More Examples of Set Proofs/Disproofs

1. If $A \subseteq B$, then $(A \cap C) \subseteq (A \cap B)$.

<u>Proof #1</u>

Essentially, we are being asked to prove that $(A \cap C) \subseteq (A \cap B)$.

Let x be an arbitrarily chosen element of $A \cap C$. We must show that $x \in A \cap B$.

Using our given information, we have that $x \in A \cap C$. From this we can deduce that $x \in A$ and $x \in C$. Now, we are given that $A \subseteq B$. Thus, by the definition of a subset and the fact that $x \in A$, we can deduce that $x \in B$. But, this means that $x \in A \cap B$, using our given information again.

Here is this proof written out in "chart" format:

We need to show that $(A \cap C) \subseteq (A \cap B)$.
Thus, we must show if $x \in A \cap C$ then $x \in A \cap B$.

1) $x \in A \cap C$	**Given**
2) $x \in A \wedge x \in C$	**Defn of $\cap$**
3) $x \in A \wedge x \in B \wedge x \in C$	**Given info + defn of $\subseteq$**
4) $x \in A \cap B$	**Defn of $\cap$**

<u>Proof #2</u>

Clearly, $(A \cap B) \subseteq A$, since all elements of the set on the right must be elements of both A and B. Furthermore, we have $A \subseteq (A \cap B)$. To see this, consider an arbitrary $x \in A$. We must show that $x \in (A \cap B)$. But since $A \subseteq B$, by the definition of subset and

the fact that $x \in A$, we can conclude that $x \in B$. But if both of these two are true, then we have $x \in (A \cap B)$.

Thus, we can conclude that $(A \cap B) = A$.

Now, we must show that $(A \cap C) \subseteq A$. For all $x \in A \cap C$, we have $x \in A$. Thus, we have shown that $(A \cap C) \subseteq A = A \cap B$, so $(A \cap C) \subseteq (A \cap B)$.

2. If $B = (A \cap B)$, then $B \subseteq A$.

<u>Proof #1</u>

We must show that if an arbitrary $x \in B$, then $x \in A$. We are given that $B = (A \cap B)$. Using this information, we know that $x \in B$ iff $x \in A \cap B$. Thus, we can conclude that $x \in A \cap B$. But this implies that $x \in A$, as we wanted to show.

Here is the proof written out in "chart" format:

We must show that $B \subset A$.
That is, if $x \in B$ then $x \in A$.

1) $x \in B$	**Given**
2) $x \in A \cap B$	**Since $A = A \cap B$ in given stmt.**
3) $x \in A$	**Defn of $\cap$**

Proof #2
We can show the contrapositive:

If $B \subseteq A$ is false, then $B \neq (A \cap B)$.

If $B \subseteq A$ is false, then we know there exists an element x such that $x \in B$ and $x \notin A$. If this is the case, then we must have that $x \notin A \cap B$, by the definition of intersection. Since we know that $x \in B$ and $x \notin A \cap B$, we can conclude that these two sets are NOT equal.

Proof #3
You can actually use a membership table to prove this:

A	B	$A \cap B$
0	0	0
0	1	0
1	0	0
1	1	1

Now, from the table you have to argue the proof in English. The given information is that $B = (A \cap B)$. Since this is the case, we can ONLY have elements in certain "parts" of the Venn Diagram so to speak. In particular, we can have NO elements designated by the 2nd row of the chart, since here the values of B and $(A \cap B)$ differ. Thus, when the given information is correct, all elements in the universe belong into the categories specified by rows 1, 3 and 4. If we look at these three rows, what we see is that if an element is present in the set B, in MUST BE present in the set A. Thus, in the situation described above, $B \subseteq A$.

3. If $(A \cup B) \subseteq (C \cup D)$,
then $((A \subseteq C) \vee (A \subseteq D) \vee (B \subseteq C) \vee (B \subseteq D))$

False. Counter Example: A= {1,2}, B={3,4}, C={1,3}, D={2,4}.

4. If $A \cap B = \varnothing$, then $B \subseteq \neg A$.

If $A \cap B = \varnothing$, for all possible x, $x \notin A \cap B$. Thus, we have $x \in \neg(A \cap B)$, which means $x \in (\neg A \cup \neg B)$, so either $x \in \neg A \vee x \in \neg B$.

Now, we must prove that if $x \in B$, then $x \in \neg A$. But, if we have $x \in B$, we know that $x \notin \neg B$. Since we know $x \in \neg A \vee x \in \neg B$ to be true, we have to conclude that $x \in \neg A$, as desired.

Here is this one in "chart" format:

We must show that $B \subset \neg A$.
That is, we must show if $x \in B$ then $x \in \neg A$.

1) $x \in B$	**Given**
2) $x \notin A$	**Using Given($A \cap B = \varnothing$) if $x \in A$, then $A \cap B$ would be empty.**
3) $x \in \neg A$	**Defn of $\neg$.**

5. If $A \times B \subseteq A \times C$, then $B \subseteq C$.

This is false. Consider $A = \varnothing$. Then we must have $A \times B = A \times C = \varnothing$, regardless of what B and C are. Now, for a counter example, set B={1} and C={2}.

6. If $A \subseteq B$, prove that $(C - B) \cap A = \varnothing$.

We must show that there does not exist an element x such that x$\in(C - B) \cap A$.

