1 + t2)

This is because the round trip has distance 2D and it took time t1 + t2.

So, $t1 = \frac{D}{40} hrs$ and , $t2 = \frac{D}{60} hrs.$

Now, plug these into the third equation:

$2D = r(\frac{D}{40} + \frac{D}{60})$

Factor out D and then divide through by it:

$2D = rD(\frac{1}{40} + \frac{1}{60})$
$2 = r(\frac{1}{40} + \frac{1}{60})$

Now, we can solve for r:

$$r = \frac{2}{\frac{1}{40} + \frac{1}{60}} = \frac{2(60)(40)}{60 + 40} = \frac{2(24)(100)}{100} = 48\ mph$$

It turns out that for two positive real numbers x and y, we define the Harmonic mean/average of the two numbers to be $\frac{2}{\frac{1}{x}+\frac{1}{y}}$. Thus, it turns out that for this specific problem, the average we want of the two speeds isn't the arithmetic mean, but the harmonic mean.

Now, let's take a look at a few examples and just apply our general principles:

Example #2: Split Trip - unknown distances

In this problem we'll have a trip split into two separate portions, where the two portions are driven at different average speeds. This is the same as the problem above. But this time, we'll make the ratio of the distances between the two segments unknown. Here is the problem:

Casey drove a total of 100 miles. For the first portion of the trip she averaged 40 miles per hour and for the second/last portion of the trip she averaged 55 miles per hour. If her average speed for the entire trip was 50 miles an hour, how long (in miles) was the first portion of her trip? Please express your answer as a fraction in lowest terms.

Although the distance of the first portion of the trip is unknown, lots of other stuff is known. It's immediately clear that we should let d = the distance of the first portion of the trip and that the second portion of the trip has distance 100 - d. Now, we can let t1 = the time for the first portion of the trip and t2 = time for the second portion of the trip and t3 = time for the full trip. Since we know the average speed and distance of the whole trip, we get the following equations for the two portions of the trip and the whole trip itself:

d = 40(t1)
100 - d = 55(t2)
100 = 50 (t3)

Also, by definition, we have:

t3 = t2 + t1

Using the last equation, we get t3 = 2. We can also solve for t2 and t1 as follows:

$$t1 = \frac{d}{40}$$

$$t2 = \frac{100-d}{55}$$

Now, just plug into that last equation to get:

$$2 = \frac{d}{40} + \frac{100-d}{55}$$

From here, it’s just algebra for one equation with one variable. Get a common denominator first:

$$2 = \frac{11d}{40(11)} + \frac{8(100-d)}{(8)55}$$

Now, multiply by that denominator:

$$2(40)(11) = 11d + 800 - 8d$$
$$(80)(11) = 3d + 80(10)$$
$$3d = 80(11) - 80(10)$$
$$3d = 80(11 - 10)$$

$3d = 80$
$d = \frac{80}{3}$

So, the key here was identifying three trips (portion1, portion2, whole) and the associated variables. Then, creating each of the relevant equations and attempting to reduce the number of variables as fast as possible. This left us with one equation in one variable, which is ideal. Also, notice some of the algebra above - multiplications were not done. Instead expressions were factored in the hope that stuff would drop out, which it did! This is key for exams w/o calculators.

<u>Example #3: Split Trip - quantities in different units</u>
A classical issue that people sometimes have with D = RT problems is forgetting to convert all the information into the same set of units. Consider the following problem:

Mr. Earl E. Bird leaves his house for work at exactly 8 AM every morning. When he averages 40 miles per hour, he arrives at his workplace three minutes late. When he averages 60 miles per hour, he arrives three minutes early. At what average speed, in miles per hour, should Mr. Bird drive to arrive at his workplace precisely on time?

Notice that the average speeds are given in miles per hour, but the timing information is given in minutes. The difference in time between the two trips is 6 minutes, which is really 1/10 hours. Let the distance from home to work be d. Let t, in hours, be the time for the faster of the two trips. Then t+1/10 represents the time for the slower of the two trips. The two corresponding equations are as follows:

$d = 60t$

$d = 40\left(t + \frac{1}{10}\right) = 40t + 4$

Since both equations showing two different ways to express d, set these two expressions equal to one another:

$60t = 40t + 4$
$20t = 4$
$t = \frac{1}{5} hours = 12\ minutes$

It follows that the correct amount of time it SHOULD take to get to work is 12 + 3 = 15 minutes, which is also 1/4 of an hour. Also, using the first equation (d = 60t), we find that work is 12 miles away. Thus, to solve for the given information, we need to consider a trip of 12 miles (trip to work) which takes 1/4 hours. Solving for the average rate we find:

$12 = r(\frac{1}{4})$
$r = 48mph$

In another solution I typed up for this problem I let t equal the correct time to get to work in hours, instead of the amount of time the faster trip took. This difference doesn't really matter. All that matters is that you set up your equations accurately based on the conceptual values each of your variables represents.

