Introduction Counting Notes

Addition

If the items we are counting can be split into several disjoint groups, then we can add up the number of items in each disjoint group. For example, consider the number of integers in between 1 and 1000 that are either (a) odd and divisible by five, or (b) divisible by 8. How many are there?

The two groups are distinct because a number can't be odd AND be divisible by 8. There are 100 numbers in the first group (5, 15, 25, …, 995) and there are 125 numbers in the second group (8, 16, 24, …, 1000), thus there are a total of 100 + 125 = 225 numbers that fit one of the two criteria.

As a quick note, in the first group, we see that the numbers are of the form 10n + 5 with n ranging from 0 to 99. In the second group, the numbers are of the form 8n with n ranging from 1 to 125. In general, there are b – a + 1 integers in the range [a, b]. ([a, b] means "a to be inclusive", which means that the range includes both a and b.)

In many problems, how to split up the counting is not obvious, the most difficult thing is figuring out which disjoint groups to split the items you are counting into. Also, as this example shows, even when you know what you're counting in a group, doing so isn't necessarily trivial. (An easier problem might be something like, there are 4 green socks, 12 red socks and 18 blue socks in a drawer. How many socks are either green or red? Answer: 4 + 12 = 16.)

Multiplication

If we want to count the number of ordered pairs (x, y) where there are A choices for x and B choices for y, then there are A x B possible ordered pairs. We can see this by listing a table:

Let x be a choice of appetizer from the list: Chips, Fries, Nachos
Let y be a choice of entrée from the list: Burger, Pizza, Spaghetti, Taco

Then the possible meals consisting of one appetizer and one entrée you could have are specified in this table:

App\Ent	B	P	S	T
C	(C, B)	(C, P)	(C, S)	(C, T)
F	(F, B)	(F, P)	(F, S)	(F, T)
N	(N, B)	(N, P)	(N, S)	(N, T)

It's clear that the number of grid squares (minus the row and column labels) is the product of the number of appetizers and number of entrees. In general, the most common error students make in counting is mistaking when to add versus when to multiply. If you can visualize the grid, then you want to multiply.

We can use multiplication to count the number of seven digit phone numbers where the first digit can't be 0 and the last digit can't be 9 as follows:

We have 9 choices for the first digit, 10 choices for digits two through six, and 9 choices for the last digit. Since we can match any first digit with any second digit (and so forth), the total number of valid phone numbers using these rules is 9 x 10 x 10 x 10 x 10 x 10 x 9 = 8.1 million. (Note: Real phone numbers are more restrictive. They can't start with 911, for example.)

Subtraction

Sometimes, it's easier to subtract out the items you don't want from what you want (from the whole) to count than to add the ones you do want to count. Consider he following problem:

How many integers in between 0 and 999 have either the digit 5 or 7?

We could start trying to list some of these: 5, 7, 15, 17, etc. This gets tricky pretty quickly though…we get to 50, 51, …, which breaks the pattern, and so on.

Consider the opposite question: How many integers in between 0 and 999 do NOT have the digit 5 and do NOT have the digit 7? Essentially, we are forming 3 digit integers (leading 0s allowed) with 8 possible digits: 0, 1, 2, 3, 4, 6, 8, and 9.

Using the multiplication, principle, we have 8 choices for each digit, so there are a total of 8^3 integers from 0 to 999 that don't have 5 as a digit and don't have 7 as a digit. It follows that the number of integers that have at least one of those digits is $1000 - 8^3 = 488$.

Inclusion-Exclusion Principle

Sometimes when you split your counting into two different groups, some items appear in both groups. This means that if we add, each of these items that were in both groups were counted twice. We can fix the mistake by subtracting out each item that was counted twice. In essence, we are "fixing" the case where the counting split into two groups wasn't a disjoint split.

Here is a sample question that uses this idea: How many integers in between 1 and 1000 inclusive are divisible by 4 or 5?

There are 1000/4 = 250 integers divisible by 4. (Note: this is integer division.)
There are 1000/5 = 200 integers divisible by 5. (Note: this is integer division.)

But, we've counted integers that are divisible by both 4 and 5 twice. To be divisible by both 4 and 5, you have to be divisible by their least common multiple, 20.

There are 1000/20 = 50 integers divisible by 4 and 5 (ie 20).

It follows that there are 250 + 200 – 50 = 400 integers from 1 to 1000 that are divisible by either 4 or 5.

Division
Consider attempting to count something, but realizing that you counted the same thing multiple times. Many times, this is an unfixable issue. (If I count something once, other things twice and a few things three times, etc. in a non-systematic way, it's pretty difficult to fix the issue.) But occasionally, if you counted each item the same number of times, we can "fix" the answer by dividing.

