Cartesian Products

A Cartesian product is defined as follows:

$A \times B = \{ (a,b) \mid a \in A \wedge b \in B \}$

This is a set of ordered pairs, hence the order here matters.

If we want the size of the Cartesian Product of two sets, we can get it as follows:

$|A \times B| = |A| \times |B|$.

One way to see that this is the case is to list out all the elements in a Cartesian product in a table. Label all the rows with elements from the set A, and all the columns with the elements from the set B. Each cell in this table will contain an unique element of the Cartesian product of A and B. Furthermore, each element of the Cartesian product can be found on the table. Thus, the total number of elements in the Cartesian product is the total number of cells in the table, $|A| \times |B|$.

Here is a problem involving cartesian products:

If $A \subseteq C$ and $B \subseteq D$, then $A \times B \subseteq C \times D$.

We must show that if $(x,y) \in A \times B$, then $(x,y) \in C \times D$.
Consider an element $(x,y) \in A \times B$.
By definition of a cartesian product, we must have that $x \in A$ and $y \in B$.
Using our given subset information, we can deduce that $x \in C$ and $y \in D$.
Finally, considering the definition of a cartesian product, we have that $(x,y) \in C \times D$, as desired.

Power Sets

A power set of a set A is defined as the set of all possible subsets of that set. So, for example, if a set A = {2,3,5}, then the power set of A, which we will denote as power(A) (in the book they use some fancy letter in italics) will be the following set:

power(A) = { ∅, {2}, {3}, {5}, {2,3}, {2,5}, {3,5}, {2,3,5} }

Thus it is a set of sets. Now, using the result we derived earlier, we find that

$|\text{power}(A)| = 2^{|A|}$.

Here is an example of an identity and proof dealing with power sets:

Prove that Power(A) ∪ Power(B) ⊆ Power(A ∪ B).

Let the set A' be an arbitrarily chosen subset of Power(A) and let the set B' be an arbitrarily chosen subset of Power(B). We must now show that

A' ∈ Power(A ∪ B) and that B' ∈ Power(A ∪ B)

Any arbitrarily chosen subset of Power(A) only contains elements from the set A. But we know that Power(A ∪ B) contains all subsets comprised of elements from the set A ∪ B. In particular, it contains every subset comprised of elements from the set A. But, A' must be one of these subsets. Hence, we have shown that A' ∈ Power(A ∪ B). The proof that B' ∈ Power(A ∪ B) is analogous to the proof above.

Here is a another problem using power sets:

For arbitraray sets A and C from a given universe, show that if $A \subseteq C$, then $Power(A) \subseteq Power(A \cup C)$

If $A \subseteq C$, then $A \cup C = C$, so we just need to show $Power(A) \subseteq Power(C)$. Consider any element of Power(A). It must be a subset with elements from A. But, this element MUST be contained in Power(C), since this set contains all subsets of elements in C. One of these subsets will contain exactly the desired elements from A, since $A \subseteq C$.

Here is a slightly different proof in chart format:

We must show that $Power(A) \subseteq Power(A \cup C)$.
If $X \in Power(A)$, then $X \in Power(A \cup C)$.

1) $X \in Power(A)$	**Given**
2) $X \subset A$	**Defn of Power Set**
3) $A \subset A \cup C$	**Defn of $\cup$**
4) $X \subset A \cup C$	**Transitivity of $\subseteq$ (This can be proved in a few steps.)**
5) $X \in Power(A \cup C)$	**Defn of Power Set**

From the intuition used in this problem, we can essentially claim a more general result:

If $A \subseteq B$, then we have $Power(A) \subseteq Power(B)$.

The Inclusion-Exclusion Principle

Let A and B denote two finite sets. Then, we have:

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

This can easily be seen by a Venn Diagram:

Logically, we can argue that since each element of $A \cup B$ belongs to either A or B, the sum $|A| + |B|$ includes a count for each of the elements of $A \cup B$, but those elements of $A \cap B$ are counted twice.

Thus, $|A| + |B| - |A \cap B|$ counts each element of $A \cup B$ exactly once, that is, it is equal to $|A \cup B|$.

Here is a more rigorous proof of the inclusion-exclusion principle:

We first claim that the following is a disjoint union, meaning that the two sets on the right of the equal sign have no elements in common.

$$A = (A - B) \cup (A \cap B)$$

Thus, by the definition of set equality, we want to prove that

1. $A \subset (A - B) \cup (A \cap B)$
2. $(A - B) \cup (A \cap B) \subset A$
3. $(A - B) \cap (A \cap B) = \varnothing$

To prove 1, let $x \in A$. Either $x \in B$ or $x \notin B$. In the first case, $x \in A \cap B$ by definition, and in the second case, we have $x \in A$ and $x \notin B$, which means $x \in A - B$ by definition.

To prove 2, note that $A - B \subset A$ because each $x \in A - B$ must also have $x \in A$ by the definition of set difference. Also, $A \cap B \subset A$ because each $x \in A \cap B$ must also have $x \in A$ by the definition of intersection. Thus, $(A - B) \cup (A \cap B) \subset A$ by the definition of set union and the subset relationship.

To prove 3, note that each $x \in A - B$ must satisfy $x \notin B$ by the definition of set difference. Also, each $x \in A \cap B$ must satisfy $x \in B$, by the definition of set intersection. Thus, it is impossible to have $x \in (A - B) \cap (A \cap B)$, meaning that the set is empty.