<u>Example #4: Split Trip - Most Necessary Quantities are Known, Read the Question!</u>
Here is a question I created with the intent of being easier than my past examples, but turned out to still be difficult (based on exam average):

Jessica is going from Orlando to Miami and she takes a 15 minute break at Vero Beach. Her goal is to average 60 miles per hour for the whole trip. The distance between Orlando and Vero Beach is 100 miles and the distance between Vero Beach and Miami is 140 miles. If her average driving speed from Orlando to Vero Beach is 50 miles per hour, how fast must her average speed be driving from Vero Beach to Miami to achieve her goal?

The reason I thought this question was easier is because there's lots of known information it in and if you set up everything correctly, you rarely have variables in the solution.

It's a split trip, but both distances are known, as well as one of the average speeds. This means that no variables are involved in the first portion of the trip!!!

First portion is 100 miles with an average speed of 50 miles per hour. So, this portion of the trip takes 2 hours. If you want, you can create the variable t1 to represent the amount of time of the first portion of the trip, or solve directly as I explained here.

This is followed by a 15 minute break in Vero Beach, which is a 1/4 hour break.

Thus, before starting the second leg of the trip, we have spend 2 hours and 15 minutes.

One other piece of information we know is that we WANT to average 60 mph for the whole trip. This is nice because we ALSO know the whole trip is 240 miles. Just like before, we can either directly calculate that the trip ought to take 240/60 = 4 hours, or we can set up a variable t, representing the whole trip time and go from there.

Putting everything together, we have 4 - 2 1/4 = 7/4 hours for the second portion of the trip, which is 140 miles. Again, we can directly solve for the average speed for this portion of the trip by dividing 140 by 7/4, yielding 80 mph, OR we can formally create a variable r for this rate, set up the equation and solve for it.

Many students forgot to incorporate the 15 minute break and others erroneously set up the equation 100/50 = 140/x, setting two ratios that don't necessarily need to be equal (these are the two times). Notice that both lead to the incorrect answer 70 mph, but the former error isn't nearly as bad as the latter error, for which there is no justification, unless you say that both portions of the trip must take 2 hours. Without that, setting the two ratios equal is completely erroneous.

Example #5: No distances involved, but the identical principle is used

Here is a problem from a past exam that I used to TEST understanding of D = RT without there being any distances involved. This question did require a bit of extra understanding of dimensions in geometry, but for the most part, is logically equivalent to other D = RT problems. In fact, real life proliferates with examples that use the same relationship as distance, rate and time in distance problems:

A swimming pool is the shape of a rectangular prism with a length of 12 feet, a width of 10 feet and a depth of 8 feet. The pool is full at Tuesday at 8 am but springs a leak from the bottom of the pool surface that leaks 1 cubic inch of water per second into the ground. (This means that slowly, the water level in the pool decreases.) How much lower (in inches) is the water level at Wednesday morning at 8 am as compared to Tuesday at 8 am when the pool was full? Which piece of information given in the problem is mostly irrelevant?

The corresponding formula that is similar to D = RT that relates to this question IS:

Total Volume Leaked = (Rate of Volume Leaked Per Unit of Time) x (# of Units of Time of Leak)

We don't know the left hand side, but if we are able to figure this out, we could figure out something about the water level. We do know the rate of volume leaked per unit of time, and we also know the number of units of time the leak has been leaking. So we get:

Total Volume Leaked = (1 cubic inch per second) x (24 x 60 x 60 seconds)
= 24 x 60 x 60 cubic inches.

Note that one would have to convert 1 day to 86,400 seconds to get the equation above. Here is the conversion for sake of completeness:

$$1day = 1day \times \frac{24\ hours}{1\ day} \times \frac{60\ minutes}{1\ hour} \times \frac{60\ seconds}{1\ minute} = 86{,}400\ seconds$$

Let the water level drop by x inches. Then the volume of water leaked is ALSO:

V = (x inches)(12 feet)(10 feet) = (x inches)(12 x 12 inches)(10 x 12 inches)
= x($12^3$)(10) inches$^3$

Setting equal the two quantities for V (volume leaked) we get:

24 x 60 x 60 = x($12^3$)(10), Divide both sides by $12^3$ to get:

2 x 5 x 5 = 10x. It follows that x = 5 inches.

Thus, in this problem, we see the same idea as D = RT problems. If we have a rate for something, and repeat it for several units of time, our total is the product of the two. In addition, matching units is extremely important in all of these types of problems.