Consider the following problem:

Mrs. Steinbeck's class has 25 students. A pair of students will represent the class for the school's student council. How many pairs of students can be chosen?

We have 25 options for the first student. Then, for the second student, since one has already been chosen, we only have 24 options. It seems like the answer might be 25 x 24 = 600. But the issue here is that we counted the ordered pair (Alice, Bob) and later counted the ordered pair (Bob, Alice). Thus, the same pair of students was counted twice, once for each order of selection. Luckily, this same error occurred for **every pair!** Thus, we can fix the answer just by taking 600 and dividing it by 2 to get 300 possible pairs of students.

Thus, any time we can prove that we got an answer of X by counting each item we wanted to count exactly Y times, the actual answer is X/Y.

Permutations
Imagine the following problem:

n students in a class are to line up. How many orders can all n students line up?

We have n choices for the front of the line.
n-1 choices left for the student to go second.
n-2 choices left for the student to go third.
…
1 choice for the last student

Each choice is independent of the previous choice and can be paired with any of the previous choices, thus, we want to multiply each of these numbers. (In general, we can multiply as long as our previous choice doesn't change the number of choices we have for the subsequent slot. But if choosing Alice leaves us 8 choices for the next slot and choosing Bob leaves us 6 choices for the next slot, we can't use the multiplication principle.)

In this case, the answer is $n \times (n-1) \times (n-2) \ldots \times 2 \times 1$.

This product is so common it's given the notation n! (read "n factorial.")

Similarly, if we only want to line up k of the n students in line, we would get the product:

$n \times (n - 1) \times (n - 2) \times \ldots \times (n - k + 1)$

(Note: be careful with your counting to get the last term…since the first term is $n - 0$, the last term has to be $n - (k - 1) = n - k + 1$.)

This more general term also has the notation ${}_nP_k = n \times (n-1) \ldots \times (n-k+1) = \frac{n!}{(n-k)!}$.

At first, the formula given might not seem obvious. But, try this with $n = 9$ and $k = 4$:

$$\frac{9!}{(9-4)!} = \frac{9 \times 8 \times 7 \times 6 \times \mathbf{5 \times 4 \times 3 \times 2 \times 1}}{\mathbf{5 \times 4 \times 3 \times 2 \times 1}}$$

Notice that in the factorial notation, all the terms in red cancel, leaving exactly the product $9 \times 8 \times 7 \times 6$, which is the original formula we derived. By hand, if we had to get a number, we would write the original definition. If we use this as a portion of a problem to do more algebra with, then the factorial form of the formula might be more useful.

Finally, we might have a situation were we are permuting all of n objects, but some of the objects are repeated. For example, consider permuting all of the letters in the word ADDING. If the two Ds were distinguishable, then there would simply be 6! permutations. We can envision these permutations as follows, where each D has a subscript, so we can temporarily distinguish it.

AD_1D_2ING
AD_2D_1ING
…

But note that the two permutations written above are really both the same permutation since everything is the same except for swapping the two Ds. Thus, this permutation (and every other one), is counted precisely twice! More generally, if there were 3 copies of a letter in a word we were permuting, then each ordering of those three letters (there are 3! such orderings) would represent the number of times each permutation in the original count was counted. Thus, for this situation, the final answer is $\frac{6!}{2!}$. More generally, the answer is $\frac{n!}{n_1!n_2!\ldots n_k!}$, where each n_i is the number of times object i appears in the original list of letters.

For example, the number of permutations of TENNESSEE is $\frac{9!}{4!2!2!1!}$. If one had to figure this out by hand, write out each of the numbers in the product and start canceling:

$$\frac{9 \times \mathbf{8} \times 7 \times 6 \times 5 \times \mathbf{4 \times 3 \times 2 \times 1}}{(\mathbf{4 \times 3 \times 2 \times 1})(\mathbf{2})(\mathbf{2})} = \mathbf{9 \times 2 \times 7} \times 6 \times \mathbf{5} = \mathbf{630} \times 6 = \mathbf{3780}$$

Combinations
A combination is a set of objects chosen out of a whole, where the order of what's selected doesn't matter. For example, if one were to choose exactly 2 candies from the following list:

Snickers
Hersheys
Twix
M+Ms
Skittles

Then here are the following possible combinations:

1. Snickers + Hersheys
2. Snickers + Twix
3. Snickers + M+Ms
4. Snickers + Skittles
5. Hersheys + Twix
6. Hersheys + M+Ms
7. Herskeys + Skittles
8. Twix + M+Ms
9. Twix + Skittles
10. M+Ms + Skittles

Notice that we don't list Hersheys + Snickers, because as a combination, this is the same set of items as Snickers + Hersheys.