Use proof by contradiction. Assume that such an element x exists. Then we have the following:

x$\in(C - B)$ and x$\in$A. This simply means, by defn of $-$ that

x$\in$C $\wedge$ x$\in\neg$B $\wedge$ x$\in$A, which implies:

x$\in$C $\wedge$ x$\notin$B $\wedge$ x$\in$A.

BUT, this contradicts our assumption that A $\subset$ B.

7. Let A, B, and C be any three sets. Prove the following:
$(C - (A \cup B)) \cup (B \cap C) \cup (A \cap C) = C$.

Probably the easiest way to show this is through a set table:

A	B	C	$A\cup B$	$C - (A\cup B)$	$B\cap C$	$A\cap C$	Whole LHS
0	0	0	0	0	0	0	0
0	0	1	0	1	0	0	1
0	1	0	1	0	0	0	0
0	1	1	1	0	1	0	1
1	0	0	1	0	0	0	0
1	0	1	1	0	0	1	1
1	1	0	1	0	0	0	0
1	1	1	1	0	1	1	1

Checking columns 3 and 8, we see that the sets are equivalent.

The Set Laws will work here as well:

$(C - (A \cup B)) \cup (B \cap C) \cup (A \cap C)$

$= (C \cap \neg(A \cup B)) \cup (B \cap C) \cup (A \cap C)$ (defn of $-$)

$= (C \cap \neg(A \cup B)) \cup (C \cap (A \cup B))$ (distributive law)

$= C \cap [\neg(A \cup B) \cup (A \cup B)]$ (distributive law)

$= \mathrm{C} \cap \mathrm{U}$ (inverse law)

$= \mathrm{C}$ (identity law)

8. Let A, B and C denote sets and suppose that $A - B \subseteq C$ and $A \subseteq C$ is false, (A is NOT a subset of C). Prove or disprove that $A \cap B \neq \varnothing$.

Proof by contradiction. Assume that $A - B \subseteq C$ and $A \subseteq C$ is false, but $A \cap B = \varnothing$. If A and B are disjoint, then any element of A does not belong to B, which means that $A \subseteq A - B$.
Then $A \subseteq A - B$ and $A - B \subseteq C$ imply $A \subseteq C$, in contradiction to $A \subseteq C$.
Since the assumption that $A - B \subseteq C$ and $A \subseteq C$, but $A \cap B = \varnothing$ results to contradiction, we can conclude that if $A - B \subseteq C$ and $A \subseteq C$, then $A \cap B \neq \varnothing$.

Direct proof. Assume that $A - B \subseteq C$ and $A \subseteq C$ is false. To prove that $A \cap B \neq \varnothing$ it is sufficient to show that there exists at least one element that belongs to both sets, A and B.
Since $A \subseteq C$ is false, there exists some element $x \in A$ and $x \notin C$.
Since $A - B \subseteq C$, we can imply that $x \in B$, otherwise it would belong to C. So, we found an element $x \in A$ and $x \in B$. This completes the proof that $A \cap B \neq \varnothing$.

9.Let A, B, and C denote 3 sets. Prove that if $A \cup C = B \cup C$ and $A \cap C = B \cap C$, then $A = B$. (Hint: Let $x \in A$, then $x \in A \cup C$ is true.)

We first prove $A \subseteq B$. Let $x \in A$ ---- (1), we need to prove $x \in B$ ---- (2). If $x \in A$, clearly, $x \in A \cup C$. Thus, $x \in A \cup C = B \cup C$. Thus, $x \in B$ or $x \in C$. There are two cases:

(Case one) Suppose $x \in B$. In this case, we have proven what we need to.

(Case two) Suppose $x \in C$. In this case, (1) implies that $x \in A \cap C = B \cap C \subseteq B$. Thus, in this case we also have that $x \in B$, as desired.

Therefore, we proved (2) in both cases.

The proof for $B \subseteq A$ is similar, due to the symmetry of A and B. Therefore, we proved $A = B$.

10. Suppose A, B and C are sets.
Prove that $A - (B - C) \subseteq (A - B) \cup C$.

We must prove the following:

Let x be an arbitrary element from the set A – (B – C).
We need to prove that x $\in (A - B) \cup C$.

$A - (B - C)$	$= A - (B \cap \neg C)$	**defn of –**
	$= A \cap \neg(B \cap \neg C)$	**defn of –**
	$= A \cap (\neg B \cup \neg\neg C)$	**DeMorgan's**
	$= A \cap (\neg B \cup C)$	**Law of Double Complement**
	$= (A \cap \neg B) \cup (A \cap C)$	**Distributive**
	$= (A - B) \cup (A \cap C)$	**defn of –**

This shows that iff $x \in A - (B - C)$, $x \in (A - B) \cup (A \cap C)$.

We can now break our proof down into two parts:

1) Prove if $x \in (A - B)$, then $x \in (A - B) \cup C$.
2) Prove if $x \in (A \cap C)$, then $x \in (A - B) \cup C$.

To show 1, notice, that by the definition of union, we MUST have $x \in (A - B) \cup C$ if $x \in (A - B)$.

To show 2, notice, that by the definition of intersection, we MUST have $x \in C$ if $x \in (A \cap C)$. If this is the case, then by definition of union, clearly $x \in (A - B) \cup C$.

Thus, we have shown that for any arbitrary x such that $x \in A - (B - C)$, that $x \in (A - B) \cup C$. It follows that $A - (B - C) \subseteq (A - B) \cup C$.