If we swap A and B in the formula $A = (A - B) \cup (A \cap B)$, we find the following:

$$B = (B - A) \cup (B \cap A)$$

Applying the Sum Principle we have the following:

$$|A| = |A - B| + |A \cap B|$$

and $$|B| = |B - A| + |B \cap A|$$

Now, we can find another disjoint union. We have already shown that $(B \cap A)$ is disjoint from the other two sets, and $A - B$ and $B - A$ can not share any elements at all since the first only has elements from A and the second only elements from B.

$$\mathbf{A \cup B = (A - B) \cup (B - A) \cup (B \cap A)}$$

which implies the following equation using the Sum Principle:

$$\mathbf{|A \cup B| = |A - B| + |B - A| + |B \cap A|}$$

Adding the equations from the top of the page we get:

$$\mathbf{|A| + |B| = |A - B| + |A \cap B| + |B - A| + |B \cap A|}$$, **so**

$$\mathbf{|A| + |B| - |B \cap A| = |A - B| + |A \cap B| + |B - A|}$$

But, we know that

$$\mathbf{|A \cup B| = |A - B| + |B - A| + |B \cap A|}$$

Thus, we have:

$$\mathbf{|A \cup B| = |A| + |B| - |A \cap B|.}$$

Inclusion-Exclusion Principle for 3 Sets

Let A, B, and C denote three finite sets. Then, we have:

$$
\begin{aligned}
|A \cup B \cup C| = {} & |A| + |B| + |C| \\
& - |A \cap B| - |B \cap C| - |A \cap C| \\
& + |A \cap B \cap C|.
\end{aligned}
$$

Applying the previous theorem to sets A and $(B \cup C)$, we have

$$
\begin{aligned}
|\mathrm{A} \cup \mathrm{B} \cup \mathrm{C}| &= |\mathrm{A} \cup (\mathrm{B} \cup \mathrm{C})| \\
&= |\mathrm{A}\,| + |\mathrm{B} \cup \mathrm{C}| - |\mathrm{A} \cap (\mathrm{B} \cup \mathrm{C})|
\end{aligned}
$$

Note that $|\mathrm{B} \cup \mathrm{C}| = |\mathrm{B}| + |\mathrm{C}| - |\mathrm{B} \cap \mathrm{C}|$

And note that $|\mathrm{A} \cap (\mathrm{B} \cup \mathrm{C})| = |\,(\mathrm{A} \cap \mathrm{B}) \cup (\mathrm{A} \cap \mathrm{C})\,|$, by the distributive law, so

$$
\begin{aligned}
|\mathrm{A} \cap (\mathrm{B} \cup \mathrm{C})| &= |\,(\mathrm{A} \cap \mathrm{B}) \cup (\mathrm{A} \cap \mathrm{C})\,| \\
&= |A \cap B| + |A \cap C| - |(A \cap B) \cap (A \cap C)| \\
&= |A \cap B| + |A \cap C\,| - |A \cap B \cap C|
\end{aligned}
$$

Thus, substituting into the above equation we find:

$$
\begin{aligned}
|\mathrm{A} \cup \mathrm{B} \cup \mathrm{C}| &= |\mathrm{A}\,| + |\mathrm{B} \cup \mathrm{C}| - |\mathrm{A} \cap (\mathrm{B} \cup \mathrm{C})| \\
&= |\mathrm{A}\,| + |\mathrm{B} \cup \mathrm{C}| - (|\mathrm{A} \cap \mathrm{B}| + |\mathrm{A} \cap \mathrm{C}\,| - |\mathrm{A} \cap \mathrm{B} \cap \mathrm{C}|) \\
&= |\mathrm{A}\,| + |\mathrm{B} \cup \mathrm{C}| - |\mathrm{A} \cap \mathrm{B}| - |\mathrm{A} \cap \mathrm{C}\,| + |\mathrm{A} \cap \mathrm{B} \cap \mathrm{C}| \\
&= |\mathrm{A}| + |\mathrm{B}| + |\mathrm{C}| - |\mathrm{B} \cap \mathrm{C}| - |\mathrm{A} \cap \mathrm{B}| - |\mathrm{A} \cap \mathrm{C}\,| + {} \\
&\quad\ |\mathrm{A} \cap \mathrm{B} \cap \mathrm{C}|
\end{aligned}
$$

Let's use the Inclusion-Exclusion Principle to deal with a counting problem involving sets:

David owns a box full of blocks which come in two colors(red,blue), two sizes(small, large), and two weights(light, heavy). He owns each possible combination of block. The total number of blocks that are red or small or light is 25. Of these, exactly 13 are small, 5 are both small and red, and 3 are red, small and light. Also, exactly 20 blocks are either red or light. But only 7 blocks are red and light. There is a total of 14 red blocks. Finally, of all the blocks 18 are not light. Find the following pieces of information:

1) Total number of blocks that are either red or small
2) Total number of light blocks
3) Total number of blocks that are small and light

Let A be the set of red blocks, B be the set of small blocks, and C be the set of light blocks.

Using the given information, we have:

$|A \cup B \cup C| = 25$
$|B| = 13$
$|A \cap B| = 5$
$|A \cap B \cap C| = 3$
$|A \cup C| = 20$
$|A \cap C| = 7$
$|A| = 14$

Using the Inclusion-Exclusion Principle with sets A and B, we have

$$
\begin{aligned}
|A \cup B| &= |A| + |B| - |A \cap B| \\
&= 14 + 13 - 5 \\
&= 22
\end{aligned}
$$

Thus there are 22 blocks that are either red or small.

Using the Inclusion-Exclusion Principle again,

$$
\begin{aligned}
|A \cup C| &= |A| + |C| - |A \cap C| \\
20 \quad &= 14 + |C| - 7
\end{aligned}
$$

So, $|C| = 13$, the total number of light blocks.

Now, apply the Inclusion-Exclusion Principle for three sets:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$25 = 14 + 13 + 13 - 5 - 7 - |B \cap C| + 3$$

$|B \cap C| = 6$, total number of small light blocks.