We previously covered the number of ways to permute k items out of n. It is

${}_nP_k = n \times (n-1) \ldots \times (n-k+1) = \frac{n!}{(n-k)!}$.

The only difference here is that each ordering of each combination is counted. So, if we can find out how many different orderings there are of each combination of k items, we can just divide by that number, because every combination of k items is being counted exactly that many times.

But this is pretty easy, the number of ways to order k items is k! Thus, if we take ${}_nP_k$ and divide it by k!, we'll get the **the number of ways to choose k items out of n**. In our previous example, each ordering of two items would have been counted twice. Similarly, if we had been allowed to choose 3 candies out of 5, our ordered list would have 5 x 4 x 3 = 60 lists, but each unique list appears on the ordered lists 3 x 2 x 1 = 6 times, so there are really 60/6 = 10 combinations of 3 items out of 5.

Combinations are so common they have two ways in which they are denoted: ${}_nC_k$ and $\binom{n}{k}$ are the two main ways to represent the number of ways to choose k items out of n. The formula is:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

One quick thing to note is that $\binom{n}{k} = \binom{n}{n-k}$. Combinatorially, each way to choose k items out of n is in one to one correspondence with each way to "not choose" n – k items out of n.

For example, the combination of Snickers + Hersheys is in one to one correspondence with the rest of the items NOT selected: Twix + M+Ms + Skittles. Thus, the two sets (all subsets of size 2 and all subsets of size 3, more generally k and n – k) must be the same size.

Algebraically, it should be easy to see that $\binom{n}{n-k} = \frac{n!}{(n-k)!(n-(n-k))!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}$. Namely, when we do the algebra, the two factorials essentially switch places.

Combinations appear in many, many problems in both direct and indirect ways.

A direct question would be: A class has 25 students. Three students must be selected as class representatives. How many ways can the representatives be selected. The answer is just $\binom{25}{3}$.

If we want to work this out by hand, notice that the 25! on the top will cancel with the (25 – 3)! on the bottom. All but 3 of the terms will cancel, leaving the following:

$$\frac{25 \times 24 \times 23}{1 \times 2 \times 3} = 25 \times 4 \times 23 = 2300$$

Thus, to calculate $\binom{n}{k}$, if k > n/2, rewrite as $\binom{n}{n-k}$. Then, the "bottom" number will be "smaller." In the numerator, multiply starting from n, counting down by 1, k terms. In the denominator, start at 1, multiplying upto k. Here is one more example:

$$\binom{10}{6} = \binom{10}{4} = \frac{10 \times 9 \times 8 \times 7}{1 \times 2 \times 3 \times 4} = 10 \times 3 \times 7 = 210$$

Note: I canceled 8 with 2x4, and then divided 9 by 3.

An indirect question is as follows:

How many initials of first, middle and last name starting letters are distinct letters in alphabetical order?

If we think about it, any choice of 3 letters out of 26 matches to exactly 1 set of initials. For example, if we choose M, E and H, then the only initials in alphabetic order this corresponds to is EHM. Thus, the answer to this question is just $\binom{26}{3}$.

Combinations also appear in Pascal's Triangle:

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            1
         1     1
      1     2     1
   1     3     3     1
1     4     6     4     1
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The outsides are 1s and to get each entry in the inside, add the two items directly above it to get it. If we treat the top row as row 0, and the first entry in each row as column 0, then the entry in row n, column k of this triangle is $\binom{n}{k}$. By the way its constructed, we see that

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

This can be proven both combinatorially and algebraically.

Here is the combinatoric proof:

Imagine counting the # of ways to select k items out of n. The LHS already has this quantity. Now, let's count this quantity in a different way, splitting our counting into two disjoint sets:

(1) All the subsets of size k that DO HAVE item 1
(2) All the subsets of size k that DO NOT have item 1.

(1) For this group, we are forced to choose item 1. This means we can only choose k – 1 items left out of the remaining n – 1 items. We can do this in $\binom{n-1}{k-1}$ ways.

(2) for this group, we can NOT select item 1. This means we have to choose all k items out of the remaining n – 1 items. We can do this in $\binom{n-1}{k}$ ways.

Since (1) and (2) are disjoint, adding them yields all the ways to choose k items out of n, because all sets of k items out of n either have item 1 or don't have item 1.

This identity can often times be useful in solving problems. More importantly, understanding this classic combinatorial proof opens the door to create and understand other combinatorial proofs.